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Interpolation on lines by ridge functions

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Abstract

In this paper we study the problem of interpolation on straight lines by linear combinations of a finite number of ridge functions with fixed directions. We analyze in totality the case of one and two directions.

Keywords: Ridge functions; Interpolation; First order difference equations

1. Introduction

A *ridge function*, in its simplest format, is a multivariate function of the form

$$f(\mathbf{a}\cdot\mathbf{x}),$$

defined for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ is a fixed non-zero vector, called a *direction*, $\mathbf{a}\cdot\mathbf{x} = \sum_{j=1}^n a_j x_j$ is the usual inner product, and f is a real-valued function defined on $\mathbb{R}$. Note that

$$f(\mathbf{a}\cdot\mathbf{x})$$

is constant on the hyperplanes $\{\mathbf{x} : \mathbf{a}\cdot\mathbf{x} = c\}$ for each $c \in \mathbb{R}$. Ridge functions are relatively simple multivariate functions. Ridge functions (formerly known as *plane waves*) were so-named in Logan, Shepp [8]. They appear in various areas and under numerous guises.

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Assume we are given pairwise linearly independent directions $\mathbf{a}^1, \ldots, \mathbf{a}^m$ in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Set

$$\mathcal{M}(\mathbf{a}^1, \ldots, \mathbf{a}^m) = \left\{ \sum_{i=1}^{m} f_i(\mathbf{a}^i \cdot \mathbf{x}) : f_i : \mathbb{R} \to \mathbb{R} \right\}.$$

That is, $\mathcal{M}(\mathbf{a}^1, \ldots, \mathbf{a}^m)$ is the subspace given by the set of all finite linear combinations of ridge functions with directions $\mathbf{a}^1, \ldots, \mathbf{a}^m$.

In this paper we are interested in the problem of interpolation by functions from $\mathcal{M}(\mathbf{a}^1, \ldots, \mathbf{a}^m)$. Interpolation at a finite number of points by such functions has been considered in various papers, see Braess, Pinkus [1], Sun [11], Reid, Sun [9], Weinmann [12] and Levesley, Sun [7]. The results therein are complete only in the case of two directions ($m = 2$), and three directions in $\mathbb{R}^2$.

In this paper we discuss the problem of interpolation by functions in $\mathcal{M}(\mathbf{a}^1, \ldots, \mathbf{a}^m)$ on straight lines. That is, assume we are given the straight lines $\{t\mathbf{b}^j + \mathbf{c}^j : t \in \mathbb{R}\}$, $\mathbf{b}^j \neq \mathbf{0}$, $j = 1, \ldots, k$. The question we ask is when, for every (or most) choice of data $g_j(t)$, $j = 1, \ldots, k$, there exists a function $G \in \mathcal{M}(\mathbf{a}^1, \ldots, \mathbf{a}^m)$ satisfying

$$G(t\mathbf{b}^j + \mathbf{c}^j) = g_j(t), \quad t \in \mathbb{R},\ j = 1, \ldots, k.$$

Let $\ell_j = \{t\mathbf{b}^j + \mathbf{c}^j : t \in \mathbb{R}\}$, $j = 1, \ldots, k$, and $L = \bigcup_{j=1}^{k} \ell_j$. The problem of interpolation of arbitrary data on these lines by linear combinations of ridge functions with fixed directions is equivalent to the problem of the representation of an arbitrarily given F defined on L by such combinations. Concerning this problem, we have the following result from Ismailov [4]. Interpolation to every function F on L from $\mathcal{M}(\mathbf{a}^1, \ldots, \mathbf{a}^m)$ is possible if and only if it is possible for any finite point set $\{\mathbf{x}^1, \ldots, \mathbf{x}^r\} \subset L$.

Let us first consider the elementary case of one direction, i.e., $\mathcal{M}(\mathbf{a})$. It is easily shown that interpolation is possible on a straight line $\{t\mathbf{b} + \mathbf{c} : t \in \mathbb{R}\}$ if and only if that straight line is not contained in any of the hyperplanes $\{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} = c\}$ for any $c \in \mathbb{R}$. In other words, $\mathbf{a} \cdot \mathbf{b} \neq 0$. Furthermore interpolation to all arbitrary functions on two distinct straight lines is never possible. By this we mean that for most functions g_1 and g_2 defined on $\mathbb{R}$, there does not exist a $G \in \mathcal{M}(\mathbf{a})$ satisfying

$$G(t\mathbf{b}^j + \mathbf{c}^j) = g_j(t), \quad t \in \mathbb{R},\ j = 1, 2,$$

where $t\mathbf{b}^j + \mathbf{c}^j$, $j = 1, 2$, define two distinct straight lines.

Proposition 1.1. (a) *We are given* $\mathbf{b} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, *the line* $\{t\mathbf{b} + \mathbf{c} : t \in \mathbb{R}\}$, *and an arbitrary function* g *defined on* $\mathbb{R}$. *Then for any given* $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ *there exists a function in*

$$\mathcal{M}(\mathbf{a}) = \{f(\mathbf{a} \cdot \mathbf{x}) : f : \mathbb{R} \to \mathbb{R}\},$$

such that

$$f(\mathbf{a} \cdot (t\mathbf{b} + \mathbf{c})) = g(t)$$

for all $t \in \mathbb{R}$ *if and only if* $\mathbf{a} \cdot \mathbf{b} \neq 0$.

(b) *Given* $\mathbf{b}^1, \mathbf{b}^2 \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, *and two distinct lines* $\{t\mathbf{b}^j + \mathbf{c}^j : t \in \mathbb{R}\}$, $j = 1, 2$, *then for almost all arbitrary functions* g_1, g_2 *defined on* $\mathbb{R}$ *there does not exist an* f *such that*

$$f(\mathbf{a} \cdot (t\mathbf{b}^j + \mathbf{c}^j)) = g_j(t)$$

for all $t \in \mathbb{R}$, *and* $j = 1, 2$.

Proof. We start with (a). We are interested in solving the interpolation problem

$$f(\mathbf{a}\cdot(t\mathbf{b}+\mathbf{c}))=g(t).$$

Set $(\mathbf{a}\cdot\mathbf{b})=B$ and $(\mathbf{a}\cdot\mathbf{c})=C$. Then the above reduces to solving

$$f(tB+C)=g(t)$$

for all t and any given g. Obviously, by a change of variable, this has a solution f if and only if $B\neq 0$.

The case (b) follows from (a). In (a) we saw that if there is a solution, then it is unique. Thus there is no room to maneuver. One can also prove it directly, as above. That is, assume we are given two lines $t\mathbf{b}^j+\mathbf{c}^j$, $j=1,2$, and two arbitrary functions g_1 and g_2. Set $(\mathbf{a}\cdot\mathbf{b}^j)=B_j$ and $(\mathbf{a}\cdot\mathbf{c}^j)=C_j$, $j=1,2$. The interpolation problem

$$f(t(\mathbf{a}\cdot\mathbf{b}^j)+(\mathbf{a}\cdot\mathbf{c}^j))=g_j(t),\quad j=1,2,$$

may be rewritten as

$$f(tB_j+C_j)=g_j(t),\quad j=1,2.$$

If $B_j=0$ for any j, then we cannot interpolate on that respective line. Assume $B_j\neq 0$, $j=1,2$. Thus we have, by a change of variable, that

$$f(s)=g_1((s-C_1)/B_1)$$

and

$$f(s)=g_2((s-C_2)/B_2)$$

implying that we must have

$$g_1((s-C_1)/B_1)=g_2((s-C_2)/B_2).$$

But for most (arbitrary) g_1 and g_2 this does not hold. There are other methods of verifying this simple result. ■

The above result illustrates why lines seem to be a natural interpolation set for ridge functions. If there exists an interpolant from $\mathcal{M}(\mathbf{a})$ to every function on a straight line, then that interpolant is unique.

We will totally analyze the interpolation/representation problem with two directions. There are two main results. The first deals with the case of two straight lines. (In the case of only one straight line we can appeal to Proposition 1.1.)

Assume we are given pairwise linearly independent directions $\mathbf{a}^1,\mathbf{a}^2$ in $\mathbb{R}^n$. Set

$$\mathcal{M}(\mathbf{a}^1,\mathbf{a}^2)=\{f_1(\mathbf{a}^1\cdot\mathbf{x})+f_2(\mathbf{a}^2\cdot\mathbf{x}):f_i:\mathbb{R}\to\mathbb{R}\}.$$

In addition, assume that we are given two distinct straight lines $\ell_j=\{t\mathbf{b}^j+\mathbf{c}^j:t\in\mathbb{R}\}$, $j=1,2$. Set

$$\mathbf{a}^i\cdot\mathbf{b}^j=B_{ij},\qquad \mathbf{a}^i\cdot\mathbf{c}^j=C_{ij},$$

for $i,j=1,2$. Then we have the following.

Theorem 1.2. *We are given pairwise linearly independent directions* $\mathbf{a}^1,\mathbf{a}^2$ *in* $\mathbb{R}^n$*, and two distinct straight lines*

$$\ell_j=\{t\mathbf{b}^j+\mathbf{c}^j:t\in\mathbb{R}\},\quad j=1,2.$$

If any of the following hold then for almost all g_1, g_2 defined on $\mathbb{R}$ there does not exist a $G \in \mathcal{M}(\mathbf{a}^1, \mathbf{a}^2)$ satisfying

$$G(t\mathbf{b}^j + \mathbf{c}^j) = g_j(t), \quad t \in \mathbb{R},\ j = 1, 2.$$

(a) $B_{11} = B_{21} = 0$,
(b) $B_{12} = B_{22} = 0$,
(c) $\operatorname{rank}\begin{bmatrix} B_{11} & B_{12} & C_{12} - C_{11} \\ B_{21} & B_{22} & C_{22} - C_{21} \end{bmatrix} = 1$,
(d) $B_{11} = B_{12} = 0$,
(e) $B_{21} = B_{22} = 0$,
(f) $B_{11}B_{22} + B_{12}B_{21} = 0$.

