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ON n-WIDTHS AND OPTIMAL RECOVERY IN $\mathcal{M}^r$

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Let BV[0,1] denote the set of all functions of bounded variation on [0,1]. For any $\lambda \in$ BV[0,1], we set $\|\lambda\|$ = total variation of λ, and define

$$\mathcal{M}^r = \mathcal{M}^r[0,1] = \{f : f(x) = \sum_{i=0}^{r-1} a_i x^i + \frac{1}{(r-1)!}\int_0^1 (x-t)_+^{r-1} d\lambda(t),$$
$$\lambda \in BV[0,1]\} .$$

In this paper, the n-width both in the sense of Kolmogorov and Gel'fand, for

$$\mathcal{B}_r = \{f : f \in \mathcal{M}^r, \|\lambda\| \le 1\}$$

is found. In addition, we solve the related problem of optimal recovery of a function $f \in \mathcal{B}_r$.

1 n-Widths

Let $d_n(\mathcal{B}_r; L^1[0,1]) = d_n(\mathcal{B}_r)$ denote the n-width in the sense of Kolmogorov of $\mathcal{B}_r$ in $L^1[0,1]$, given explicitly by

$$d_n(\mathcal{B}_r) = \inf_{X_n} \sup_{f \in \mathcal{B}_r} \inf_{g \in X_n} \|f - g\|_1 ,$$

where the infimum is taken over all n-dimensional linear subspaces X_n of $L^1[0,1]$ and $\|f\|_1 = \int_0^1 |f(t)|\, dt$. The n-width in the sense of Gel'fand for $\mathcal{B}_r$ in C[0,1] is given as

$$d^n(\mathcal{B}_r) = d^n(\mathcal{B}_r; C[0,1]) = \inf_{L_n} \sup_{f \in \mathcal{B}_r \cap L_n} \|f\|_1 ,$$

where the infimum is taken over all subspaces L_n of C[0,1] of codimension n. An extremal subspace for either $d_n(\mathcal{B}_r)$ or $d^n(\mathcal{B}_r)$ is any subspace for which the respective infimum is attained.

A perfect spline P of degree r with s knots at $x_1, \ldots, x_s$, $0 = x_0 < x_1 < \ldots < x_s < x_{s+1} = 1$ is any function expressible as

$$P(x) = \sum_{i=0}^{r-1} a_i x^i + c \sum_{j=0}^{s} (-1)^j \int_{x_j}^{x_{j+1}} (x - t)_+^{r-1} dt.$$

The following result is essentially contained in Karlin [1].

THEOREM 1. <u>For</u> $n \geq r$, <u>there exists a perfect spline</u> $Q_{n,r} = Q$, <u>unique up to multiplication by</u> -1, <u>of degree</u> r <u>with</u> n <u>knots such that</u> $\|Q^{(r)}\|_\infty = 1$ <u>which satisfies the boundary conditions</u> $Q^{(i)}(0) = Q^{(i)}(1) = 0$, $i = 0, 1, \ldots, r-1$, <u>and equioscillates at</u> $n - r + 1$ <u>points in</u> $(0,1)$, i.e., $Q(\eta_i) = (-1)^i \sigma \|Q\|_\infty$, $i = 1, 2, \ldots, n-r+1$, $0 < \eta_1 < \ldots < \eta_{n-r+1} < 1$, $\sigma = +1$ <u>or</u> -1, <u>fixed</u>.

Utilizing Theorem 1, we obtain the following result

THEOREM 2. <u>For</u> $r \geq 2$,

$$(1)\quad d_n(\mathcal{B}_r) = d^n(\mathcal{B}_r) = \begin{cases} \infty, & n < r \\ \|Q_{n,r}\|_\infty, & n \geq r \end{cases}$$

(2) $Q_{n,r}$ <u>has exactly</u> $n - r$ <u>simple zeros</u> $\{\zeta_i\}_{i=1}^{n-r}$ <u>in</u> $(0,1)$, <u>and the linear space spanned by the functions</u> $\{1, x, \ldots, x^{r-1}, (x - \zeta_1)_+^{r-1}, \ldots, (x - \zeta_{n-r})_+^{r-1}\}$ <u>is an extremal subspace for</u> $d_n(\mathcal{B}_r)$.

