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Rank restricting functions

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Abstract

In this paper we characterize, for given positive integers k and d, the class of functions $f:\mathbb{R}\to\mathbb{R}$ such that for every $n\times m$ real-valued matrix $A=(a_{ij})_{i=1}^{n}{}_{j=1}^{m}$ (arbitrary n and m) of rank at most k, the matrix $f(A)=(f(a_{ij}))_{i=1}^{n}{}_{j=1}^{m}$ has rank at most d, as well as the class of functions $g:\mathbb{C}\to\mathbb{C}$ such that for every $n\times m$ complex-valued matrix $A=(a_{ij})_{i=1}^{n}{}_{j=1}^{m}$ (arbitrary n and m) of rank at most k, the matrix $g(A)=(g(a_{ij}))_{i=1}^{n}{}_{j=1}^{m}$ has rank at most d. For $k\geqslant 2$ each such function f is a polynomial of an appropriate form which we shall exactly delineate, while each g is a polynomial in z and $\bar{z}$, also of an explicitly delineated form. For $k=1$ the class of such functions, in each case, is significantly different. Nonetheless it is via the study of the case $k=1$ that we are able to characterize such functions where $k\geqslant 2$.

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1. Introduction

In this paper we prove two main results, one for real-valued functions, the other for complex-valued functions.

Let f denote a real-valued function defined on all of $\mathbb{R}$. We say that f *takes matrices of rank k to matrices of rank d* if for every real-valued matrix $A=(a_{ij})_{i=1}^{n}{}_{j=1}^{m}$

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(arbitrary n and m) of rank at most k, the real-valued matrix $f(A) = (f(a_{ij}))_{i=1}^{n}{}_{j=1}^{m}$ has rank at most d. We similarly let g denote a complex-valued function defined on $\mathbb{C}$, and say that *g takes matrices of rank k to matrices of rank d* if for every complex-valued matrix $A = (a_{ij})_{i=1}^{n}{}_{j=1}^{m}$ (arbitrary n and m) of rank at most k, the complex-valued matrix $g(A) = (g(a_{ij}))_{i=1}^{n}{}_{j=1}^{m}$ has rank at most d.

The two main results of this paper are the following.

Theorem 1. *Assume $k \geqslant 2$, and $f : \mathbb{R} \to \mathbb{R}$ is measurable. Then f takes matrices of rank k to matrices of rank d if and only if*

$$f(t) = \sum_{\ell=1}^{p} \alpha_\ell t^{m_\ell},$$

where the m_ℓ are distinct nonnegative integers, $\alpha_\ell \neq 0$, $\alpha_\ell \in \mathbb{R}$, $\ell = 1, \ldots, p$, and

$$\sum_{\ell=1}^{p} \binom{k + m_\ell - 1}{m_\ell} \leqslant d.$$

The characterization in the case $k = 1$ is somewhat different. The class of functions is substantially larger and is given explicitly in Theorem 5. It is essentially all appropriate solutions of an Euler equation of degree d on all of $\mathbb{R}$. Despite the difference in the character of the solutions for $k = 1$ and $k \geqslant 2$, the analysis of the case $k = 1$ will be used to prove Theorem 1.

Theorem 2. *Assume $k \geqslant 2$, and $g : \mathbb{C} \to \mathbb{C}$ is continuous. Then g takes matrices of rank k to matrices of rank d if and only if*

$$g(z) = \sum_{\ell=1}^{p} \beta_\ell z^{m_\ell} \bar{z}^{n_\ell},$$

where the (m_ℓ, n_ℓ) are distinct pairs of nonnegative integers, $\beta_\ell \neq 0$, $\ell = 1, \ldots, p$, and

$$\sum_{\ell=1}^{p} \binom{k + m_\ell - 1}{m_\ell} \binom{k + n_\ell - 1}{n_\ell} \leqslant d.$$

We also characterize all $g : \mathbb{C} \to \mathbb{C}$ which take matrices of rank 1 to matrices of rank d. Here the class of functions is significantly different and larger. This result may be found in Theorem 8. The proof of Theorem 2 depends upon this result.

This paper is organized as follows. In Section 2 we consider the case $k = 1$ for real f. In Section 3 we prove Theorem 1. The complex case is considered in Section 4.

There is a fairly extensive literature on operators which influence rank. See, for example, the many articles in the 1992 volume 33 of *Linear and Multilinear Algebra* devoted to linear preserver problems. However the problems considered in these articles are different in character. They consider functions which operate on the full matrix, e.g., $f(A) = PAQ$ where P, Q are appropriate matrices, rather than our "entrywise" functions $f(A) = (f(a_{ij}))_{i=1}^{n}{}_{j=1}^{m}$. The type of function operation on matrices which we deal with has been considered in the literature in connection with a variety of problems. Functions taking positive definite or Hermitian matrices to positive definite or Hermitian matrices have been explicitly characterized (see e.g. [3,7,13]). Functions which preserve and generalize spectral radii inequalities in connection with Hadamard-like products have been studied (see e.g. [5,8]). The problem considered in this paper was motivated in part by a paper of Pinkus [12] (see also [11]), wherein are characterized real-valued functions f for which $f(\langle x, y\rangle)$ is a strictly positive definite kernel. Here $\langle\cdot,\cdot\rangle$ is a real inner product, and x and y are elements of some real inner product space. This problem is equivalent to that of characterizing those f which take all symmetric positive definite matrices of rank $\leqslant k$, with no two rows identical, to matrices of full rank.

2. Real f and the case $k = 1$

We start by considering the simplest case of $k = d = 1$, i.e., real-valued functions taking matrices of rank 1 to matrices of rank 1.

Proposition 3. *Assume $f : \mathbb{R} \to \mathbb{R}$ is measurable. Then f takes matrices of rank 1 to matrices of rank 1 if and only if f is one of the following functions, where $C, c \in \mathbb{R}$:*

(a) C,
(b) $C|t|^c,\ f(0) = 0$,
(c) $C|t|^c(\operatorname{sgn} t),\ f(0) = 0$.