Assuming that (a)–(f) *do not hold there always exists a $G \in \mathcal{M}(\mathbf{a}^1, \mathbf{a}^2)$ satisfying*

$$G(t\mathbf{b}^j + \mathbf{c}^j) = g_j(t), \quad t \in \mathbb{R},\ j = 1, 2,$$

under the proviso that if $B_{11}B_{22} - B_{12}B_{21} \neq 0$, then we must impose a condition of the form

$$g_1(t_1) = g_2(t_2)$$

where t_1, t_2 satisfy

$$\begin{bmatrix} B_{11} & -B_{12} \\ B_{21} & -B_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} C_{12} - C_{11} \\ C_{22} - C_{21} \end{bmatrix}.$$

Theorem 1.2 can be restated in a simpler and more geometric form when we are in $\mathbb{R}^2$.

Theorem 1.3. *We are given pairwise linearly independent directions $\mathbf{a}^1, \mathbf{a}^2$ in $\mathbb{R}^2$, and two distinct straight lines*

$$\ell_j = \{t\mathbf{b}^j + \mathbf{c}^j : t \in \mathbb{R}\}, \quad j = 1, 2.$$

Assume that if ℓ_1 and ℓ_2 intersect, then g_1 and g_2 agree at this point of intersection. Then for almost all such g_1, g_2 defined on $\mathbb{R}$ there does not exist a $G \in \mathcal{M}(\mathbf{a}^1, \mathbf{a}^2)$ satisfying

$$G(t\mathbf{b}^j + \mathbf{c}^j) = g_j(t), \quad t \in \mathbb{R},\ j = 1, 2,$$

if and only if there exist $(k_1, k_2) \in \mathbb{R}^2 \backslash \{(0, 0)\}$ for which

$$(k_1\mathbf{a}^1 - k_2\mathbf{a}^2) \cdot \mathbf{b}^1 = 0$$

and

$$(k_1\mathbf{a}^1 + k_2\mathbf{a}^2) \cdot \mathbf{b}^2 = 0.$$

It should be noted that the continuity of g_1 and g_2 does not guarantee the continuity of the associated f_1 and f_2. Even in $\mathbb{R}^2$ we can have the interpolation problem solvable for all g_1 and g_2 that agree at their point of intersection, and yet for certain continuous (and bounded) g_1 and g_2 on $\mathbb{R}$ there exist no continuous f_1 and f_2 satisfying

$$f_1(\mathbf{a}^1 \cdot (t\mathbf{b}^j + \mathbf{c}^j)) + f_2(\mathbf{a}^2 \cdot (t\mathbf{b}^j + \mathbf{c}^j)) = g_j(t), \quad t \in \mathbb{R},\ j = 1, 2.$$

The second major result of this paper deals with the case of three (or more) distinct straight lines.

Theorem 1.4. *We are given pairwise linearly independent directions* $\mathbf{a}^1, \mathbf{a}^2$ *in* $\mathbb{R}^n$, *and three distinct straight lines*

$$\ell_j = \{t\mathbf{b}^j + \mathbf{c}^j : t \in \mathbb{R}\}, \quad j = 1, 2, 3.$$

Then for almost all g_1, g_2, g_3 *defined on* $\mathbb{R}$ *there does not exist a* $G \in \mathcal{M}(\mathbf{a}^1, \mathbf{a}^2)$ *satisfying*

$$G(t\mathbf{b}^j + \mathbf{c}^j) = g_j(t), \quad t \in \mathbb{R},\ j = 1, 2, 3.$$

We do not know what happens when we have three or more directions. The difficulties become so much greater. We conjecture that, paralleling Theorems 1.2 and 1.4, for given

$$\mathcal{M}(\mathbf{a}^1, \dots, \mathbf{a}^m) = \left\{ \sum_{i=1}^m f_i(\mathbf{a}^i \cdot \mathbf{x}) : f_i : \mathbb{R} \to \mathbb{R} \right\},$$

with pairwise linearly independent directions $\mathbf{a}^1, \dots, \mathbf{a}^m$ in $\mathbb{R}^n \backslash \{\mathbf{0}\}$, it should be possible, except in certain specific cases, to interpolate along m straight lines. And it should be impossible to interpolate arbitrary data on any $m + 1$ or more straight lines.

This paper is organized as follows. In Section 2 we present some preliminary material needed for the proof of Theorem 1.2. Two proofs of Theorem 1.2 are given. The first proof of Theorem 1.2 is detailed in Section 3. We also present in Section 3 a proof of Theorem 1.3 and an example of the phenomenon of noncontinuity of the f_1 and f_2 despite the continuity of g_1 and g_2, as explained above. In Section 4 we prove Theorem 1.2 via a somewhat different method and then use this proof and an analysis of certain first order difference equations to prove Theorem 1.4. We believe that the two different methods of the proof of Theorem 1.2 provide additional insight into this result.

2. Preliminaries

We start with some general remarks concerning interpolation from $\mathcal{M}(\mathbf{a}^1, \dots, \mathbf{a}^m)$.

Definition 2.1. Given directions $\{\mathbf{a}^i\}_{i=1}^m \subset \mathbb{R}^n$, we say that the set of points $\{\mathbf{x}^j\}_{j=1}^r \subset \mathbb{R}^n$ has the *NI-property* (non-interpolation property) with respect to the $\{\mathbf{a}^i\}_{i=1}^m$ if there exist $\{g_j\}_{j=1}^r \subset \mathbb{R}$ such that we *cannot* find $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, m$, satisfying

$$\sum_{i=1}^m f_i(\mathbf{a}^i \cdot \mathbf{x}^j) = g_j, \quad j = 1, \dots, r.$$

We say that the set of points $\{\mathbf{x}^j\}_{j=1}^r \subset \mathbb{R}^n$ has the *MNI-property* (minimal non-interpolation property) with respect to the $\{\mathbf{a}^i\}_{i=1}^m$, if $\{\mathbf{x}^j\}_{j=1}^r$ has the NI-property and no proper subset of the $\{\mathbf{x}^j\}_{j=1}^r$ has the NI-property.

The following result, to be found in Braess, Pinkus [1], easily follows from the above definitions.

Proposition 2.1. *Given directions* $\{\mathbf{a}^i\}_{i=1}^m \subset \mathbb{R}^n$, *the set of points* $\{\mathbf{x}^j\}_{j=1}^r \subset \mathbb{R}^n$ *has the NI-property if and only if there exists a vector* $\boldsymbol{\beta} = (\beta_1, \dots, \beta_r) \in \mathbb{R}^r \backslash \{\mathbf{0}\}$ *such that*

$$\sum_{j=1}^r \beta_j f_i(\mathbf{a}^i \cdot \mathbf{x}^j) = 0$$

for all $f_i : \mathbb{R} \to \mathbb{R}$ *and each* $i = 1, \ldots, m$. *The set* $\{\mathbf{x}^j\}_{j=1}^r \subset \mathbb{R}^n$ *has the MNI-property if and only if the vector* $\boldsymbol{\beta} \in \mathbb{R}^r \backslash \{\mathbf{0}\}$ *satisfying the above is unique, up to multiplication by a constant, and has no zero component.*

Remark. Note that the existence of $\boldsymbol{\beta} \neq \mathbf{0}$ satisfying the above is the existence of a non-trivial linear functional supported on the points $\{\mathbf{x}^j\}_{j=1}^k$ annihilating all functions from $\mathcal{M}(\mathbf{a}^1, \ldots, \mathbf{a}^m)$.

For the case $m = 2$, i.e, two directions, we recall from point interpolation the concept of an ordered closed path. We start with a definition.

Definition 2.2. The set of points $\{\mathbf{v}^i\}_{i=1}^p$ forms a *closed path* with respect to the distinct directions $\mathbf{a}^1$ and $\mathbf{a}^2$ if $p = 2q$, and for some permutation of the $\{\mathbf{v}^i\}_{i=1}^{2q}$ (which we assume to be as given) we have

$$\mathbf{a}^1 \cdot \mathbf{v}^{2j-1} = \mathbf{a}^1 \cdot \mathbf{v}^{2j}, \quad j = 1, \ldots, q$$

and

$$\mathbf{a}^2 \cdot \mathbf{v}^{2j} = \mathbf{a}^2 \cdot \mathbf{v}^{2j+1}, \quad j = 1, \ldots, q$$

where we set $\mathbf{v}^{2q+1} = \mathbf{v}^1$.

Some authors use the term *lightening bolt* rather than *closed path*. See Khavinson [5, p. 55] for many references to where this concept is used. It is also to be found in Dyn, Light, and Cheney [3].

Geometrically, having a closed path as in the above definition means that the points $\mathbf{v}^1, \ldots, \mathbf{v}^p$, and $\mathbf{v}^1$ again, form the vertices of a closed figure with edges in directions parallel to $\{\mathbf{x} : \mathbf{a}^i \cdot \mathbf{x} = 0\}$, $i = 1, 2$. An example, in $\mathbb{R}^2$, with directions parallel to the axes and $p = 10$, is given in Fig. 1.

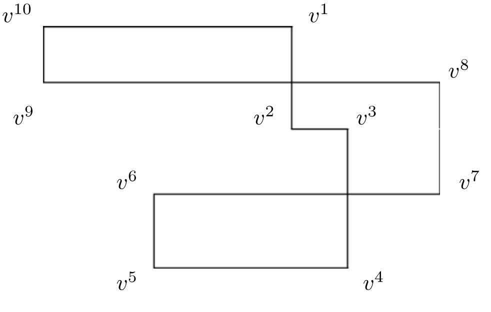

Fig. 1.

