Spectral Properties of Totally Positive Kernels and Matrices

Allan Pinkus

Abstract. We detail the history and present complete proofs of the spectral properties of totally positive kernels and matrices.

§1. Introduction

There seem to have been four central topics which historically led to the development of the theory of total positivity. These included the variation diminishing (VD) properties, the study of which was initiated by I. J. Schoenberg in 1930 (see [30]), and the study of Pólya frequency functions also initiated by I. J. Schoenberg in the late 1940's and early 1950's (see the relevant papers in [31]). These two topics are extensively investigated and expanded upon in Karlin [20]. In addition we have research connected with ordinary differential equations whose Green's function is totally positive (work started by M. G. Krein and some of his colleagues in the mid 1930's), and finally the study of the spectral properties of totally positive kernels and matrices. This last topic was actually the first studied. There are various misconceptions relating to the history and priority of results concerning spectral properties of totally positive kernels and matrices. The main aim of this paper is to review this history and provide complete proofs of the main results.

There are, as we shall see, differences between the matrix and kernel cases. These differences are not so much conceptual, but more in terms of history and proofs. The analysis of the spectral properties of totally positive matrices is simpler, well-understood, and easily documented. It is essentially all laid out in the 1937 paper Gantmacher, Krein [11] (an announcement of which appears in 1935 in Gantmacher, Krein [10]). This paper contains most of the known results relating to spectral properties of totally positive matrices, and many other important results concerning totally positive matrices (except that they were then ignorant of the variation diminishing properties connected with such matrices). This paper, in slightly expanded form, is most of Chapter II

Total Positivity and its Applications
M. Gasca and C. A. Micchelli (eds.), pp. 1–35.

ISBN xxx.

of the book Gantmacher, Krein [13]. A review of only the spectral properties of totally positive matrices would not be warranted.

The more interesting case is that of the spectral properties of totally positive continuous kernels. As is often the case, the study of this problem predates that of the associated matrix problem. By 1918 O. D. Kellogg [24] had proved the main spectral properties of a continuous *symmetric* totally positive kernel. In 1936, F. R. Gantmacher [6] announced the analogous result in the continuous *non-symmetric* case. He also indicated how the proof should go, and it follows very essentially the same lines as the proof in the matrix case. Surprisingly he does not reference in this paper either the announcement which appeared in 1935 or the paper which appeared in 1937, mentioned above, by Krein and himself which dealt with the totally analogous problems for matrices. Both the 1935 announcement and the 1937 paper of Gantmacher and Krein do, however, mention this paper of Gantmacher. But they only mention it in passing and say nothing about what is in the paper. In the book of Gantmacher, Krein [13], this paper of Gantmacher is mentioned briefly but twice. This all seems rather odd. One can only speculate as to the possible reasons for this neglect. The work of Kellogg and of Gantmacher on this subject certainly deserves more recognition.

We organize this review paper as follows. In Section 2 we collect some of the notation and basic facts which we will subsequently use. Section 3 is devoted to the results of Kellogg concerning spectral properties of continuous symmetric totally positive kernels. The non-symmetric case, with proofs of the theorems of Jentzsch and Schur, is considered in Section 4. In Section 5 we quickly, and mainly for completeness, present and prove the central result in the matrix case.

§2. Basic Facts and Notation

For a given positive integer n we set

$$I_p = \{\mathbf{i} = (i_1, \ldots, i_p) \,:\, 1 \le i_1 < \cdots < i_p \le n\}$$

where $1 \le p \le n$, and $\mathbf{i} \in \mathbb{Z}^p$. If $B = (b_{ij})_{i,j=1}^n$ is an $n \times n$ matrix, then we set for $\mathbf{i}, \mathbf{j} \in I_p$

$$B_{[p]}(\mathbf{i}, \mathbf{j}) = B\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix} = \det\left(b_{i_k j_\ell}\right)_{k,\ell=1}^p .$$

The pth *compound matrix* of B is denoted by $B_{[p]}$ and is defined as the $\binom{n}{p} \times \binom{n}{p}$ matrix with entries

$$\left(B_{[p]}(\mathbf{i}, \mathbf{j})\right)_{\mathbf{i},\mathbf{j} \in I_p}$$

where the $\mathbf{i} \in I_p$ are arranged in lexicographic order, i.e., $\mathbf{i} \ge \mathbf{j}$ ($\mathbf{i} \ne \mathbf{j}$) if the first non-zero term in the sequence $i_1 - j_1, \ldots, i_p - j_p$ is positive.