Proof. Obviously the constant function, i.e., $f(t) = C$ for all t, takes every matrix to a matrix of rank at most 1. Let us assume that f is not the constant function. The matrix

$$\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$$

is of rank 1 for any $a \neq 0$. Thus

$$\begin{pmatrix} f(0) & f(0) \\ f(a) & f(0) \end{pmatrix}$$

must also be of rank at most 1. As its determinant is $f(0)(f(0) - f(a))$ and there exists an a such that $f(0) - f(a) \neq 0$ (f is not the constant function), we must have $f(0) = 0$.

Now consider the matrix

$$\begin{pmatrix} xy & y \\ x & 1 \end{pmatrix}$$

of rank 1, and set $f(1) = C$. Thus

$$\begin{pmatrix} f(xy) & f(y) \\ f(x) & C \end{pmatrix}$$

is of rank 1 implying

$$Cf(xy) = f(x)f(y)$$

for all $x, y \in \mathbb{R}$. If $C = 0$, then $f(x) = 0$ for all $x \in \mathbb{R}$, a contradiction. This is a Cauchy equation. From Aczél [1, p. 41] we have that if, for example, f is measurable (less will suffice) and not the constant function, then f is necessarily of one of the following forms with arbitrary $C, c \in \mathbb{R}$:

(b) $C|t|^c$,
(c) $C|t|^c(\operatorname{sgn} t)$,

for $t \neq 0$, and satisfies $f(0) = 0$.

The converse direction, i.e., if f has the form (a), (b) or (c), then f takes matrices of rank 1 to matrices of rank 1, easily follows using the fact that matrices are of rank at most 1 if and only if they are of the explicit form $A = (b_i c_j)_{i=1}^{n}{}_{j=1}^{m}$. □

The above proposition also follows as a consequence of these next two results which characterize functions taking matrices of rank 1 to matrices of rank d.

Proposition 4. *Let $f : \mathbb{R} \to \mathbb{R}$. Then f takes matrices of rank 1 to matrices of rank d if and only if f is an element of a dilation invariant subspace of dimension at most d.*

A *dilation invariant subspace* X over $\mathbb{R}$ is a linear subspace with the property that if $g \in X$ then $g(a\cdot) \in X$ for every $a \in \mathbb{R}$.

Proof. Let us assume that $f : \mathbb{R} \to \mathbb{R}$ takes matrices of rank 1 to matrices of rank d and, without loss of generality, that there exists a real matrix $A = (b_i c_j)_{i,j=1}^{d}$ of rank 1 for which $f(A)$ is of exact rank d, i.e., $\det f(A) \neq 0$. Consider for each $x, y \in \mathbb{R}$ the $(d+1) \times (d+1)$ matrix

$$B = \begin{pmatrix} xy & xc_1 & \cdots & xc_d \\ b_1 y & b_1 c_1 & \cdots & b_1 c_d \\ \vdots & \vdots & \ddots & \vdots \\ b_d y & b_d c_1 & \cdots & b_d c_d \end{pmatrix}$$

of rank 1. By assumption we have

$$\det f(B) = 0.$$

Since $\det f(A) \neq 0$, expanding $\det f(B)$ by its first row and column gives us the equation

$$f(xy) = \sum_{i,j=1}^{d} e_{ij} f(xc_j) f(b_i y). \tag{1}$$

Note that this can be rewritten in various ways in the form

$$f(xy) = \sum_{k=1}^{d} r_k(x) s_k(y).$$

Let

$$\mathscr{R} = \mathrm{span}\{f(a\cdot) \colon a \in \mathbb{R}\}.$$

$\mathscr{R}$ is a dilation invariant subspace. From (1) we see that it is of dimension at most d, since each $f(a\cdot)$ is a linear combination of the $\{f(c_j\cdot)\}_{j=1}^{d}$.

On the other hand, if f is an element of a dilation invariant subspace $\mathscr{R}$ of dimension at most d, then we can write

$$f(xy) = \sum_{k=1}^{d} r_k(x) s_k(y),$$

for some functions $\{r_k\}_{k=1}^{d}$ and $\{s_k\}_{k=1}^{d}$. If A is a rank 1 matrix, then $A = (b_i c_j)_{i=1}^{n}{}_{j=1}^{m}$. Therefore

$$f(b_i c_j) = \sum_{k=1}^{d} r_k(b_i) s_k(c_j) = \sum_{k=1}^{d} B_{ik} C_{kj},$$

i.e., $f(A)$ is the pointwise sum of d matrices of rank at most 1, or alternatively, the $n \times m$ matrix $f(A)$ is the product of an $n \times d$ and a $d \times m$ matrix, and thus $f(A)$ is a matrix of rank at most d. $\square$

We can now explicitly characterize functions taking matrices of rank 1 to matrices of rank d.

Theorem 5. *Assume $f : \mathbb{R} \to \mathbb{R}$ takes matrices of rank 1 to matrices of rank d and f is measurable. Then f is a C^∞ function on $\mathbb{R}\backslash\{0\}$. It is a solution thereon of a nontrivial Euler equation of degree at most d, i.e., satisfies*

$$\sum_{\ell=0}^{d} c_\ell t^\ell f^{(\ell)}(t) = 0, \tag{2}$$

for $t \neq 0$, where the $c_0, \ldots, c_d \in \mathbb{R}$ are not all zero.

To be more exact it is a solution of (2) *of the precise form*

$$
\begin{aligned}
f(t) = \sum_{j=1}^{m} & p_{1j}(\ln|t|)|t|^{\lambda_j} + p_{2j}(\ln|t|)|t|^{\lambda_j}(\operatorname{sgn} t) \\
& + \sum_{j=1}^{n} \big[q_{1j}(\ln|t|)|t|^{\mu_j}\cos(\nu_j \ln|t|) + r_{1j}(\ln|t|)|t|^{\mu_j}\sin(\nu_j \ln|t|) \\
& \qquad + q_{2j}(\ln|t|)|t|^{\mu_j}\cos(\nu_j \ln|t|)(\operatorname{sgn} t) \\
& \qquad + r_{2j}(\ln|t|)|t|^{\mu_j}\sin(\nu_j \ln|t|)(\operatorname{sgn} t)\big], \qquad (3)
\end{aligned}
$$

where the $\{\lambda_j\}_{j=1}^m$ *are distinct values in* $\mathbb{R}$, *the* $\{\mu_j \pm \mathrm{i}\nu_j\}_{j=1}^n$ *distinct values in* $\mathbb{C}\backslash\mathbb{R}$, *the* p_{1j}, p_{2j}, q_{1j}, q_{2j}, r_{1j} *and* r_{2j} *are polynomials*, *and*

$$
\begin{aligned}
\sum_{j=1}^{m} & (\partial p_{1j} + 1) + (\partial p_{2j} + 1) + 2\sum_{j=1}^{n}(\max\{\partial q_{1j}, \partial r_{1j}\} + 1) \\
& + (\max\{\partial q_{2j}, \partial r_{2j}\} + 1) \leqslant d,
\end{aligned}
$$

where ∂p *denotes the degree of the polynomial* p (*with* $\partial p = -1$ *if* $p = 0$).