In this case of $m = 2$ in $\mathbb{R}^n$ a set of points has the NI-property if and only if a subset thereof forms a closed path. In Braess and Pinkus [1], this theorem is only stated in $\mathbb{R}^2$, but it holds in $\mathbb{R}^n$ by the same reasoning. In this theorem, we set

$$\Gamma_{\mathbf{a}^i}(\lambda) := \{\mathbf{x} : \mathbf{a}^i \cdot \mathbf{x} = \lambda\}.$$

Theorem 2.2. *Assume we are given two distinct directions* $\mathbf{a}^1$ *and* $\mathbf{a}^2$ *in* $\mathbb{R}^n$. *Then the following are equivalent.*

(a) *The set of points* $\{\mathbf{x}^j\}_{j=1}^r$ *has the NI-property.*
(b) *There exists a subset* $\{\mathbf{y}^j\}_{j=1}^s$ *of the* $\{\mathbf{x}^j\}_{j=1}^r$ *such that*

$$\left|\Gamma_{\mathbf{a}^i}(\lambda) \cap \{\mathbf{y}^j\}_{j=1}^s\right| \neq 1$$

for $i = 1, 2$ *and every* $\lambda \in \mathbb{R}$.
(c) *There exists a subset of the* $\{\mathbf{x}^j\}_{j=1}^r$ *which forms a closed path.*
(d) *There exists a subset* $\{\mathbf{z}^j\}_{j=1}^t$ *of the* $\{\mathbf{x}^j\}_{j=1}^r$ *and* $\varepsilon_j \in \{-1, 1\}$, $j = 1, \ldots, t$, *such that*

$$\sum_{j=1}^{t} \varepsilon_j f_i(\mathbf{a}^i \cdot \mathbf{z}^j) = 0$$

for every $f_i : \mathbb{R} \to \mathbb{R}$ *and* $i = 1, 2$.

The relevance of the above result is the following special case of a result from Ismailov [4, Theorem 2.6].

Theorem 2.3. *Assume we are given two distinct directions* $\mathbf{a}^1$ *and* $\mathbf{a}^2$ *in* $\mathbb{R}^n$. *Let* X *be any subset of* $\mathbb{R}^n$. *Then every function defined on* X *is in* $\mathcal{M}(\mathbf{a}^1, \mathbf{a}^2)$ *if and only if there are no finite set of points in* X *that form a closed path with respect to the directions* $\mathbf{a}^1$ *and* $\mathbf{a}^2$.

Similar results, in a slightly different framework, may be found in Khavinson [5] and Sternfeld [10]. These results answer the question of when, for a given subset $X \subset \mathbb{R}^n$ and for every bounded or continuous function G on X there exist bounded or continuous functions f_1 and f_2 satisfying

$$f_1(\mathbf{a}^1 \cdot \mathbf{x}) + f_2(\mathbf{a}^2 \cdot \mathbf{x}) = G(\mathbf{x})$$

for all $\mathbf{x} \in X$. In the case of continuous functions the set X is assumed to be compact.

3. Proofs of Theorems 1.2 and 1.3

Recall that we are given pairwise linearly independent directions $\mathbf{a}^1, \mathbf{a}^2$ in $\mathbb{R}^n$ and

$$\mathcal{M}(\mathbf{a}^1, \mathbf{a}^2) = \{f_1(\mathbf{a}^1 \cdot \mathbf{x}) + f_2(\mathbf{a}^2 \cdot \mathbf{x}) : f_i : \mathbb{R} \to \mathbb{R}\}.$$

In addition, we have two distinct straight lines $\ell_j = \{t\mathbf{b}^j + \mathbf{c}^j : t \in \mathbb{R}\}$, $j = 1, 2$. Set

$$\mathbf{a}^i \cdot \mathbf{b}^j = B_{ij}, \qquad \mathbf{a}^i \cdot \mathbf{c}^j = C_{ij},$$

for $i, j = 1, 2$.

Based on Theorem 2.3, we will search for closed paths on $X = \ell_1 \bigcup \ell_2$. We consider, sequentially, two-point, four-point, and $2r$-point ($r \geq 3$) closed paths with respect to $\mathbf{a}^1$ and $\mathbf{a}^2$. In fact, we will show that the union of two straight lines cannot contain $2r$-point ($r \geq 3$) closed paths with respect to any $\mathbf{a}^1$ and $\mathbf{a}^2$ without containing two-point or four-point closed subpaths. This is a geometric statement, although our proof is totally analytic.

Two-point closed paths. By definition $\{\mathbf{v}^1, \mathbf{v}^2\}$ is a two-point closed path if

$$\mathbf{a}^i \cdot \mathbf{v}^1 = \mathbf{a}^i \cdot \mathbf{v}^2 \tag{3.1}$$

for both $i = 1$ and $i = 2$. (We assume the points $\mathbf{v}^1$ and $\mathbf{v}^2$ are distinct.)

If the two points $\{\mathbf{v}^1, \mathbf{v}^2\}$ of the closed path lie on the line ℓ_1, then $\mathbf{v}^j = t_j\mathbf{b}^1 + \mathbf{c}^1$ for $t_1 \neq t_2$, and it easily follows from (3.1) that we must have

$$\mathbf{a}^1 \cdot \mathbf{b}^1 = \mathbf{a}^2 \cdot \mathbf{b}^1 = 0,$$

i.e., $B_{11} = B_{21} = 0$. In this case

$$f_1(\mathbf{a}^1 \cdot (t\mathbf{b}^1 + \mathbf{c}^1)) + f_2(\mathbf{a}^2 \cdot (t\mathbf{b}^1 + \mathbf{c}^1)) = f_1(\mathbf{a}^1 \cdot \mathbf{c}^1) + f_2(\mathbf{a}^2 \cdot \mathbf{c}^1)$$

for all t, i.e., $f_1(\mathbf{a}^1 \cdot \mathbf{x}) + f_2(\mathbf{a}^2 \cdot \mathbf{x})$ is a constant function on the line ℓ_1, and thus cannot interpolate to any non-constant function g_1 thereon. Conversely if $B_{11} = B_{21} = 0$ then any two distinct points $\{\mathbf{v}^1, \mathbf{v}^2\}$ of ℓ_1 is a two-point closed path. Similarly, the two points $\{\mathbf{v}^1, \mathbf{v}^2\}$ form a closed path on the line ℓ_2 if and only if

$$\mathbf{a}^1 \cdot \mathbf{b}^2 = \mathbf{a}^2 \cdot \mathbf{b}^2 = 0,$$

i.e., $B_{12} = B_{22} = 0$.

Assume that the two points $\{\mathbf{v}^1, \mathbf{v}^2\}$ forming a closed path are not on the same line. We assume, without loss of generality, that $\mathbf{v}^1 \in \ell_1$ and $\mathbf{v}^2 \in \ell_2$. Set $\mathbf{v}^1 = t_1\mathbf{b}^1 + \mathbf{c}^1$ and $\mathbf{v}^2 = t_2\mathbf{b}^2 + \mathbf{c}^2$. Thus our conditions are:

$$\begin{aligned} \mathbf{a}^1 \cdot (t_1\mathbf{b}^1 + \mathbf{c}^1) &= \mathbf{a}^1 \cdot (t_2\mathbf{b}^2 + \mathbf{c}^2) \\ \mathbf{a}^2 \cdot (t_1\mathbf{b}^1 + \mathbf{c}^1) &= \mathbf{a}^2 \cdot (t_2\mathbf{b}^2 + \mathbf{c}^2) \end{aligned} \tag{3.2}$$

that we rewrite as:

$$\begin{aligned} B_{11}t_1 + C_{11} &= B_{12}t_2 + C_{12} \\ B_{21}t_1 + C_{21} &= B_{22}t_2 + C_{22}. \end{aligned} \tag{3.3}$$

When do there exist solutions to this problem with distinct $\mathbf{v}^1$ and $\mathbf{v}^2$? The first possibility is that the matrix

$$\begin{bmatrix} B_{11} & -B_{12} \\ B_{21} & -B_{22} \end{bmatrix} \tag{3.4}$$

is nonsingular. In this case there is a unique t_1 and t_2 such that the above ((3.2) or (3.3)) holds. Moreover from (3.2) we see that for any f_1 and f_2

$$f_1(\mathbf{a}^1 \cdot (t_1\mathbf{b}^1 + \mathbf{c}^1)) + f_2(\mathbf{a}^2 \cdot (t_1\mathbf{b}^1 + \mathbf{c}^1)) = f_1(\mathbf{a}^1 \cdot (t_2\mathbf{b}^2 + \mathbf{c}^2)) + f_2(\mathbf{a}^2 \cdot (t_2\mathbf{b}^2 + \mathbf{c}^2)),$$

and thus we get the condition

$$g_1(t_1) = g_2(t_2). \tag{3.5}$$

This is a generic case. That is, in general the g_1 and g_2 are not absolutely arbitrary. There is a condition of the form (3.5) that must be satisfied by the given data. It may be that $\mathbf{v}^1 = \mathbf{v}^2$. For example, in $\mathbb{R}^2$ the matrix (3.4) is non-singular if and only if the two lines ℓ_1 and ℓ_2 are not parallel and meet at the common point $\mathbf{v}^1 = \mathbf{v}^2$ (see the proof of Theorem 1.3). In this case we certainly must have that $g_1(t_1) = g_2(t_2)$. In $\mathbb{R}^n$, $n > 2$, we need not have $\mathbf{v}^1 = \mathbf{v}^2$. The second possibility is that the matrix (3.4) is singular. Note that since we assume that there does not exist a two-point closed path on either one of the two lines, then it follows that the rank of this matrix is 1. In this case there exists a solution if and only if

$$(C_{12} - C_{11}, C_{22} - C_{21})^T$$

is in the range of the matrix (3.4). And if this is the case, then there are in fact an affine set of dimension one of such solutions (t_1, t_2). This is the condition

$$\text{rank} \begin{bmatrix} B_{11} & -B_{12} & C_{12} - C_{11} \\ B_{21} & -B_{22} & C_{22} - C_{21} \end{bmatrix} = 1.$$

To summarize: we have two-point closed paths and definitely cannot interpolate to arbitrarily given functions on ℓ_1 and ℓ_2 if we have any of:

(a) $B_{11} = B_{21} = 0$
(b) $B_{12} = B_{22} = 0$
(c) rank $\begin{bmatrix} B_{11} & B_{12} & C_{12} - C_{11} \\ B_{21} & B_{22} & C_{22} - C_{21} \end{bmatrix} = 1.$

If

$$\text{rank} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = 2$$

then we have a condition of the form

$$g_1(t_1) = g_2(t_2)$$

where (t_1, t_2) satisfy

$$\begin{bmatrix} B_{11} & -B_{12} \\ B_{21} & -B_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} C_{12} - C_{11} \\ C_{22} - C_{21} \end{bmatrix}.$$

We call this "Condition Z".