(3) $L_n^* = \{g : g \in C[0,1],\ g(\xi_i) = 0,\ i = 1, 2, \ldots, n\}$, <u>where</u> $\{\xi_i\}_{i=1}^n$ <u>are the</u> n <u>knots of</u> $Q_{n,r}$ <u>is an extremal subspace for</u> $d^n(\mathcal{B}_r)$.

The proof of Theorem 2 is based upon duality and analysis similar to that found in [2].

2 Optimal Recovery

Let $\underline{x} = (x_1, \ldots, x_n)$, $x_0 = 0 < x_1 < \ldots < x_n < 1 = x_{n+1}$, be given and denote the vector $(f(x_1), \ldots, f(x_n))$ by $\underline{f}$. The problem of optimal recovery is one of determining a rule for best recovering $f \in \mathcal{B}_r$, $n \geq r \geq 2$, based on the information $\underline{f}$ (for the L^∞ analogue of this problem see [3]).

Any transformation R from $\{\underline{f}: f \in \mathcal{B}_r\}$ into $L^1[0,1]$ determines a recovery scheme $Sf = R\underline{f}$ for $f \in \mathcal{B}_r$. The error for recovery based on S is defined as

$$\|I - S\|_1 = \sup_{f \in \mathcal{B}_r} \|f - Sf\|_1,$$

and

$$E(\underline{x}) = \inf\{\|I - S\|_1 : S \text{ a recovery scheme}\}$$

is the minimum error in recovering $f \in \mathcal{B}_r$ from the information $\underline{f}$. S^* is called an optimal recovery scheme provided that $\|I - S^*\|_1 = E(\underline{x})$.

THEOREM 3. *For every* $n \geq r \geq 2$, *and each* $\underline{x}$ *as above, there exists a function*

$$P_{\underline{x}}(x) = \sum_{j=0}^{n} \frac{(-1)^j}{(r-1)!} \int_{x_j}^{x_{j+1}} (x-t)_+^{r-1} dt + \sum_{j=1}^{n} c_j (x - x_j)_+^{r-1}$$

which satisfies the boundary conditions $P_{\underline{x}}^{(i)}(1) = 0$, $i = 0, 1, \ldots, r-1$, *and equioscillates* $n - r + 1$ *times on* $(0,1)$. $P_{\underline{x}}$ *has* $n - r$ *simple zeros* $\{\zeta_i(\underline{x})\}_{i=1}^{n-r}$, $0 < \zeta_1(\underline{x}) < \ldots < \zeta_{n-r}(\underline{x}) < 1$, *and* $x_i < \zeta_i(\underline{x}) < x_{i+r}$, $i = 1, \ldots, n-r$.

Based on Theorem 3, we define a recovery scheme S^* by interpolating the function f at the values $\{x_i\}_{i=1}^{n}$ by linear combinations of $1, x, \ldots, x^{r-1}$, $(x - \zeta_1(\underline{x}))_+^{r-1}, \ldots,$

$(x - \zeta_{n-r}(\underline{x}))_+^{r-1}$, i.e., $(S^*f)(x_i) = f(x_i)$, $i = 1, \ldots, n$. Since $x_i < \zeta_i(\underline{x}) < x_{i+r}$, $i = 1, \ldots, n - r$, such a recovery scheme is well-defined.

We prove the following theorem by using Theorems 1-3, and analysis paralleling that used in the proof of Theorem 2.

THEOREM 4. S^* is an optimal recovery scheme for $\mathcal{B}_r$, and $E(\underline{x}) = \|P_{\underline{x}}\|_\infty$. Furthermore, if $\underline{\xi} = (\xi_1, \ldots, \xi_n)$ are the knots of $Q_{n,r}$, then $\min_{\underline{x}} E(\underline{x}) = E(\underline{\xi}) = \|Q_{n,r}\|_\infty = \min_{\underline{x}}\|P_{\underline{x}}\|_\infty$.

Full details and extensions of all the above results will appear elsewhere.

References

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