Proof. As $f : \mathbb{R} \to \mathbb{R}$ takes matrices of rank 1 to matrices of rank d it follows from Proposition 4 that f belongs to a dilation invariant subspace $\mathscr{R}$ of dimension at most d. It is well known (see e.g. [2,6,10,14] and references therein) that every *translation invariant* subspace of dimension at most d of measurable functions on all of $\mathbb{R}$ is exactly the space of solutions of

$$
\sum_{\ell=0}^{d} a_\ell g^{(\ell)}(s) = 0 \qquad (4)
$$

for all $s \in \mathbb{R}$, $a_0, \ldots, a_d \in \mathbb{R}$ not all zero. A change of variable $x = \mathrm{e}^s$ (or $x = -\mathrm{e}^s$) implies that each $f \in \mathscr{R}$ must satisfy an equation of the form (2) on $\mathbb{R}_+$ and on $\mathbb{R}_-$. In fact we can assume that it is the same equation on both $\mathbb{R}_+$ and $\mathbb{R}_-$. This follows from the dilation invariance, but can also be proven directly as follows.

$\mathscr{R}$ is finite dimensional and therefore closed under uniform convergence on compact sets. Furthermore

$$
\frac{f(x(y+h)) - f(xy)}{h} \in \mathscr{R}
$$

for every $h \neq 0$. As all solutions of (4) are in $C^\infty(\mathbb{R})$, it follows that f is in $C^\infty(\mathbb{R}\backslash\{0\})$, and thus

$$
xf'(x) = \frac{\mathrm{d}}{\mathrm{d}y} f(xy)\Big|_{y=1} \in \mathscr{R}
$$

for $x \in \mathbb{R}\backslash\{0\}$. Repeated applications of the above imply that

$$x^k f^{(k)}(x) = \frac{\mathrm{d}^k}{\mathrm{d}y^k} f(xy)\Big|_{y=1} \in \mathscr{R}$$

for $k = 0, 1, \ldots, d$. As $\mathscr{R}$ is a space of dimension at most d, these $d + 1$ functions are linearly dependent. Thus f must satisfy an equation of the form (2) on $\mathbb{R}\backslash\{0\}$.

We know the solutions of (2) (see e.g. [4, p. 148]). Solutions of (2) on $(0, \infty)$ are obtained as linear combinations of the following functions. We consider the polynomial

$$p(x) = \sum_{\ell=0}^{d} c_\ell x(x-1)\cdots(x-\ell+1). \tag{5}$$

Associated with each real root λ of p of multiplicity m we have the m solutions $(\ln t)^k t^\lambda$, $k = 0, \ldots, m-1$. Associated with each pair of complex conjugate roots $\mu \pm \mathrm{i}\nu$, each of multiplicity m, we have the $2m$ solutions $(\ln t)^k t^\mu \cos(\nu \ln t)$ and $(\ln t)^k t^\mu \sin(\nu \ln t)$, $k = 0, \ldots, m-1$. (We are simply taking a real basis for $\mathrm{span}\{(\ln t)^k t^{\mu \pm \mathrm{i}\nu}\}$.) On $(-\infty, 0)$ we have the same class of solutions where t is replaced by $|t|$. Our function f is in the linear space generated by these functions. These are the functions which appear in (3), with an important difference. Solutions of (2), assuming $c_d \neq 0$, form a d dimensional subspace on $\mathbb{R}_+$, a d dimensional subspace on $\mathbb{R}_-$, and thus a $2d$ dimensional subspace of $C(\mathbb{R}\backslash\{0\})$. However f is an element of $\mathscr{R}$, a dilation invariant subspace of dimension at most d of the solution space of (2).

We claim that each element of a dilation invariant subspace of dimension at most d which satisfies (2) is of the form (3). To see this, first note that $\mathscr{R}$ is both positive and negative dilation invariant. That is, if $f(\cdot) \in \mathscr{R}$ then we also have $f(-\cdot) \in \mathscr{R}$. Thus any $f \in \mathscr{R}$ can be decomposed into the sum of an even and odd function, both of which belong to $\mathscr{R}$. That is,

$$\mathscr{R} = \mathscr{R}_\mathrm{e} \oplus \mathscr{R}_\mathrm{o}$$

where $\mathscr{R}_\mathrm{e}$ and $\mathscr{R}_\mathrm{o}$ consist of the even and odd functions in $\mathscr{R}$, respectively. Furthermore $\mathscr{R}_\mathrm{e}$ and $\mathscr{R}_\mathrm{o}$ are dilation invariant and

$$\dim \mathscr{R} = \dim \mathscr{R}_\mathrm{e} + \dim \mathscr{R}_\mathrm{o}.$$

Let $f = g + h$ where $g \in \mathscr{R}_\mathrm{e}$ and $h \in \mathscr{R}_\mathrm{o}$. Then g is necessarily of the form

$$g(t) = \sum_{j=1}^{m} p_{1j}(\ln|t|)|t|^{\lambda_j} + \sum_{j=1}^{n} \big[q_{1j}(\ln|t|)|t|^{\mu_j} \cos(\nu_j \ln|t|) + r_{1j}(\ln|t|)|t|^{\mu_j} \sin(\nu_j \ln|t|)\big],$$

where the p_{1j}, q_{1j} and r_{1j} are polynomials, the $\{\lambda_j\}_{j=1}^m$ in $\mathbb{R}$ and the $\{\mu_j \pm \mathrm{i}\nu_j\}_{j=1}^n$ in $\mathbb{C}\backslash\mathbb{R}$ are the roots of (5) of appropriate multiplicity (at least $\partial p_{1j} + 1$, $j = 1, \ldots, m$, and $\max\{\partial q_{1j}, \partial r_{1j}\} + 1$, $j = 1, \ldots, n$, respectively), and

$$\sum_{j=1}^{m}(\partial p_{1j} + 1) + 2\sum_{j=1}^{n}(\max\{\partial q_{1j}, \partial r_{1j}\} + 1) \leqslant \dim \mathscr{R}_\mathrm{e}.$$