Four-point closed paths. We assume that (a)–(c) do not hold, i.e., there are no two-point closed paths, but there is a four-point closed path with distinct points $\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3, \mathbf{v}^4\}$. If three of these points lie on the same line ℓ_j, then we claim that there is a two-point closed path of the form (a) or (b). To see this, assume without loss of generality that $\mathbf{v}^1$, $\mathbf{v}^2$ and $\mathbf{v}^3$ lie on ℓ_1. Since

$$\mathbf{a}^1 \cdot (t_1\mathbf{b}^1 + \mathbf{c}^1) = \mathbf{a}^1 \cdot \mathbf{v}^1 = \mathbf{a}^1 \cdot \mathbf{v}^2 = \mathbf{a}^1 \cdot (t_2\mathbf{b}^1 + \mathbf{c}^1)$$

and $\mathbf{v}^1$ and $\mathbf{v}^2$ are distinct, i.e., $t_1 \neq t_2$, it follows that $B_{11} = \mathbf{a}^1 \cdot \mathbf{b}^1 = 0$. Similarly from

$$\mathbf{a}^2 \cdot \mathbf{v}^2 = \mathbf{a}^2 \cdot \mathbf{v}^3$$

it follows that $B_{21} = \mathbf{a}^2 \cdot \mathbf{b}^1 = 0$. Thus (a) holds and the two points $\{\mathbf{v}^1, \mathbf{v}^2\}$ are on a two-point closed path. If three points lie on ℓ_2, then (b) will hold.

By a suitable permutation, we may therefore assume that we have either:

(i) $\mathbf{v}^1, \mathbf{v}^2 \in \ell_1$ and $\mathbf{v}^3, \mathbf{v}^4 \in \ell_2$, or
(ii) $\mathbf{v}^1, \mathbf{v}^3 \in \ell_1$ and $\mathbf{v}^2, \mathbf{v}^4 \in \ell_2$.

Assume (i) holds. Then we obtain the equations

$$\begin{aligned} \mathbf{a}^1 \cdot (t_1\mathbf{b}^1 + \mathbf{c}^1) &= \mathbf{a}^1 \cdot (t_2\mathbf{b}^1 + \mathbf{c}^1) \\ \mathbf{a}^2 \cdot (t_2\mathbf{b}^1 + \mathbf{c}^1) &= \mathbf{a}^2 \cdot (t_3\mathbf{b}^2 + \mathbf{c}^2) \\ \mathbf{a}^1 \cdot (t_3\mathbf{b}^2 + \mathbf{c}^2) &= \mathbf{a}^1 \cdot (t_4\mathbf{b}^2 + \mathbf{c}^2) \\ \mathbf{a}^2 \cdot (t_4\mathbf{b}^2 + \mathbf{c}^2) &= \mathbf{a}^2 \cdot (t_1\mathbf{b}^1 + \mathbf{c}^1). \end{aligned}$$

Consider the first equation. We see that $(t_1 - t_2)B_{11} = 0$. But as $\mathbf{v}^1, \mathbf{v}^2 \in \ell_1\mathbf{v}^1 \neq \mathbf{v}^2$, we have $t_1 - t_2 \neq 0$. Thus $B_{11} = \mathbf{a}^1 \cdot \mathbf{b}^1 = 0$. Similarly from the third equation we obtain $B_{12} = \mathbf{a}^1 \cdot \mathbf{b}^2 = 0$. In this case our original interpolation problem

$$f_1(\mathbf{a}^1 \cdot (t\mathbf{b}^1 + \mathbf{c}^1)) + f_2(\mathbf{a}^2 \cdot (t\mathbf{b}^1 + \mathbf{c}^1)) = g_1(t)$$
$$f_1(\mathbf{a}^1 \cdot (t\mathbf{b}^2 + \mathbf{c}^2)) + f_2(\mathbf{a}^2 \cdot (t\mathbf{b}^2 + \mathbf{c}^2)) = g_2(t),$$

reduces to

$$f_1(\mathbf{a}^1 \cdot \mathbf{c}^1) + f_2(\mathbf{a}^2 \cdot (t\mathbf{b}^1 + \mathbf{c}^1)) = g_1(t)$$
$$f_1(\mathbf{a}^1 \cdot \mathbf{c}^2) + f_2(\mathbf{a}^2 \cdot (t\mathbf{b}^2 + \mathbf{c}^2)) = g_2(t).$$

Note that the function f_1 does not properly enter into the analysis and based on the proof of Proposition 1.1 (the case $m = 1, k = 2$) it easily follows that we cannot interpolate to almost any given g_1 and g_2. If we assume, say $\mathbf{v}^1, \mathbf{v}^2 \in \ell_2$ and $\mathbf{v}^3, \mathbf{v}^4 \in \ell_1$, then we get $B_{21} = B_{22} = 0$ and a similar analysis holds. Conversely if $B_{11} = B_{12} = 0$ or $B_{21} = B_{22} = 0$, then we can construct many four-point closed paths. Thus we also have the conditions

(d) $B_{11} = B_{12} = 0$
(e) $B_{21} = B_{22} = 0$.

Assuming (ii) we have

$$\begin{aligned}
\mathbf{a}^1 \cdot (t_1\mathbf{b}^1 + \mathbf{c}^1) &= \mathbf{a}^1 \cdot (t_2\mathbf{b}^2 + \mathbf{c}^2)\\
\mathbf{a}^2 \cdot (t_2\mathbf{b}^2 + \mathbf{c}^2) &= \mathbf{a}^2 \cdot (t_3\mathbf{b}^1 + \mathbf{c}^1)\\
\mathbf{a}^1 \cdot (t_3\mathbf{b}^1 + \mathbf{c}^1) &= \mathbf{a}^1 \cdot (t_4\mathbf{b}^2 + \mathbf{c}^2)\\
\mathbf{a}^2 \cdot (t_4\mathbf{b}^2 + \mathbf{c}^2) &= \mathbf{a}^2 \cdot (t_1\mathbf{b}^1 + \mathbf{c}^1).
\end{aligned} \tag{3.6}$$

Subtracting the first from the third equation and the second from the fourth, we obtain

$$B_{11}(t_3 - t_1) = B_{12}(t_4 - t_2)$$
$$B_{21}(t_3 - t_1) = -B_{22}(t_4 - t_2).$$

Since $\mathbf{v}^1, \mathbf{v}^3 \in \ell_1, \mathbf{v}^1 \neq \mathbf{v}^3$, we have $t_3 - t_1 \neq 0$, and similarly $t_4 - t_2 \neq 0$. Thus, if there is a solution (with distinct $\mathbf{v}^j$) then the associated determinant is zero, i.e.,

(f) $B_{11}B_{22} + B_{12}B_{21} = 0$.

If we assume that the B_{ij} are nonzero (the other cases are covered by (a), (b) and (d)), then it may be verified that, say, given any t_1 there exist t_2, t_3, t_4 such that (3.6) holds. The converse also holds. That is, (f) implies many solutions to (3.6).

$2r$-point closed paths, $r > 2$. We claim that the union of two straight lines cannot contain six-point, eight-point, etc. closed paths with respect to any pairwise linearly independent $\mathbf{a}^1$ and $\mathbf{a}^2$ without containing two-point or four-point closed paths satisfying at least one of (a)–(f). We first prove this for six-point closed paths, and then present the general analysis.

Assume $\{\mathbf{v}^1, \ldots, \mathbf{v}^6\}$ form a six-point closed path on ℓ_1 and ℓ_2, but no subset is a two or four-point closed path (other than satisfying Condition Z). If there are three consecutive points on any one line then, by the previous analysis, both $\mathbf{a}^1$ and $\mathbf{a}^2$ are orthogonal to $\mathbf{b}^1$ or $\mathbf{b}^2$. That is, we either have $B_{11} = B_{21} = 0$ or $B_{12} = B_{22} = 0$, i.e., (a) or (b) holds, and we have a two-point closed path. If three consecutive points are not on any one line, but two consecutive points are on one line, then by parity considerations we must have two other consecutive points on one line. These can be on the same line or on different lines. Let us consider both situations. Recall that if we have two consecutive points on one line, e.g. $\mathbf{v}^1, \mathbf{v}^2 \in \ell_1$, then $B_{11} = \mathbf{a}^1 \cdot \mathbf{b}^1 = 0$. Now if we have two pairs of two consecutive points on the same line (but no three consecutive points on one line), then we can assume, up to permutation, that $\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^4, \mathbf{v}^5 \in \ell_1$. But from $\mathbf{v}^1, \mathbf{v}^2 \in \ell_1$ we get $B_{11} = 0$ and from $\mathbf{v}^4, \mathbf{v}^5 \in \ell_1$ we get $B_{21} = 0$, i.e., (a) holds. This implies that we have a two-point closed path. If we have two pairs of consecutive points on different lines (but no three consecutive points on one line), and if we have $\mathbf{v}^1, \mathbf{v}^2 \in \ell_1$ then we must have either $\mathbf{v}^3, \mathbf{v}^4 \in \ell_2$ or $\mathbf{v}^5, \mathbf{v}^6 \in \ell_2$. In both cases we have $B_{11} = 0$ and $B_{12} = 0$, i.e., (d) holds and we have a four-point closed path.