Assume $B = CD$ where B, C and D are $n \times n$ matrices. The *Cauchy-Binet formula* (which also holds for non-square matrices) (see e.g., Karlin [20, p. 1]), may be written as follows. For each $\mathbf{i}, \mathbf{j} \in I_p$

$$B_{[p]}(\mathbf{i}, \mathbf{j}) = \sum_{\mathbf{k} \in I_p} C_{[p]}(\mathbf{i}, \mathbf{k}) D_{[p]}(\mathbf{k}, \mathbf{j}) .$$

Definition 2.1. *An $n \times n$ matrix A is said to be totally positive* (TP) *if*

$$A_{[p]}(\mathbf{i}, \mathbf{j}) \geq 0 \tag{2.1}$$

for all $\mathbf{i}, \mathbf{j} \in I_p$, and all $p = 1, \ldots, n$. It is said to be strictly totally positive (STP) *if strict inequality always holds in (2.1).*

We say that $\lambda^* \in \mathbb{C}$ is an eigenvalue, and $\mathbf{u} \in \mathbb{C}^n$ ($\mathbf{u} \neq \mathbf{0}$) an associated eigenvector for A if

$$A\mathbf{u} = \lambda^* \mathbf{u} .$$

To each $n \times n$ matrix the sum total of the algebraic multiplicities of all its eigenvalues (multiplicities of the zeros of the characteristic polynomial) is n. Assume λ^* is an eigenvalue with algebraic multiplicity m. There exist at most m linearly independent eigenvectors. Moreover there exists an m-dimensional subspace of "generalized eigenvectors". That is, we can find linearly independent $\mathbf{u}^1, \ldots, \mathbf{u}^m$ (whose union over all distinct eigenvalues spans $\mathbb{C}^n$) such that

$$A\mathbf{u}^j = \sum_{\ell=1}^{m} b_{j\ell} \mathbf{u}^\ell , \qquad j = 1, \ldots, m ,$$

where the $m \times m$ matrix $(b_{j\ell})_{j,\ell=1}^m$ has characteristic polynomial $(\lambda - \lambda^*)^m$.

For a non-zero vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$, we define two notions of the number of sign changes of the vector $\mathbf{c}$. These are:

$S^-(\mathbf{c})$ — the number of sign changes in the sequence $c_1, \ldots, c_n$ with zero terms discarded.

$S^+(\mathbf{c})$ — the maximum number of sign changes in the sequence $c_1, \ldots, c_n$ where zero terms are arbitrarily assigned values $+1$ and -1.

For any p vectors $\mathbf{u}^1, \ldots, \mathbf{u}^p \in \mathbb{C}^n$ we define

$$\left(\mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^p\right)(\mathbf{i}) = \det \left(u^j_{i_\ell}\right)_{j,\ell=1}^p$$

for each $\mathbf{i} = (i_1, \ldots, i_p) \in I_p$. This vector (in $\mathbb{C}^{\binom{n}{p}}$) is called the *Grassman product* or *wedge product* or *exterior product* of $\mathbf{u}^1, \ldots, \mathbf{u}^p$. Obviously $\mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^p = \mathbf{0}$ if and only if the $\mathbf{u}^1, \ldots, \mathbf{u}^p$ are linearly dependent. It is also well-known, and easily proven, that if the $\mathbf{u}^1, \ldots, \mathbf{u}^m$ are linearly independent, then the $\binom{m}{p}$ vectors $\{\mathbf{u}^{j_1} \wedge \cdots \wedge \mathbf{u}^{j_p}\}$ $(1 \le j_1 < \cdots < j_p \le m)$ are linearly independent. From the Cauchy-Binet formula

$$A_{[p]}(\mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^p) = A\mathbf{u}^1 \wedge \cdots \wedge A\mathbf{u}^p \,.$$