Similarly, h is necessarily of the form

$$h(t) = \sum_{j=1}^{m} p_{2j}(\ln|t|)|t|^{\lambda_j}(\operatorname{sgn} t) + \sum_{j=1}^{n} \big[q_{2j}(\ln|t|)|t|^{\mu_j}\cos(\nu_j \ln|t|)(\operatorname{sgn} t) + r_{2j}(\ln|t|)|t|^{\mu_j}\sin(\nu_j \ln|t|)(\operatorname{sgn} t)\big],$$

where

$$\sum_{j=1}^{m}(\partial p_{2j} + 1) + 2\sum_{j=1}^{n}(\max\{\partial q_{2j}, \partial r_{2j}\} + 1) \leqslant \dim \mathscr{R}_{\text{o}}.$$

This proves that f is of the form (3). □

Every f of the form (3) takes matrices of rank 1 to matrices of rank d, modulo the fact that we have not defined f at 0. From (1) it easily follows (substitute $y = 0$ therein) that if the constant function is not in $\mathscr{R}$, then $f(0) = 0$.

3. Real f and the case $k \geqslant 2$

Based on Theorem 5 we first prove that in the case $k \geqslant 2$ our functions f are necessarily polynomials. We will then delineate their possible explicit form, dependent upon k and d.

Proposition 6. *Assume $f : \mathbb{R} \to \mathbb{R}$ is measurable and f takes matrices of rank k, $k \geqslant 2$, to matrices of rank d. Then f is a polynomial.*

Proof. As f takes matrices of rank k to matrices of rank d, it also takes matrices of rank 1 to matrices of rank d. Thus f has the form given in Theorem 5. More specifically, it is a C^∞ function on $\mathbb{R}\backslash\{0\}$.

Now if $A = (a_{ij})$ is a matrix of rank 1, then for each real constant c the matrix $(a_{ij} - c)$ is of rank at most 2. As $k \geqslant 2$ the matrix $(f(a_{ij} - c))$ is of rank at most d. Thus for each constant c the function

$$f_c(t) = f(t - c)$$

take matrices of rank 1 to matrices of rank at most d. Thus f_c is also of the form given in Theorem 5.

There is no a priori reason to assume that the different f_c, as c varies, satisfy Eq. (2) with one and the same coefficients. They simply satisfy equations of the form (2). However one property which is a result of the previous analysis is that every solution of (2) is a C^∞ function on $\mathbb{R}\backslash\{0\}$. Saying that f_c is a C^∞ function on $\mathbb{R}\backslash\{0\}$ for any $c \neq 0$ implies that f is a C^∞ function also at zero and thus on all of $\mathbb{R}$. It is easily verified that the only functions in our solution set of (3) which are C^∞ functions on all of $\mathbb{R}$ (and thus at zero) are the powers x^m, m a nonnegative integer, and linear combinations thereof. Thus f is a polynomial. □

Now that we know that our fs are necessarily polynomials, we can characterize them explicitly.

Theorem 7. *Let*

$$f(t)=\sum_{\ell=1}^{p}\alpha_\ell t^{m_\ell},$$

where the m_ℓ are distinct nonnegative integers and $\alpha_\ell \neq 0$, $\alpha_\ell \in \mathbb{R}$, $\ell = 1,\dots,p$. For each matrix $A=(a_{ij})_{i=1}^{n}{}_{j=1}^{m}$ of rank at most k, the matrix

$$f(A)=(f(a_{ij}))_{i=1}^{n}{}_{j=1}^{m}$$

is of rank at most

$$\sum_{\ell=1}^{p}\binom{k+m_\ell-1}{m_\ell}.$$

Furthermore there exists a matrix A of rank k for which

$$\operatorname{rank} f(A)=\sum_{\ell=1}^{p}\binom{k+m_\ell-1}{m_\ell}.$$

Before presenting the proof, we recall some notation. For $\mathbf{x}=(x_1,\dots,x_k)\in\mathbb{R}^k$ and $\mathbf{n}=(n_1,\dots,n_k)\in\mathbb{Z}_+^k$ let

$$\mathbf{x}^{\mathbf{n}}=x_1^{n_1}\cdots x_k^{n_k},$$

and $|\mathbf{n}|=n_1+\cdots+n_k$. For $\mathbf{b},\mathbf{c}\in\mathbb{R}^k$ we use the standard inner product

$$\mathbf{b}\cdot\mathbf{c}=\sum_{i=1}^{k}b_ic_i.$$

Finally we recall that for any $q\in\mathbb{Z}_+$

$$(\mathbf{b}\cdot\mathbf{c})^q=\sum_{|\mathbf{n}|=q}\binom{|\mathbf{n}|}{\mathbf{n}}\mathbf{b}^{\mathbf{n}}\cdot\mathbf{c}^{\mathbf{n}},$$

where

$$\mathbf{b}^{\mathbf{n}}\cdot\mathbf{c}^{\mathbf{n}}=b_1^{n_1}\cdots b_k^{n_k}c_1^{n_1}\cdots c_k^{n_k},$$

and

$$\binom{|\mathbf{n}|}{\mathbf{n}}=\frac{(|\mathbf{n}|)!}{n_1!\cdots n_k!}.$$

Proof. A matrix $A=(a_{ij})_{i=1}^{n}{}_{j=1}^{m}$ is of rank at most k if and only if it is the sum of k rank 1 matrices, i.e., can be written in the form

$$a_{ij}=\sum_{t=1}^{k}b_{it}c_{jt},\quad i=1,\dots,n;\quad j=1,\dots,m. \tag{6}$$