What remains to analyze is, up to permutation, the case where $\mathbf{v}^1, \mathbf{v}^3, \mathbf{v}^5 \in \ell_1$ and $\mathbf{v}^2, \mathbf{v}^4, \mathbf{v}^6 \in \ell_2$. Writing down the resulting equations we have

$$\begin{aligned}
t_1 B_{11} + C_{11} &= t_2 B_{12} + C_{12} \\
t_2 B_{22} + C_{22} &= t_3 B_{21} + C_{21} \\
t_3 B_{11} + C_{11} &= t_4 B_{12} + C_{12} \\
t_4 B_{22} + C_{22} &= t_5 B_{21} + C_{21} \\
t_5 B_{11} + C_{11} &= t_6 B_{12} + C_{12} \\
t_6 B_{22} + C_{22} &= t_1 B_{21} + C_{21}.
\end{aligned}$$

Since $\mathbf{v}^1, \mathbf{v}^3$ and $\mathbf{v}^5$ are distinct points on ℓ_1, it follows that t_1, t_3 and t_5 are distinct values. Similarly, t_2, t_4 and t_6 are distinct values. In the above equations take differences of the equations containing the B_{11}, and also those containing the B_{22} to obtain

$$\begin{aligned}
(t_3 - t_1) B_{11} &= (t_4 - t_2) B_{12} \\
(t_5 - t_1) B_{11} &= (t_6 - t_2) B_{12} \\
(t_5 - t_3) B_{11} &= (t_6 - t_4) B_{12}
\end{aligned}$$

and

$$\begin{aligned}
(t_4 - t_2) B_{22} &= (t_5 - t_3) B_{21} \\
(t_6 - t_2) B_{22} &= -(t_3 - t_1) B_{21} \\
(t_6 - t_4) B_{22} &= -(t_5 - t_1) B_{21}.
\end{aligned}$$

From here we see that $B_{11} = 0$ if and only if $B_{12} = 0$ and (d) holds, while $B_{22} = 0$ if and only if $B_{21} = 0$ and (e) holds. As such, let us assume that $B_{ij} \neq 0$ for all i, j. There are many ways of proving that the above cannot hold. For example, it follows (dividing by B_{11} and B_{22}, as appropriate) that

$$\begin{aligned}
(t_3 - t_1) &= C(t_5 - t_3) \\
(t_5 - t_1) &= -C(t_3 - t_1) \\
(t_5 - t_3) &= -C(t_5 - t_1)
\end{aligned}$$

where $C \neq 0$. Multiplying the above equations we have $C = 1$. Thus each of the t_1, t_3, t_5 is an average of the other two, contradicting the fact that they are distinct.

What about closed paths of more points? The above argument may be extended as follows. Assume $\{\mathbf{v}^1, \ldots, \mathbf{v}^{2r}\}$ form a $2r$-point closed path on ℓ_1 and ℓ_2, $r > 3$, but no subset is a two or four-point closed path (other than satisfying Condition Z). If there are three consecutive points on any one line then, by the previous analysis, both $\mathbf{a}^1$ and $\mathbf{a}^2$ are orthogonal to $\mathbf{b}^1$ or $\mathbf{b}^2$. That is, we either have $B_{11} = B_{21} = 0$ or $B_{12} = B_{22} = 0$, i.e., (a) or (b), and we have a two-point closed path. Assume three consecutive points are not on any one line, but two consecutive points are on one line, namely $\mathbf{v}^1, \mathbf{v}^2 \in \ell_1$. As noted, this implies that $B_{11} = 0$. From parity considerations we must have two other consecutive points on one line. Starting at $\mathbf{v}^3 \in \ell_2$ consider the first time we have $\mathbf{v}^k, \mathbf{v}^{k+1}$ on the same line. If k is even then they must lie in ℓ_1 and $B_{21} = 0$. If k is odd then they must lie in ℓ_2 and $B_{12} = 0$. Thus it follows that (a) or (d) holds, a contradiction.

Thus we must analyze, up to permutations, the case where $\mathbf{v}^1, \mathbf{v}^3, \ldots, \mathbf{v}^{2r-1} \in \ell_1$ and $\mathbf{v}^2, \mathbf{v}^4, \ldots, \mathbf{v}^{2r} \in \ell_2$. Writing down the resulting equations we have

$$\begin{aligned} t_1 B_{11} + C_{11} &= t_2 B_{12} + C_{12} \\ t_2 B_{22} + C_{22} &= t_3 B_{21} + C_{21} \\ t_3 B_{11} + C_{11} &= t_4 B_{12} + C_{12} \\ \cdots &= \cdots \\ t_{2r-1} B_{11} + C_{11} &= t_{2r} B_{12} + C_{12} \\ t_{2r} B_{22} + C_{22} &= t_1 B_{21} + C_{21}. \end{aligned}$$

Since $\mathbf{v}^1, \mathbf{v}^3, \ldots, \mathbf{v}^{2r-1}$ are distinct points on ℓ_1, it follows that $t_1, t_3, \ldots, t_{2r-1}$ are distinct values. Similarly, $t_2, t_4, \ldots, t_{2r}$ are distinct values. In the above equations take differences of the equations containing the B_{11}, and also those containing the B_{22} to obtain

$$\begin{aligned} (t_3 - t_1) B_{11} &= (t_4 - t_2) B_{12} \\ (t_5 - t_1) B_{11} &= (t_6 - t_2) B_{12} \\ (t_5 - t_3) B_{11} &= (t_6 - t_4) B_{12} \\ \cdots &= \cdots \\ (t_{2r-1} - t_{2r-3}) B_{11} &= (t_{2r} - t_{2r-2}) B_{12} \end{aligned}$$

and

$$\begin{aligned} (t_4 - t_2) B_{22} &= (t_5 - t_3) B_{21} \\ (t_6 - t_2) B_{22} &= (t_7 - t_3) B_{21} \\ (t_6 - t_4) B_{22} &= (t_7 - t_5) B_{21} \\ \cdots &= \cdots \\ (t_{2r} - t_{2r-2}) B_{22} &= -(t_{2r-1} - t_1) B_{21}. \end{aligned}$$

From here we see that $B_{11} = 0$ if and only if $B_{12} = 0$ and (d) holds, while $B_{22} = 0$ if and only if $B_{21} = 0$ and (e) holds. As such, we may assume that $B_{ij} \neq 0$ for all i, j. It now follows (dividing by B_{11} and B_{22}, as appropriate) that

$$\begin{aligned} (t_3 - t_1) &= C(t_5 - t_3) \\ (t_5 - t_1) &= C(t_7 - t_3) \\ (t_5 - t_3) &= C(t_7 - t_5) \\ \cdots &= \cdots \\ (t_{2r-1} - t_{2r-3}) &= -C(t_{2r-1} - t_1) \end{aligned}$$

where $C \neq 0$. Multiplying the above equations we obtain $\pm C^k = 1$ for some $\pm$ and k. If we have $-C^k = 1$ where k is even, then we immediately obtain a contradiction. Otherwise we can have $C = 1$ or $C = -1$. If $C = -1$, then from the first equation we obtain $t_1 = t_5$ which is a contradiction. If $C = 1$, then each of the t_{2i-1} is an average of t_{2i-3} and t_{2i+1}, $i = 1, \ldots, r$, where $t_{-1} = t_{2r-1}$ and $t_{2r+1} = t_1$, i.e., we consider t_{2i-1} cyclically. In any case, as these t_{2i-1} are all distinct we have arrived at a contradiction.

Applying Theorem 2.3 we have therefore proved Theorem 1.2. ■

Proof of Theorem 1.3. Let us consider each of the conditions (a)–(f), as well as Condition Z. In $\mathbb{R}^2$, as $\mathbf{a}^1$ and $\mathbf{a}^2$ are linearly independent, their span is all of $\mathbb{R}^2$. It therefore follows that neither (a) nor (b) can hold. That is, it cannot be that both $\mathbf{a}^1$ and $\mathbf{a}^2$ are orthogonal to any non-zero vector in $\mathbb{R}^2$. Condition (d) holds if and only if $\mathbf{a}^1$ is orthogonal to both $\mathbf{b}^1$ and $\mathbf{b}^2$ (implying that ℓ_1 and ℓ_2 are parallel lines). Thus the desired condition holds with $k_1 = 1, k_2 = 0$. Similarly (e) holds if and only if $\mathbf{a}^2$ is orthogonal to both $\mathbf{b}^1$ and $\mathbf{b}^2$ and the desired condition holds with $k_1 = 0, k_2 = 1$.

We claim that (c) cannot hold. Since the span of $\mathbf{a}^1$ and $\mathbf{a}^2$ is all of $\mathbb{R}^2$, the first two columns of the matrix

$$\begin{bmatrix} B_{11} & B_{12} & C_{12} - C_{11} \\ B_{21} & B_{22} & C_{22} - C_{21} \end{bmatrix},$$

are non-zero. If

$$\operatorname{rank} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = 1,$$

then it follows that $\mathbf{b}^1 = \alpha \mathbf{b}^2$ for some $\alpha \neq 0$. If (c) holds and

$$\operatorname{rank} \begin{bmatrix} B_{11} & B_{12} & C_{12} - C_{11} \\ B_{21} & B_{22} & C_{22} - C_{21} \end{bmatrix} = 1,$$

it now also follows that $\mathbf{c}^2 - \mathbf{c}^1 = \beta \mathbf{b}^2$. Substituting we see that $\ell_1 = \ell_2$, a contradiction.