For each $p = 1, 2, \ldots$, we define the p-simplex

$$S_p = \{\mathbf{x} = (x_1, \ldots, x_p) \,:\, 0 \le x_1 \le \cdots \le x_p \le 1\} \,.$$

For convenience we also define

$$S_p^* = \{\mathbf{x} = (x_1, \ldots, x_p) \,:\, 0 \le x_1 < \cdots < x_p \le 1\} \,.$$

Given a kernel $K \in C([0,1] \times [0,1])$, we define in an analogous fashion the pth *compound kernel* $K_{[p]} \in C(S_p \times S_p)$ by

$$K_{[p]}(\mathbf{x}, \mathbf{y}) = K\begin{pmatrix} x_1, \ldots, x_p \\ y_1, \ldots, y_p \end{pmatrix} = \det\left(K(x_i, y_j)\right)_{i,j=1}^p \,.$$

It will sometimes be convenient to consider this kernel as having been defined on $R_p \times R_p$ where $R_p = [0,1]^p$.

If $L, M \in C([0,1] \times [0,1])$ and

$$K(x, y) = \int_0^1 L(x, z) M(z, y)\, dz \,,$$

then the *Basic Composition Formula* (a direct generalization of Cauchy-Binet) (see Karlin [20, p. 17]) may be written as

$$K_{[p]}(\mathbf{x}, \mathbf{y}) = \int_{S_p} L_{[p]}(\mathbf{x}, \mathbf{z}) M_{[p]}(\mathbf{z}, \mathbf{y})\, d\mathbf{z} \,.$$

Definition 2.2. *A kernel $K \in C([0,1] \times [0,1])$ is said to be totally positive (TP) if*

$$K_{[p]}(\mathbf{x}, \mathbf{y}) \ge 0 \tag{2.2}$$

for all $\mathbf{x}, \mathbf{y} \in S_p^*$, and all $p = 1, 2, \ldots$. It is said to be strictly totally positive (STP) if strict inequality holds in (2.2) for all $\mathbf{x}, \mathbf{y} \in S_p^*$, and all $p = 1, 2, \ldots$.

Aside from the pth compound kernel defined above, we also have the rth *iterated kernel* K_r defined by $K_1(x, y) = K(x, y)$, and

$$K_r(x, y) = \int_0^1 K_{r-1}(x, z) K(z, y)\, dz \,.$$

(The matrix analogue of this is simply the power of a matrix.) It is a consequence of the Basic Composition Formula that the operations of iteration and compound are interchangeable, i.e., for positive integers p and r,

$$(K_{[p]})_r = (K_r)_{[p]} \,.$$

We say that λ^* is an eigenvalue and ϕ ($\phi \neq 0$) is an eigenfunction of K if

$$\int_0^1 K(x,y)\phi(y)\,dy = \lambda^*\phi(x)\,.$$

Let $d_0 = 1$, and

$$d_p = \frac{1}{p!}\int_{R_p} K_{[p]}(\mathbf{x},\mathbf{x})\,d\mathbf{x} = \left(\int_{S_p} K_{[p]}(\mathbf{x},\mathbf{x})\,d\mathbf{x}\right).$$