Set $\mathbf{b}_i = (b_{i1}, \dots, b_{ik})$, $i = 1, \dots, n$, and $\mathbf{c}_j = (c_{j1}, \dots, c_{jk})$, $j = 1, \dots, m$. Thus

$$a_{ij} = \mathbf{b}_i \cdot \mathbf{c}_j.$$

Furthermore for each $q \in \mathbb{Z}_+$

$$(\mathbf{b}_i \cdot \mathbf{c}_j)^q = \sum_{|\mathbf{n}|=q} \binom{|\mathbf{n}|}{\mathbf{n}} \mathbf{b}_i^{\mathbf{n}} \cdot \mathbf{c}_j^{\mathbf{n}}. \tag{7}$$

The number of summands in (7) is the dimension of the space of homogeneous polynomials of degree q in k variables, namely

$$\binom{k+q-1}{q}.$$

(This is easily checked by setting $b_{it}c_{jt} = x_t$ in (7).) It therefore follows from (6) and (7) that the matrix whose entries are

$$\alpha_\ell (a_{ij})^{m_\ell}$$

is of rank at most

$$\binom{k+m_\ell-1}{m_\ell},$$

and thus

$$\operatorname{rank} f(A) \leqslant \sum_{\ell=1}^{p} \binom{k+m_\ell-1}{m_\ell}.$$

Let P_q^k denote the space of homogeneous polynomials of degree q in k variables. Set $P = \bigoplus_{\ell=1}^{p} P_{m_\ell}^k$ and

$$h = \dim P = \sum_{\ell=1}^{p} \binom{k+m_\ell-1}{m_\ell}.$$

We wish to prove the existence of a matrix A of rank k for which $f(A)$ is of rank h. In other words, we claim there exist vectors $\mathbf{b}_i, \mathbf{c}_j$, $i, j = 1, \dots, h$, in $\mathbb{R}^k$ such that the $h \times h$ matrix with (i, j) entry

$$\sum_{\ell=1}^{p} \alpha_\ell (\mathbf{b}_i \cdot \mathbf{c}_j)^{m_\ell}, \quad i, j = 1, \dots, h,$$

is of rank h. To this end it suffices to prove the existence of vectors $\mathbf{b}_i$, $i = 1, \dots, h$, for which the polynomials

$$\sum_{\ell=1}^{p} \alpha_\ell (\mathbf{b}_i \cdot \mathbf{x})^{m_\ell}, \quad i = 1, \dots, h,$$

are linearly independent. For if true, then these h polynomials are also linearly independent over some h points $\mathbf{c}_1, \dots, \mathbf{c}_h$.

We will prove the existence of the required vectors. Assume such vectors do not exist. Then for each choice of $\mathbf{b}_i$, $i = 1, \ldots, h$, in $\mathbb{R}^k$ there exist $\beta_1, \ldots, \beta_h$, not all zero, such that for all $\mathbf{x} \in \mathbb{R}^k$

$$\sum_{i=1}^{h} \beta_i \left(\sum_{\ell=1}^{p} \alpha_\ell (\mathbf{b}_i \cdot \mathbf{x})^{m_\ell} \right) = 0.$$

Thus

$$\sum_{\ell=1}^{p} \alpha_\ell \left(\sum_{i=1}^{h} \beta_i (\mathbf{b}_i \cdot \mathbf{x})^{m_\ell} \right) = 0. \tag{8}$$

Each

$$\sum_{i=1}^{h} \beta_i (\mathbf{b}_i \cdot \mathbf{x})^{m_\ell}$$

is a polynomial in $P^k_{m_\ell}$ (homogeneous of degree m_ℓ). For (8) to hold it is necessary, since each $\alpha_\ell \neq 0$, that

$$\sum_{i=1}^{h} \beta_i (\mathbf{b}_i \cdot \mathbf{x})^{m_\ell} = 0, \quad \ell = 1, \ldots, p,$$

which from (7) implies

$$\sum_{|\mathbf{n}|=m_\ell} \binom{|\mathbf{n}|}{\mathbf{n}} \sum_{i=1}^{h} \beta_i \mathbf{b}_i^{\mathbf{n}} \cdot \mathbf{x}^{\mathbf{n}} = 0, \quad \ell = 1, \ldots, p. \tag{9}$$

As the set of functions in $\{\mathbf{x}^{\mathbf{n}}\}_{|\mathbf{n}|=m_\ell}$ are linearly independent in $P^k_{m_\ell}$ (this is just the monomial basis for $P^k_{m_\ell}$), (9) implies that

$$\sum_{i=1}^{h} \beta_i \mathbf{b}_i^{\mathbf{n}} = 0,$$

for each $\mathbf{n} \in \mathbb{Z}^k_+$, $|\mathbf{n}| = m_\ell$, and $\ell = 1, \ldots, p$. The number of such equations is exactly h. Moreover as the

$$\left\{\mathbf{x}^{\mathbf{n}} \colon \mathbf{n} \in \mathbb{Z}^k_+, |\mathbf{n}| = m_\ell, \ell = 1, \ldots, p\right\}$$

form a basis for P (of dimension h), it follows that there exist vectors $\mathbf{b}_1, \ldots, \mathbf{b}_h \in \mathbb{R}^k$ for which the matrix

$$\left\{\mathbf{b}_i^{\mathbf{n}} \colon \mathbf{n} \in \mathbb{Z}^k_+, |\mathbf{n}| = m_\ell, \ell = 1, \ldots, p, i = 1, \ldots, h\right\}$$

is nonsingular. For such a choice of vectors $\mathbf{b}_i$, $i = 1, \ldots, h$, the requisite $\beta_1, \ldots, \beta_h$, not all zero, do not exist. This proves that for these $\mathbf{b}_i$, $i = 1, \ldots, h$, the polynomials

$$\sum_{\ell=1}^{p} \alpha_\ell (\mathbf{b}_i \cdot \mathbf{x})^{m_\ell}, \quad i = 1, \ldots, h,$$

are linearly independent, and thus there exist $\mathbf{c}_1, \ldots, \mathbf{c}_h$ for which the matrix

$$\sum_{\ell=1}^{p} \alpha_\ell (\mathbf{b}_i \cdot \mathbf{c}_j)^{m_\ell}, \quad i, j = 1, \ldots, h,$$

is of rank h. $\square$

Proposition 6 and Theorem 7 immediately imply Theorem 1.