What about Condition Z? If t_1, t_2 satisfy

$$\begin{bmatrix} B_{11} & -B_{12} \\ B_{21} & -B_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} C_{12} - C_{11} \\ C_{22} - C_{21} \end{bmatrix},$$

it then follows that both $\mathbf{a}^1$ and $\mathbf{a}^2$ are orthogonal to

$$(t_1 \mathbf{b}^1 + \mathbf{c}^1) - (t_2 \mathbf{b}^2 + \mathbf{c}^2).$$

But in $\mathbb{R}^2$ this implies that

$$t_1 \mathbf{b}^1 + \mathbf{c}^1 = t_2 \mathbf{b}^2 + \mathbf{c}^2.$$

In other words, Condition Z simply says that g_1 and g_2 agree at the point of intersection of the lines ℓ_1 and ℓ_2.

Consider condition (f). By the previous analysis we may assume that $B_{21}, B_{22} \neq 0$. Thus we can rewrite (f) as

$$\frac{\mathbf{a}^1 \cdot \mathbf{b}^1}{\mathbf{a}^2 \cdot \mathbf{b}^1} = -\frac{\mathbf{a}^1 \cdot \mathbf{b}^2}{\mathbf{a}^2 \cdot \mathbf{b}^2}.$$

Let this common value be k. Then

$$(\mathbf{a}^1 - k\mathbf{a}^2) \cdot \mathbf{b}^1 = 0 \tag{3.7}$$

and

$$(\mathbf{a}^1 + k\mathbf{a}^2) \cdot \mathbf{b}^2 = 0. \tag{3.8}$$

This proves Theorem 1.3. ■

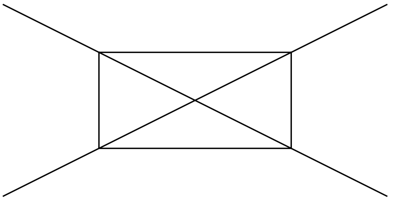

Fig. 2.

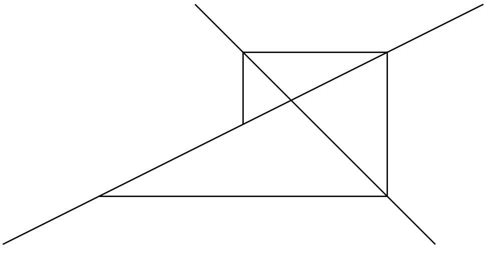

Fig. 3.

Figs. 2 and 3 illustrate what happens, geometrically, in the case where $\mathbf{a}^1 = (1, 0)$, $\mathbf{a}^2 = (0, 1)$ and neither $\mathbf{b}^1$ nor $\mathbf{b}^2$ are multiples of $\mathbf{a}^1$ or $\mathbf{a}^2$. In Fig. 2 we have that (3.7) and (3.8) hold and there are four point closed paths. In Fig. 3, (3.7) and (3.8) do not hold and we cannot have a closed path.

We now present an example where interpolation is always possible, but for some given continuous (and bounded) g_1 and g_2 there are no continuous (or bounded) f_1 and f_2 satisfying

$$f_1(\mathbf{a}^1 \cdot (t\mathbf{b}^j + \mathbf{c}^j)) + f_2(\mathbf{a}^2 \cdot (t\mathbf{b}^j + \mathbf{c}^j)) = g_j(t), \quad t \in \mathbb{R},\ j = 1, 2. \tag{3.9}$$

Example. Set $\mathbf{a}^1 = (1, -1)$, $\mathbf{a}^2 = (1, 1)$,

$$\ell_1 = \{t(1, 1/3) : t \in \mathbb{R}\}$$

and

$$\ell_2 = \{t(1, -1/3) + (0, 4/3) : t \in \mathbb{R}\}.$$

Thus, in our terminology, $\mathbf{b}^1 = (1, 1/3)$, $\mathbf{b}^2 = (1, -1/3)$, $\mathbf{c}^1 = (0, 0)$ and $\mathbf{c}^2 = (0, 4/3)$. It is readily verified that none of the conditions (a)–(f) of Theorem 1.2 holds or, equivalently as we are in $\mathbb{R}^2$, there exist no $(k_1, k_2) \in \mathbb{R}^2 \backslash \{(0, 0)\}$ for which

$$(k_1\mathbf{a}^1 - k_2\mathbf{a}^2) \cdot \mathbf{b}^1 = 0$$

and

$$(k_1\mathbf{a}^1 + k_2\mathbf{a}^2) \cdot \mathbf{b}^2 = 0.$$

Furthermore, the lines ℓ_1 and ℓ_2 intersect at the point $(2, 2/3)$ where $t_1 = t_2 = 2$. Thus for all g_1, g_2 satisfying $g_1(2) = g_2(2)$ we know from Theorem 1.2 that there exists a $G \in \mathcal{M}(\mathbf{a}^1, \mathbf{a}^2)$ satisfying

$$G(t(1, 1/3)) = g_1(t)$$

and

$$G(t(1,-1/3)+(0,4/3))=g_2(t).$$

Now

$$G(\mathbf{x})=f_1(\mathbf{a}^1\cdot\mathbf{x})+f_2(\mathbf{a}^2\cdot\mathbf{x})$$

for some f_1, f_2 defined on $\mathbb{R}$. We construct continuous (and bounded) g_1, g_2 for which any f_1, f_2 satisfying the above cannot be continuous (and bounded) on $\mathbb{R}$.

To this end, let $\{c_n\}_{n=1}^{\infty}$ be any decreasing sequence of positive numbers tending to zero for which

$$\sum_{n=1}^{\infty} c_n=\infty.$$

Set $g_1(t)=0$ on all $\mathbb{R}$, and define g_2 to satisfy

$$g_2\left(\sum_{k=0}^{2n}\frac{1}{2^k}\right)=c_n,$$

$g_2(t)=0$ for $t\geq 2$, $g_2(t)=c_1$ for $t\leq 7/4$, and g_2 is continuous and piecewise linear on $[7/4,2)$. That is, on the interval

$$\left[\sum_{k=0}^{2n}\frac{1}{2^k},\sum_{k=0}^{2n+2}\frac{1}{2^k}\right]$$

g_2 is a linear function with endpoint values c_n and c_{n+1}. Since the $c_n\downarrow 0$ it follows that g_2 is continuous (and bounded) on all $\mathbb{R}$, and $g_1(2)=g_2(2)$.

Consider the following set of points in $\mathbb{R}^2$. Let

$$\mathbf{x}^n=\left(\sum_{k=0}^{2n-1}\frac{1}{2^k},\sum_{k=0}^{2n-1}\frac{(-1)^k}{2^k}\right),$$

$n=1,2,\ldots$ and

$$\mathbf{y}^n=\left(\sum_{k=0}^{2n}\frac{1}{2^k},\sum_{k=0}^{2n}\frac{(-1)^k}{2^k}\right),$$

$n=1,2,\ldots$. It is a simple exercise to verify that $\mathbf{x}^n\in\ell_1$ and $\mathbf{y}^n\in\ell_2$ for all n, and, in addition, that

$$\mathbf{a}^1\cdot\mathbf{x}^n=\mathbf{a}^1\cdot\mathbf{y}^n,\quad n=1,2,\ldots,\tag{3.10}$$

and

$$\mathbf{a}^2\cdot\mathbf{x}^{n+1}=\mathbf{a}^2\cdot\mathbf{y}^n,\quad n=1,2,\ldots.\tag{3.11}$$

Assume f_1, f_2 satisfy (3.9). Thus we have

$$f_1(\mathbf{a}^1 \cdot \mathbf{x}^n) + f_2(\mathbf{a}^2 \cdot \mathbf{x}^n) = 0, \quad n = 1, 2, \ldots,$$

and

$$f_1(\mathbf{a}^1 \cdot \mathbf{y}^n) + f_2(\mathbf{a}^2 \cdot \mathbf{y}^n) = g_2\left(\sum_{k=0}^{2n} \frac{1}{2^k}\right) = c_n, \quad n = 1, 2, \ldots.$$

From (3.10) and taking differences in the above two equalities we obtain

$$f_2(\mathbf{a}^2 \cdot \mathbf{y}^n) - f_2(\mathbf{a}^2 \cdot \mathbf{x}^n) = c_n, \quad n = 1, 2, \ldots.$$

Thus

$$\sum_{n=1}^{r} \left(f_2(\mathbf{a}^2 \cdot \mathbf{y}^n) - f_2(\mathbf{a}^2 \cdot \mathbf{x}^n) \right) = \sum_{n=1}^{r} c_n.$$

From (3.11) we see that this is a telescoping sum and thus the left-hand-side equals

$$f_2(\mathbf{a}^2 \cdot \mathbf{y}^r) - f_2(\mathbf{a}^2 \cdot \mathbf{x}^1).$$

Since $\sum_{n=1}^{\infty} c_n = \infty$, it follows that

$$\lim_{r\to\infty} f_2(\mathbf{a}^2 \cdot \mathbf{y}^r) = \infty.$$

Now $\lim_{r\to\infty} \mathbf{y}^r = (2, 2/3)$ (the intersection point of the lines) and we have that f_2 is unbounded in a neighborhood of $t = 8/3$. As such, f_2 is not continuous at $t = 8/3$. The same must therefore hold for f_1 at the point $t = 4/3$.

The analysis of the first order difference equations in the next section gives an insight into the reason for this phenomenon.