Then

$$D(\mu) := \sum_{p=0}^{\infty} d_p(-\mu)^p$$

is called the *Fredholm determinant* of K. This series converges for all $\mu \in \mathbb{C}$, and thus $D(\mu)$ is an entire function of μ. $D(\mu^*) = 0$ if and only if $\lambda^* = 1/\mu^*$ is an eigenvalue of K. Since $D(\mu)$ is entire each zero μ^* is of finite multiplicity, and this multiplicity is called the algebraic multiplicity of the eigenvalue λ^* (as a root of the Fredholm determinant of K). Assume λ^* is an eigenvalue with algebraic multiplicity m. The number of linearly independent eigenfunctions (the geometric multiplicity) with eigenvalue λ^* is at most m. Moreover there does exist an m-dimensional subspace of "generalized eigenfunctions". That is, there exist m (and no more than m) linearly independent functions $\phi_1,\ldots,\phi_m$ such that

$$\int_0^1 K(x,y)\phi_i(y)\,dy = \sum_{j=1}^{m} b_{ij}\phi_j(x)\,, \quad i = 1,\ldots,m,$$

where the $m \times m$ matrix $(b_{ij})_{i,j=1}^m$ has characteristic polynomial $(\lambda^* - \lambda)^m$. (For a review of this theory, see e.g., Goursat [16] or Smithies [33].)

If $\lambda_0, \lambda_1, \ldots,$ are all the eigenvalues of K, listed to their algebraic multiplicity as roots of the Fredholm determinant of K, then

$$s_r = \int_0^1 K_r(x,x)\,dx = \sum_{i=0}^{\infty} \lambda_i^r\,,$$

which converges for $r \geq 2$.

For $f \in C[0,1]$, we let $Z(f)$ count the number of distinct zeros of f on $[0,1]$. We will use $Z_{(0,1)}(f)$ to denote the number of distinct zeros on $(0,1)$. $S(f)$ and $S_{(0,1)}(f)$ will denote the number of sign changes of f in $[0,1]$ and $(0,1)$, respectively.

For any p functions $\psi_1,\ldots,\psi_p \in C[0,1]$ we define

$$(\psi_1 \wedge \cdots \wedge \psi_p)(\mathbf{x}) = \det\left(\psi_i(x_j)\right)_{i,j=1}^p$$

for $\mathbf{x} \in S_p$ (or R_p). This is called the *Grassman product* or *wedge product* or *exterior product* of $\psi_1,\ldots,\psi_p$. Obviously $\psi_1 \wedge \cdots \wedge \psi_p = 0$ if and only if the $\psi_1, \ldots, \psi_p$ are linearly dependent. It is also well-known, and easily proven, that if the $\psi_1, \ldots, \psi_m$ are linearly independent, then the $\binom{m}{p}$ functions $\{\psi_{j_1} \wedge \cdots \wedge \psi_{j_p}\}$ $(1 \leq j_1 < \cdots < j_p \leq m)$ are linearly independent.

From the Basic Composition Formula

$$\int_{S_p} K_{[p]}(\mathbf{x}, \mathbf{y})\,(\psi_1 \wedge \cdots \wedge \psi_p)\,(\mathbf{y})\,d\mathbf{y} = ((K\psi_1) \wedge \cdots \wedge (K\psi_p))\,(\mathbf{x})\,,$$

where

$$(K\psi)(x) = \int_0^1 K(x,y)\psi(y)\,dy\,.$$

§3. The Beginning: O. D. Kellogg

In this section we will try to review some of the contributions of O. D. Kellogg to the development of the spectral theory of totally positive kernels. Oliver Dimon Kellogg (1878–1932) was an American who obtained his Ph. D. from Göttingen in 1903 under the supervision of D. Hilbert. (He was Hilbert's first Ph. D. student to do a thesis on integral equations.) In 1905 Kellogg accepted a position at the University of Missouri. He moved to Harvard in 1919, where he remained until his death (see G. D. Birkhoff [3]). He is best known for his work in potential theory, and his book *Foundations of Potential Theory* is still in print today. Kellogg wrote two papers relevant to the topic under discussion, Kellogg [23], and [24]. The first has to do with orthogonal sets of functions, while the second deals with integral equations. A third paper, Kellogg [25], is devoted to the related topic of Sturm-Liouville ordinary differential equations. We will not discuss this third paper here.