Note that Theorem 1 states that $f : \mathbb{R} \to \mathbb{R}$ takes matrices of rank k to matrices of rank $< k$ ($k \geqslant 2$) if and only if f is the constant function, while it takes matrices of rank k to matrices of rank k ($k \geqslant 2$) if and only if f is the constant function or $f(t) = \alpha t$ for some $\alpha \in \mathbb{R}$. These simple facts can also be proven directly as follows.

Assume f takes matrices of rank k to matrices of rank $< k$ ($k \geqslant 2$). Consider the $k \times k$ matrix

$$\begin{pmatrix} b & b & \cdots & b & b \\ a & b & \cdots & b & b \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & \cdots & a & b \end{pmatrix}.$$

That is, the (i, j)th element is equal to b for $i \leqslant j$, and to a for $i > j$. The determinant of this matrix is $b(b-a)^{k-1}$. Since f takes matrices of rank k to matrices of rank $< k$ we must therefore have

$$f(b)(f(b) - f(a))^{k-1} = 0$$

for every choice of $a, b \in \mathbb{R}$. But this implies that f is the constant function.

Now assume f takes matrices of rank k to matrices of rank k ($k \geqslant 2$), and f is not the constant function. The $(k+1) \times (k+1)$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ a & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & \cdots & a & 0 \end{pmatrix}$$

is of rank k for $a \neq 0$. Thus

$$\begin{pmatrix} f(0) & f(0) & \cdots & f(0) & f(0) \\ f(a) & f(0) & \cdots & f(0) & f(0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f(a) & f(a) & \cdots & f(a) & f(0) \end{pmatrix}$$

has rank at most k. We know that its determinant $f(0)(f(0) - f(a))^k$ must therefore equal zero. If $f(0) \neq 0$ then as f is not the constant function, a contradiction ensues. So $f(0) = 0$. (This is the same argument as that found at the beginning of the proof of Proposition 3.) Choose a so that $f(a) = c \neq 0$. Consider the $(k+1) \times (k+1)$ matrix

$$\begin{pmatrix} a & \cdots & 0 & 0 & a \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & a & 0 & a \\ 0 & \cdots & 0 & a & a \\ 0 & \cdots & x & y & x+y \end{pmatrix}.$$

This matrix is of rank k. (The last column is the sum of the previous columns.) As f takes matrices of rank k to matrices of rank k, the matrix

$$\begin{pmatrix} c & \cdots & 0 & 0 & c \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & c & 0 & c \\ 0 & \cdots & 0 & c & c \\ 0 & \cdots & f(x) & f(y) & f(x+y) \end{pmatrix}$$

must be singular. This implies that we must have for all $x, y \in \mathbb{R}$

$$c^k(f(x+y) - f(x) - f(y)) = 0.$$

Thus

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. This is a well-known Cauchy equation. Under any one of the conditions on f such as measurability, continuity at a point, or boundedness on an interval, this implies that

$$f(t) = \alpha t$$

for some $\alpha \in \mathbb{R}$.

4. Complex *g*

What we did in Sections 2 and 3 for real-valued functions we will now do for complex-valued functions.

Theorem 8. *Assume $g : \mathbb{C} \to \mathbb{C}$ is continuous on $\mathbb{C}\backslash\{0\}$ and takes matrices of rank 1 to matrices of rank d. Then g is of the form*

$$g(z) = \sum_{\ell=1}^{p} p_\ell(\ln|z|)\mathrm{e}^{c_\ell \ln|z|} z^{k_\ell} \tag{10}$$

on $\mathbb{C}\backslash\{0\}$, where p_ℓ is a polynomial of degree ∂p_ℓ, $c_\ell \in \mathbb{C}$, $k_\ell \in \mathbb{Z}$ and $\sum_{\ell=1}^{p}(\partial p_\ell + 1) \leqslant d$.

Proof. We mimic the proofs of Proposition 4 and Theorem 5. We assume that there exists a complex matrix $A = (b_i c_j)_{i,j=1}^d$ of rank 1 for which $g(A)$ is of exact rank d, i.e., $\det g(A) \neq 0$. Consider for each $z, w \in \mathbb{C}$ the $(d+1) \times (d+1)$ matrix

$$B = \begin{pmatrix} zw & zc_1 & \cdots & zc_d \\ b_1 w & b_1 c_1 & \cdots & b_1 c_d \\ \vdots & \vdots & \ddots & \vdots \\ b_d w & b_d c_1 & \cdots & b_d c_d \end{pmatrix}$$

of rank 1. By assumption we have

$$\det g(B) = 0.$$

Since $\det g(A) \neq 0$, expanding $\det g(B)$ by its first row and column gives us the equation

$$g(zw) = \sum_{i,j=1}^{d} e_{ij} g(zc_j) g(b_i w), \tag{11}$$

which we can rewrite in various ways as

$$g(zw) = \sum_{\ell=1}^{d} r_\ell(z) s_\ell(w). \tag{12}$$

Any g which satisfies (12) takes matrices of rank 1 to matrices of rank d.

Set

$$\mathscr{V} = \operatorname{span}\{g(w\cdot) : w \in \mathbb{C}\}.$$

From (11) we have that $\mathscr{V}$ is of dimension at most d. $\mathscr{V}$ is invariant under dilation by $w \in \mathbb{C}$. We may therefore regard $\mathscr{V}$ as a subspace invariant under rotation (multiplication by $w = \mathrm{e}^{\mathrm{i}t}$) and positive dilation (multiplication by $w = \rho > 0$). As $\mathscr{V}$ is of finite dimension, it is also closed under uniform convergence on compact sets. Define for each fixed $n \in \mathbb{Z}$ and $k \in \mathbb{N}$

$$\phi_k(z) = \frac{1}{k} \sum_{j=1}^{k} g\left(\mathrm{e}^{-\frac{2\pi \mathrm{i} j}{k}} z\right) \mathrm{e}^{\frac{2\pi \mathrm{i} jn}{k}}.$$