4. Theorem 1.4 and first order difference equations

There is another approach to a proof of Theorem 1.2 that we now explain. Assume (a)–(f) do not hold, and if Condition Z holds then $g_1(t_1) = g_2(t_2)$, where t_1, t_2 are defined as previously in the statement of Theorem 1.2. We want to solve the equations

$$f_1(tB_{1j} + C_{1j}) + f_2(tB_{2j} + C_{2j}) = g_j(t), \quad j = 1, 2. \tag{4.1}$$

We start with the simpler case where we assume that at least one of the $B_{ij} = 0$. As all these cases are the same let us assume, without loss of generality, that $B_{11} = 0$. Since (a)–(f) do not hold, we have $B_{12}, B_{21} \neq 0$. Solving (4.1) for $j = 1$ with the change of variable $s = tB_{21} + C_{21}$ gives

$$f_2(s) = g_1((s - C_{21})/B_{21}) - f_1(C_{11}).$$

Substituting this into (4.1) with $j = 2$ and setting $tB_{12} + C_{12} = C_{11}$ give us

$$g_1\left(\frac{(C_{11} - C_{12})B_{22} + (C_{22} - C_{21})B_{12}}{B_{12}B_{21}}\right) = g_2\left(\frac{C_{11} - C_{12}}{B_{12}}\right). \tag{4.2}$$

That is, we have Condition Z with

$$t_1 = \frac{(C_{11} - C_{12})B_{22} + (C_{22} - C_{21})B_{12}}{B_{12}B_{21}}$$

and

$$t_2 = \frac{C_{11} - C_{12}}{B_{12}}.$$

Let $f_1(C_{11}) = \alpha$ for any $\alpha \in \mathbb{R}$. Solving in (4.1) with $j = 1$ gives us

$$f_2(s) = g_1((s - C_{21})/B_{21}) - \alpha. \tag{4.3}$$

From (4.1) with $j = 2$ and a change of variable we get

$$f_1(s) = g_2((s - C_{12})/B_{12}) - g_1\left(\frac{(s - C_{12})B_{22} + (C_{22} - C_{21})B_{12}}{B_{12}B_{21}}\right) + \alpha.$$

This is well-defined since on setting $s = C_{11}$ we obtain, by (4.2), that we have $f_1(C_{11}) = \alpha$. Thus we have determined solutions f_1 and f_2 for (4.1) in this case.

Let us now assume that $B_{ij} \neq 0$, $i, j = 1, 2$. By a change of variable we rewrite (4.1) as

$$f_1(s) + f_2\left(\frac{(s - C_{1j})B_{2j}}{B_{1j}} + C_{2j}\right) = g_j\left(\frac{s - C_{1j}}{B_{1j}}\right), \quad j = 1, 2. \tag{4.4}$$

Taking the difference between these equations we get

$$\begin{aligned} &f_2\left(\frac{(s - C_{11})B_{21}}{B_{11}} + C_{21}\right) - f_2\left(\frac{(s - C_{12})B_{22}}{B_{12}} + C_{22}\right) \\ &\quad = g_1\left(\frac{s - C_{11}}{B_{11}}\right) - g_2\left(\frac{s - C_{12}}{B_{12}}\right). \end{aligned} \tag{4.5}$$

Note that if

$$\frac{(s - C_{11})B_{21}}{B_{11}} + C_{21} = \frac{(s - C_{12})B_{22}}{B_{12}} + C_{22} \tag{4.6}$$

then the left hand-side of (4.5) is the zero function and this does not solve for f_2. Since g_1 and g_2 are arbitrarily taken, this leads to a contradiction. However, if (4.6) holds then (c) holds which contradicts our assumption. Note that if we solve (4.5) to obtain f_2, then from (4.4) we get f_1.

In this case we have reduced our problem, via (4.5), to that of solving

$$f_2(sD_1 + E_1) - f_2(sD_2 + E_2) = g(s)$$

for some almost arbitrary g where $D_1, D_2 \neq 0$ (since $B_{ij} \neq 0$, all i, j), $(D_1, E_1) \neq (D_2, E_2)$ (since (c) does not hold), and $D_1 \neq -D_2$ (since (f) does not hold). What about Condition Z as it applies to g? If $D_1 = D_2$ there is no condition on g. Otherwise we must have $g((E_2 - E_1)/(D_1 - D_2)) = 0$. By the change of variable $u = sD_1 + E_1$ and writing f in place of f_2 we obtain the more easily stated difference equation

$$f(u) - f(uD + E) = g(u) \tag{4.7}$$

where $D \neq 0$, $(D, E) \neq (1, 0)$, and $D \neq -1$. For $D \neq 1$ we have arbitrary g that must satisfy

$$g(E/(1 - D)) = 0. \tag{4.8}$$

From Theorem 1.2 we know that there exist solutions. We will now exhibit solutions to Eqs. (4.7) and (4.8), and also discuss their continuity properties and the extent to which these solutions are unique. We will then have the tools to prove Theorem 1.4. We refer the reader to Buck [2] and Kuczma [6] for related results. We highlight the main results as propositions.

We start with the case $D = 1$, i.e.,

$$f(u) - f(u+E) = g(u) \tag{4.9}$$

with $E \neq 0$. By the change of variable $v = u + E$, if necessary, we may assume $E > 0$.

Proposition 4.1. *Given any $c \in \mathbb{R}$ and arbitrary h defined on $[c, c+E)$, there is a unique f satisfying $f(u) = h(u)$ for $u \in [c, c+E)$ and Eq. (4.9). This function f is given by*

$$f(u) = h(u - kE) - \sum_{r=1}^{k} g(u - rE)$$

for $u \in [c + kE, c + (k+1)E)$, $k = 1, 2, \ldots,$ and by

$$f(u) = h(u + kE) + \sum_{r=0}^{k-1} g(u + rE)$$

for $u \in [c - kE, c - (k-1)E)$, $k = 1, 2 \ldots.$ In addition, if g is continuous on $\mathbb{R}$, h is continuous on $[c, c+E]$ and

$$h(c) - h(c+E) = g(c),$$

then f is continuous on all of $\mathbb{R}$.

Proof. Set $f(u) = h(u)$ on $[c, c+E)$, any h and any $c \in \mathbb{R}$. We rewrite (4.9) as

$$f(u) = f(u+E) + g(u) \tag{4.10}$$

or

$$f(u) = f(u-E) - g(u-E). \tag{4.11}$$

From (4.11) we have for $u \in [c+E, c+2E)$

$$f(u) = h(u-E) - g(u-E),$$

and for $u \in [c+2E, c+3E)$

$$f(u) = h(u-2E) - g(u-2E) - g(u-E).$$

Thus for $u \in [c + kE, c + (k+1)E)$, $k = 1, 2, \ldots,$

$$f(u) = h(u - kE) - \sum_{r=1}^{k} g(u - rE).$$

For $u < c$ we use (4.10). From (4.10) we have for $u \in [c - E, c)$

$$f(u) = h(u+E) + g(u),$$

and thus for $u \in [c-2E, c-E)$

$$f(u) = h(u+2E) + g(u+E) + g(u).$$

This gives us for $u \in [c - kE, c - (k-1)E)$, $k = 1, 2, \ldots,$

$$f(u) = h(u + kE) + \sum_{r=0}^{k-1} g(u + rE).$$

From the above equations defining f it easily follows that if g is continuous on $\mathbb{R}$, h is continuous on $[c, c+E]$ and

$$h(c) - h(c+E) = g(c),$$

then f is continuous on all of $\mathbb{R}$. ■

Remark. Note that if g is bounded, then f is not necessarily bounded no matter what the choice of h. (Take, for example, $g(u) = 1$ for all u.)

Remark. In the above and in the next propositions we will have uniqueness of the form "given arbitrary h and assuming $f = h$ on an interval(s)" then f is uniquely defined on all $\mathbb{R}$. Considering an interval is convenient for us, but it can be replaced by any set of points for which the orbits of the points u under the mapping $uD + E$ and its inverse mapping exactly cover $\mathbb{R}\backslash\{E/(1-D)\}$.

We now consider the case where $D \neq 1, -1, 0$. By the change of variable $v = uD + E$, if necessary, we may assume that $|D| > 1$. We first assume that $D > 1$.

Proposition 4.2. *Assume $D > 1$ in (4.7). Given c_1 such that $c_1 D + E > c_1$ and c_2 such that $c_2 D + E < c_2$, an arbitrary h_1 defined on $[c_1, c_1 D + E)$, and an arbitrary h_2 defined on $[c_2 D + E, c_2)$, then there exists a unique f satisfying (4.7) on $\mathbb{R}\backslash\{E/(1-D)\}$, where $f(u) = h_1(u)$ for $u \in [c_1, c_1 D + E)$, and $f(u) = h_2(u)$ for $u \in [c_2 D + E, c_2)$. In addition, if g is continuous on $(E/(1-D), \infty)$, h_1 is continuous on $[c_1, c_1 D + E]$ and*

$$h_1(c_1) - h_1(c_1 D + E) = g(c_1)$$

then f is continuous on $(E/(1-D), \infty)$. Similarly, if g is continuous on $(-\infty, E/(1-D))$, h_2 is continuous on $[c_2 D + E, c_2]$ and

$$h_2(c_2) - h_2(c_2 D + E) = g(c_2)$$

then f is continuous on $(-\infty, E/(1-D))$.