The first paper, Kellogg [23], is short (5 pages) and simple. I quote the first three sentences from the paper. *The sets of orthogonal functions which occur in mathematical physics have, in general, the property that each changes sign in the interior of the interval on which they are orthogonal once more than its predecessor. So universal is this property that such sets are frequently referred to as sets of "oscillating functions." The question arises, is this property of oscillation inherent in that of orthogonality?* Kellogg then shows, by simple example, that the "oscillating" property does not solely depend on the orthogonality. Moreover he then introduces an additional condition (which he later terms Property (D) in Kellogg [24]) which is the following.

Definition 3.1. *We say that the real continuous functions $\phi_0, \ldots, \phi_n, \ldots$, orthogonal on $(0,1)$ satisfy Property (D) if for every $n = 0, 1, 2, \ldots$, and any $0 < x_0 < \cdots < x_n < 1$ we have*

$$\det(\phi_i(x_j))_{i,j=0}^n > 0\,.$$

In other words, the $\{\phi_i\}$ are orthogonal and constitute what we today call a *CT*-system (complete Chebyshev system). Kellogg then goes on to prove, for such functions, a series of properties which we assemble together.

Proposition 3.1. *Let $\{\phi_i\}$ be an orthogonal set of continuous functions on $(0,1)$ satisfying Property (D), and set*

$$\Phi_n = \operatorname{span}\{\phi_0, \ldots, \phi_n\}\,.$$

Then:

1) *Given any $n+1$ points $0 < x_0 < x_1 < \cdots < x_n < 1$ and data $c_0, c_1, \ldots, c_n$, there exists a $\phi \in \Phi_n$ satisfying $\phi(x_j) = c_j$, $j = 0, \ldots, n$.*
2) *For each $\phi \in \Phi_n \backslash \{0\}$, $Z(\phi) \le n$.*
3) *If $\phi \in \Phi_n \backslash \{0\}$ and $Z(\phi) = n$, then ϕ changes sign at each of its zeros.*
4) *If $\psi \in C(0,1)$ is orthogonal to Φ_n, then ψ has at least $n+1$ sign changes.*
5) *For every choice of non-trivial $(a_m, \ldots, a_n)$*

$$m \le S_{(0,1)}(\sum_{i=m}^{n} a_i\phi_i) \le Z_{(0,1)}(\sum_{i=m}^{n} a_i\phi_i) \le n\,.$$

6) *The zeros of ϕ_{n-1} and ϕ_n strictly interlace.*

We do not prove this result here. The proof is fairly simple, and we will eventually prove a slightly more general result. The properties listed in Proposition 3.1 are familiar. They are (aside from the orthogonality) properties associated with eigenfunctions of an integral equation whose kernel is strictly totally positive.

Kellogg, in fact, ends this paper by expressing his hope of finding conditions on the kernel of an integral equation so that the eigenfunctions will satisfy Property (D). In other words he seems to have known where he was going, because this hope is fulfilled in Kellogg [24]. The main result of that paper is the following result, which we precede with a definition.

Definition 3.2. *Let K be real, continuous and symmetric on $[0,1] \times [0,1]$. We say that K satisfies Property (K) if*

$$a)\ K\begin{pmatrix} x_0, x_1, \ldots, x_n \\ y_0, y_1, \ldots, y_n \end{pmatrix} \ge 0, \qquad n = 0, 1, \ldots,$$

for any $0 < x_0 < x_1 < \cdots < x_n < 1$, and $0 < y_0 < y_1 < \cdots < y_n < 1$.

$$b)\ K\begin{pmatrix} x_0, x_1, \ldots, x_n \\ x_0, x_1, \ldots, x_n \end{pmatrix} > 0, \qquad n = 0, 1, \ldots,$$

for any $0 < x_0 < x_1 < \cdots < x_n < 1$.