Since $\mathscr{V}$ is rotation invariant we have that $\phi_k \in \mathscr{V}$ for each $k \in \mathbb{N}$. However the ϕ_k are Riemann sums associated with the integral

$$\frac{1}{2\pi} \int_0^{2\pi} g(\mathrm{e}^{-\mathrm{i}s} z) \mathrm{e}^{\mathrm{i}ns} \, \mathrm{d}s,$$

and converge to it uniformly in z on compact subsets of $\mathbb{C}\backslash\{0\}$. Thus this integral is also in $\mathscr{V}$. Letting $z = r\mathrm{e}^{\mathrm{i}\theta}$, and applying the change of variable $t = \theta - s$, it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} g(r\mathrm{e}^{\mathrm{i}t}) \mathrm{e}^{\mathrm{i}n(\theta - t)} \, \mathrm{d}t$$

is in $\mathscr{V}$. This integral equals

$$g_n(r) \mathrm{e}^{\mathrm{i}n\theta},$$

where

$$g_n(r) = \frac{1}{2\pi}\int_0^{2\pi} g(re^{it})e^{-int}\,dt.$$

The functions $g_n(r)$ are in $C(0,\infty)$.

From the linear independence of the $\{e^{in\theta}\}_{n\in\mathbb{Z}}$, we have that those $\{g_n(r)e^{in\theta}\}$ with g_n not identically zero, are also linearly independent. As each of these functions is in $\mathscr{V}$, and $\mathscr{V}$ is of dimension at most d, this implies that there are at most d distinct integers $n \in \mathbb{Z}$ for which g_n is not identically zero. By the unicity theorem for Fourier series (see e.g. [9, p. 4]), we have

$$g(re^{i\theta}) = \sum_{\ell=1}^{d} g_{n_\ell}(r)e^{in_\ell\theta}.$$

The space $\mathscr{V}$ is also invariant under positive dilation. As the $\{e^{in_\ell\theta}\}_{\ell=1}^d$ are linearly independent and $\mathscr{V}$ is of finite dimension, then for each $n \in \{n_1,\ldots,n_d\}$ the space

$$\operatorname{span}\{g_n(\rho\cdot)\colon \rho > 0\}$$

is a finite dimensional subspace of $C(0,\infty)$. Therefore by a change of variable (see the proof of Theorem 5) using the known result characterizing finite dimensional translation invariant subspaces of complex-valued functions in $C(\mathbb{R})$ it follows that each g_n is a linear combination of a finite number of functions of the form $(\ln r)^k r^\lambda$ for $k = 0,\ldots,m$, and some $\lambda \in \mathbb{C}$. Combining the above facts implies that g is necessarily of the form

$$g(z) = g(re^{i\theta}) = \sum_{\ell=1}^{p} p_\ell(\ln r)r^{d_\ell}e^{ik_\ell\theta}$$

on $\mathbb{C}\backslash\{0\}$, where $z = re^{i\theta}$, each p_ℓ is a polynomial, $d_\ell \in \mathbb{C}$ and $k_\ell \in \mathbb{Z}$. Since $r = |z|$ and $e^{ik_\ell\theta} = z^{k_\ell}|z|^{-k_\ell} = z^{k_\ell}e^{-k_\ell \ln|z|}$, the above may be rewritten as

$$g(z) = \sum_{\ell=1}^{p} p_\ell(\ln|z|)e^{c_\ell \ln|z|}z^{k_\ell},$$

where p_ℓ is a polynomial, $c_\ell \in \mathbb{C}$ and $k_\ell \in \mathbb{Z}$. As $g \in \mathscr{V}$, $\dim\mathscr{V} \leqslant d$ and $\mathscr{V}$ is invariant under dilations by positive constants and under rotation, it follows that $\sum_{\ell=1}^{p}(\partial p_\ell + 1) \leqslant d$. □

Remark 1. It is also not difficult to show that for any g of the form (10) with $\sum_{\ell=1}^{p}(\partial p_\ell + 1) \leqslant d$ the associated linear subspace generated by dilations of g is dilation invariant of dimension at most d. Thus, if we also set $g(0) = 0$, then g will take matrices of rank 1 to matrices of rank d.

Remark 2. In Theorem 5 we assumed that f was measurable. Here we demanded that g be continuous on $\mathbb{C}\backslash\{0\}$. It is possible to prove this result for measurable functions. However the proof then becomes rather detailed and complicated. The interested reader may wish to consult the methods in [6].

Based on Theorem 8 we first prove that for $k \geqslant 2$ the associated g are necessarily polynomials in z and $\bar{z}$. We then delineate the possible explicit forms, dependent upon k and d.

Proposition 9. *Assume* $g : \mathbb{C} \to \mathbb{C}$ *is continuous and* g *takes matrices of rank* k, $k \geqslant 2$, *to matrices of rank* d. *Then* g *is a polynomial in* z *and* $\bar{z}$.

Proof. We mirror the proof of Proposition 6. Since g takes matrices of rank k to matrices of rank d, and $k \geqslant 2$, it follows as in the proof of Proposition 6 that $g_c(t) = g(t - c)$ takes matrices of rank 1 to matrices of rank d for every constant $c \in C$.

Thus g is necessarily of the form (10) and its translates also have the same general form. Note that any g which satisfies (10), i.e., is of the form

$$g(z) = \sum_{\ell=1}^{p} p_\ell(\ln|z|)\mathrm{e}^{c_\ell \ln|z|} z^{k_\ell}$$

is a C^∞ function in x and y ($z = x + \mathrm{i}y$) for $(x, y) \neq (0, 0)$. If its translates have this same property, then g must be a C^∞ function at (0, 0). It is readily verified that this implies that each of the above polynomials p_ℓ is a constant function, and that the c_ℓ and $2k_\ell + c_\ell$ are nonnegative even integers implying

$$\mathrm{e}^{c_\ell \ln|z|} z^{k_\ell} = z^{k_\ell + c_\ell/2} \bar{z}^{c_\ell/2}.$$

Thus g has the desired form, namely

$$g(z) = \sum_{\ell=1}^{p} \beta_\ell z^{m_\ell} \bar{z}^{n_\ell},$$

where $\beta_\ell \in \mathbb{C}$ and the (m_ℓ, n_ℓ) are distinct pairs of nonnegative integers. □

Now we can explicitly characterize the g of Theorem 2.