Proof. We rewrite (4.7) as

$$f(u) = f((u - E)/D) - g((u - E)/D). \tag{4.12}$$

Set $f(u) = h_1(u)$ on $[c_1, c_1 D + E)$. Thus for $u \in [c_1 D + E, (c_1 D + E)D + E)$ we have

$$f(u) = h_1((u - E)/D) - g((u - E)/D).$$

Continuing, for $u \in [c_1 D^2 + E(1 + D), c_1 D^3 + E(1 + D + D^2))$ we have

$$f(u) = h_1((u - E(1 + D))/D^2) - g((u - E(1 + D))/D^2) - g((u - E)/D)$$

etc. Now at the nth stage the right endpoint of the interval of definition equals

$$c_1 D^n + E(1 + D + \cdots + D^{n-1})$$

that tends to infinity as $n \uparrow \infty$. To see this, note that it equals

$$c_1 D^n + E\left(\frac{D^n-1}{D-1}\right) = \frac{D^n(c_1 D + E - c_1)}{D-1} - \frac{E}{D-1}$$

and, by assumption, $c_1 D + E - c_1 > 0$, $D - 1 > 0$. Thus the above process defines f on all of $[c_1, \infty)$. We now go in the reverse direction, i.e., write (4.7) as

$$f(u) = f(uD + E) + g(u). \tag{4.13}$$

Thus for $u \in [(c_1 - E)/D, c_1)$ we then have

$$f(u) = h_1(uD + E) + g(u).$$

For $u \in [(c_1 - E - DE)/D^2, (c_1 - E)/D)$ we have

$$f(u) = h_1(uD^2 + DE + E) + g(uD + E) + g(u),$$

etc. After the nth stage we are considering the interval whose left endpoint is

$$\frac{c_1 - E(1 + D + \cdots + D^{n-1})}{D^n}.$$

Since this equals

$$\frac{c_1(D-1) - E(D^n - 1)}{(D-1)D^n}$$

it follows that this decreases monotonically to

$$\frac{E}{1-D}$$

as n tends to infinity. Thus we have determined f on $(E/(1-D), \infty)$. We also easily see that if g is continuous thereon, h_1 is continuous on $[c, cD + E]$ and

$$h_1(c) - h_1(c_1 D + E) = g(c_1)$$

then f is continuous on $(E/(1-D), \infty)$.

On the interval $(-\infty, E/(1-D))$ we have the same analysis. Recall that c_2 satisfies $c_2 D + E < c_2$. Set $f(u) = h_2(u)$ on $[c_2 D + E, c_2)$. Thus for $u \in [(c_2 D + E)D + E, c_2 D + E)$ and by (4.12) we have

$$f(u) = h_2((u - E)/D) - g((u - E)/D).$$

Continuing, for $u \in [c_2 D^3 + E(1 + D + D^2), c_2 D^2 + E(1 + D))$ we have

$$f(u) = h_2((u - E(1 + D))/D^2) - g((u - E(1 + D))/D^2) - g((u - E)/D)$$

etc. Now

$$c_2 D^n + E(1 + D + \cdots + D^{n-1})$$

tends to minus infinity as $n \uparrow \infty$ since, by assumption, $c_2 D + E - c_2 < 0$, $D - 1 > 0$. Thus the above process defines f on all of $(-\infty, c_2]$. We now go in the reverse direction using (4.13). Thus for $u \in [c_2, (c_2 - E)/D)$ we then have

$$f(u) = h_2(uD + E) + g(u).$$

For $u \in [(c_2 - E)/D, (c_2 - E - DE)/D^2)$ we have

$$f(u) = h_2(uD^2 + DE + E) + g(uD + E) + g(u),$$

etc. After the nth stage we are considering the interval whose left endpoint is

$$\frac{c_2 - E(1 + D + \cdots + D^{n-1})}{D^n},$$

that increases to

$$\frac{E}{1 - D}$$

as n tends to infinity. Thus we have determined f on $(-\infty, E/(1 - D))$. We also easily see that if g is continuous thereon, h_2 is continuous on $[c_2 D + E, c_2]$ and

$$h_2(c) - h_2(cD + E) = g(c)$$

then f is continuous on $(-\infty, E/(1 - D))$. ■

There remains the case $D < -1$. This case is slightly different because the transformation u to $uD + E$ flips us back and forth. We have the following.

Proposition 4.3. *Assume $D < -1$ in (4.7). Choose any c satisfying $(cD + E)D + E > c$. For an arbitrary h defined on $[c, (cD + E)D + E)$, there exists a unique f defined on $\mathbb{R}\backslash\{E/(1 - D)\}$ satisfying (4.7), where $f(u) = h(u)$ for $u \in [c, (cD + E)D + E)$. In addition, if g is continuous on $\mathbb{R}\backslash\{E/(1 - D)\}$, h is continuous on $[c, (cD + E)D + E]$ and*

$$h(c) - h((cD + E)D + E) = g(c) + g(cD + E),$$

then f is continuous on $\mathbb{R}\backslash\{E/(1 - D)\}$.

Proof. Note that $(cD + E)D + E > c$ if and only if $c > E/(1 - D)$, since $D < -1$. We will first use

$$f(u) = f((u - E)/D) - g((u - E)/D).$$

Since $f(u) = h(u)$ on $[c, (cD + E)D + E)$, we have

$$f(u) = h((u - E)/D) - g((u - E)/D)$$

for $u \in ((cD + E)D^2 + DE + E, cD + E]$. Continuing, for $u \in [(cD + E)D + E, (cD + E)D^3 + D^2E + DE + E)$ we have

$$f(u) = h((u - E(1 + D))/D^2) - g((u - E(1 + D))/D^2) - g((u - E)/D)$$

etc. These intervals flip from side to side under the transformation since $D < 0$ and they also grow outwards. The right endpoints of the right more intervals are of the form

$$(cD + E)D^{2n-1} + E(D^{2n-2} + \cdots + D + 1).$$

This equals

$$\frac{D^{2n-1}}{D - 1}[(cD + E)(D - 1) + E] - \frac{E}{D - 1} = \frac{D^{2n-1}}{D - 1}[(cD + E)D - cD] - \frac{E}{D - 1}.$$

Since $c > E/(1-D)$ and $D < -1$ it follows that $(cD+E)D-cD > 0$. Furthermore $D^{2n-1} < 0$ and $D-1 < 0$. Thus this value tends to infinity as $n \uparrow \infty$. The left endpoints of the left more intervals are of the form

$$(cD+E)D^{2n} + E(D^{2n-1} + \cdots + D + 1).$$

This equals

$$\frac{D^{2n}}{D-1}[(cD+E)D - Dc] - \frac{E}{D-1}.$$

For the same reasons as above, except that the power $2n$ replaces $2n-1$, this tends to minus infinity as $n \uparrow \infty$. Note that the two sets of intervals uniquely define f on $[c, \infty)$ and $(-\infty, cD+E]$.

We now go the other way using (4.7). We obtain

$$f(u) = h(uD+E) + g(u)$$

for $u \in (cD+E, c/D - E/D]$. Continuing we have for $u \in [c/D^2 - E/D^2 - E/D, c)$ that

$$f(u) = h(uD^2 + DE + E) + g(uD+E) + g(u).$$

We continue in this way. The left endpoints of the right more intervals are of the form

$$\frac{c}{D^{2n}} - E\left(\frac{1}{D} + \frac{1}{D^2} + \cdots + \frac{1}{D^{2n}}\right).$$

This equals

$$\frac{c(D-1)+E}{D^{2n}(D-1)} - \frac{E}{D-1},$$

that, as $n \uparrow \infty$, tends to $E/(1-D)$ from above. The right endpoints of the left more intervals are of the form

$$\frac{c}{D^{2n-1}} - E\left(\frac{1}{D} + \frac{1}{D^2} + \cdots + \frac{1}{D^{2n-1}}\right).$$

This equals

$$\frac{c(D-1)+E}{D^{2n-1}(D-1)} - \frac{E}{D-1},$$

that, as $n \uparrow \infty$, tends to $E/(1-D)$ from below. This uniquely defines f on $(E/(1-D), c)$ and $(cD+E, E/(1-D))$. In summary, the function f is uniquely defined everywhere except at the point $E/(1-D)$.

If we want continuity of f on $\mathbb{R}\backslash\{E/(1-D)\}$, then we will attain it if we have the continuity g thereon, h on $[c, (cD+E)D+E]$ and

$$h(c) - h((cD+E)D+E) = g(c) + g(cD+E). \quad \blacksquare$$

Remark. The value of f at $E/(1-D)$ is immaterial, nor do we know anything about the behavior of f near $E/(1-D)$. In fact, by our Example at the end of Section 3 it follows that there exist continuous and bounded g for which every solution f is unbounded about $E/(1-D)$.

Proof of Theorem 1.4. Consider the interpolation problem

$$f_1(tB_{1j} + C_{1j}) + f_2(tB_{2j} + C_{2j}) = g_j(t), \quad j = 1, 2, 3. \tag{4.14}$$

Assume that this interpolation problem can be solved for most g_1, g_2, g_3. Then it can also be solved on any two of the three lines, and thus we can apply Theorem 1.2 with respect to any two of these three lines. The analysis of this section therefore holds for any two of these three lines. We claim that this leads to a contradiction.

Assume some of the B_{ij} are equal to zero. Without loss of generality, assume $B_{11} = 0$. From (4.3) we have

$$f_2(s) = g_1((s - C_{21})/B_{21}) - \alpha$$

for some arbitrary constant α. This satisfies (4.14) when $j = 1$. Substituting into (4.14) with $j = 2$ and $j = 3$ we get that

$$f_1(tB_{1j} + C_{1j}) = g_j(t) - g_1((tB_{2j} + C_{2j} - C_{21})/B_{21}) + \alpha, \quad j = 2, 3.$$

Obviously, for general g_2, g_3 there is no f_1 that simultaneously satisfies both equations.

Assume the B_{ij} are all not zero, $i = 1, 2$, $j = 1, 2, 3$. Then by the previous analysis we are led to f_2 simultaneously satisfying

$$f_2(u) - f_2(uD_j + E_j) = \widetilde{g}_j(u), \quad j = 1, 2,$$

for D_j, E_j satisfying the conditions $D_j \neq 0$, $(D_j, E_j) \neq (1, 0)$ and $D_j \neq -1$, $j = 1, 2$. Here $\widetilde{g}_1, \widetilde{g}_2$ are arbitrary functions. Start with $j = 1$. Then from Propositions 4.1–4.3 we see that we can arbitrarily define f_2 on a finite interval (or pair of intervals) depending on D_1, E_1. But on the rest of $\mathbb{R}$ the function f_2 is uniquely determined by this arbitrary function and $\widetilde{g}_1$. But then for almost all $\widetilde{g}_2$ we cannot solve (4.14) for $j = 2$. There is simply insufficient flexibility. This proves the result. ∎

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