Theorem 10. *Let*

$$g(z) = \sum_{\ell=1}^{p} \beta_\ell z^{m_\ell} \bar{z}^{n_\ell},$$

where the (m_ℓ, n_ℓ) *are distinct pairs of nonnegative integers*, $\beta_\ell \neq 0$, $\ell = 1, \ldots, p$. *For each complex-valued matrix* $A = (a_{ij})_{i=1}^{n}{}_{j=1}^{m}$ *of rank at most* k, *the matrix*

$$g(A) = (g(a_{ij}))_{i=1}^{n}{}_{j=1}^{m}$$

is of rank at most

$$\sum_{\ell=1}^{p} \binom{k + m_\ell - 1}{m_\ell} \binom{k + n_\ell - 1}{n_\ell}.$$

Furthermore there exists a matrix A of rank k for which

$$\operatorname{rank} g(A) = \sum_{\ell=1}^{p} \binom{k+m_\ell-1}{m_\ell}\binom{k+n_\ell-1}{n_\ell}.$$

Proof. We parallel the proof of Theorem 7. Let $A = (a_{ij})_{i=1}^{n}{}_{j=1}^{m}$ be a matrix of rank k. Then it may be written in the form

$$a_{ij} = \sum_{t=1}^{k} b_{it}c_{jt}, \quad i = 1, \ldots, n; \quad j = 1, \ldots, m, \tag{13}$$

for some choice of constants b_{it} and c_{jt}. Set $\mathbf{b}_i = (b_{i1}, \ldots, b_{ik})$, $i = 1, \ldots, n$, and $\mathbf{c}_j = (c_{j1}, \ldots, c_{jk})$, $j = 1, \ldots, m$. Thus

$$a_{ij} = \mathbf{b}_i \cdot \mathbf{c}_j.$$

Furthermore for each $p, q \in \mathbb{Z}_+$

$$a_{ij}^{p}\bar{a}_{ij}^{q} = (\mathbf{b}_i \cdot \mathbf{c}_j)^{p}\overline{(\mathbf{b}_i \cdot \mathbf{c}_j)}^{q} = \sum_{|\mathbf{m}|=p} \binom{|\mathbf{m}|}{\mathbf{m}} \mathbf{b}_i^{\mathbf{m}} \cdot \mathbf{c}_j^{\mathbf{m}} \sum_{|\mathbf{n}|=q} \binom{|\mathbf{n}|}{\mathbf{n}} \bar{\mathbf{b}}_i^{\mathbf{n}} \cdot \bar{\mathbf{c}}_j^{\mathbf{n}}. \tag{14}$$

The number of summands in (14) is

$$\binom{k+p-1}{p}\binom{k+q-1}{q}.$$

It therefore follows from (13) and (14) that the matrix whose entries are

$$\beta_\ell (a_{ij})^{m_\ell} (\bar{a}_{ij})^{n_\ell}$$

is of rank at most

$$\binom{k+m_\ell-1}{m_\ell}\binom{k+n_\ell-1}{n_\ell},$$

and thus

$$\operatorname{rank} g(A) \leqslant \sum_{\ell=1}^{p} \binom{k+m_\ell-1}{m_\ell}\binom{k+n_\ell-1}{n_\ell}.$$

We now follow the remaining arguments in the proof of Theorem 7 replacing $\mathbf{x}^{\mathbf{n}}$ by $\mathbf{z}^{\mathbf{m}}\bar{\mathbf{z}}^{\mathbf{n}}$, where we note that the

$$\left\{\mathbf{z}^{\mathbf{m}}\bar{\mathbf{z}}^{\mathbf{n}} \colon \mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^k,\ |\mathbf{m}| = m_\ell,\ |\mathbf{n}| = n_\ell,\ \ell = 1, \ldots, p\right\}$$

are linearly independent functions which span a subspace of $\mathbb{C}^k$ of dimension

$$\sum_{\ell=1}^{p} \binom{k+m_\ell-1}{m_\ell}\binom{k+n_\ell-1}{n_\ell}.$$

The proof of Theorem 10 then follows. □

Remark 3. A study of the proof of the above results shows that if $g : \mathbb{C} \to \mathbb{C}$ takes matrices of rank k to matrices of rank d, then g satisfies the functional equation

$$g(\mathbf{z} \cdot \mathbf{w}) = \sum_{\ell,m=1}^{d} a_{\ell m} g(\mathbf{c}^m \cdot \mathbf{z}) g(\mathbf{b}^\ell \cdot \mathbf{w}), \tag{15}$$

for all $\mathbf{z}, \mathbf{w} \in \mathbb{C}^k$, some $\mathbf{b}^\ell, \mathbf{c}^m \in \mathbb{C}^k$ and $a_{\ell m} \in \mathbb{C}$, $\ell, m = 1, \ldots, d$. Similarly if g solves an equation of the form (15), then for any $\mathbf{d}^i, \mathbf{e}^j \in \mathbb{C}^k$, $i, j = 1, \ldots, n$,

$$\begin{aligned} g(\mathbf{d}^i \cdot \mathbf{e}^j) &= \sum_{\ell,m=1}^{d} a_{\ell m} g(\mathbf{c}^m \cdot \mathbf{d}^i) g(\mathbf{b}^\ell \cdot \mathbf{e}^j) \\ &= \sum_{\ell=1}^{d} \left(\sum_{m=1}^{d} a_{\ell m} g(\mathbf{c}^m \cdot \mathbf{d}^i) \right) g(\mathbf{b}^\ell \cdot \mathbf{e}^j) \\ &= \sum_{\ell=1}^{d} D_{i\ell} E_{\ell j} \end{aligned}$$

which implies that rank $(g(\mathbf{d}^i \cdot \mathbf{e}^j))_{i,j=1}^n \leqslant d$. That is, g takes matrices of rank k to matrices of rank d if and only if g solves (15). We solved (15) for $k = 1$ directly, and for $k \geqslant 2$ indirectly. This same result holds for $f : \mathbb{R} \to \mathbb{R}$ taking matrices of rank k to matrices of rank d.

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