Theorem 3.2. (Kellogg [24]) *Assume K satisfies Property (K). Then the eigenvalues of K are positive and simple, and for each $n = 0, 1, \ldots,$ the $n+1$ eigenfunctions associated with the largest $n+1$ eigenvalues of K may be chosen to satisfy Property (D).*

Kellogg's proof of Theorem 3.2 is different from that which we will give in Section 4, but has many of the same ideas. As such we will present much of it in detail. The kernel K is continuous, real, symmetric and positive definite (from (b)). Since K is symmetric, the algebraic and geometric multiplicity of each eigenvalue is the same. Since K is positive definite each eigenvalue is positive. We list them to their multiplicity and order them by

$$\lambda_0 \geq \lambda_1 \geq \cdots > 0\,.$$

To each λ_i we may choose an eigenfunction ϕ_i such that the $\{\phi_i\}$ are orthonormal. By Mercer's Theorem

$$K(x,y) = \sum_{i=0}^{\infty} \lambda_i \phi_i(x)\phi_i(y)$$

where the series converges absolutely and uniformly on $[0,1] \times [0,1]$. In this case we also have that the series

$$s_1 = \int_0^1 K(x,x)\,dx = \sum_{i=0}^{\infty} \lambda_i\,,$$

converges.

We divide the "proof" of Theorem 3.2 into a series of steps.

Proposition 3.3. *The largest eigenvalue λ_0 of K is simple, and ϕ_0 is strictly of one sign on $(0,1)$.*

Proof: Assume (without loss of generality) that $1 = \lambda_0 = \lambda_1 = \cdots = \lambda_m > \lambda_{m+1} \geq \cdots$. Taking the limit of

$$K_r(x,y) = \sum_{i=0}^{\infty} \lambda_i^r \phi_i(x)\phi_i(y)\,,$$

as $r \to \infty$, we obtain

$$F(x,y) = \sum_{i=0}^{m} \phi_i(x)\phi_i(y) \tag{3.1}$$

where F is continuous, symmetric, and positive on $(0,1) \times (0,1)$, and is easily seen to satisfy

$$F(x,y) = \int_0^1 F(x,z)F(y,z)\,dz\,. \tag{3.2}$$

Kellogg now distinguishes between three possible cases which follow from (3.2), and from the fact that F is symmetric and strictly positive on $(0,1) \times (0,1)$. In the first case, F is strictly positive on all of the boundary of the unit square. In the second case, F vanishes on all of the boundary. In the third case, F vanishes on two symmetric sides of the boundary, but is strictly positive on the interior of the other two. As in Kellogg [24], we will present proofs in the first two cases only. The third case is proven by a similar method.

Assume F is strictly positive on $[0,1]\times[0,1]$. For any fixed point $y \in [0,1]$, the function $\psi_0(x) = F(x,y)$ is a continuous eigenfunction of F, with eigenvalue $\lambda = 1$ (see (3.2)), which is strictly positive on $[0,1]$. We can thus redefine the $\{\phi_0, \dots, \phi_m\}$ so that ϕ_0 is strictly positive on $[0,1]$. Assume $m \geq 1$, and let c be any constant such that $c\phi_0 - \phi_1 \geq 0$ on $[0,1]$, but vanishes at some point z therein. Then

$$0 = c\phi_0(z) - \phi_1(z) = \int_0^1 F(z,y)\,[c\phi_0(y) - \phi_1(y)]\,dy\,.$$

This is a contradiction, since F is strictly positive and $c\phi_0 - \phi_1 \geq 0$ and not identically zero. Thus $m = 0$.

Let us now assume that F vanishes on all of the boundary of the unit square. By arguing as above we may assume that ϕ_0 is strictly positive on $(0,1)$. It is easily seen that all eigenfunctions must vanish at 0 and at 1. We will prove that we can still apply the method of proof used above. Since F is continuous and vanishes on the boundary we have, for given $\varepsilon > 0$, a $\delta > 0$ such that $0 \leq F(x,y) < \varepsilon$ for all $x \in [0,\delta] \cup [1-\delta, 1]$ and all $y \in [0,1]$ (and by symmetry for all $y \in [0,\delta] \cup [1-\delta,1]$ and all $x \in [0,1]$). Choose ε and δ such that $2\delta\varepsilon < 1$.

For $x \in [0,\delta] \cup [1-\delta,1]$,

$$|\phi_i(x)| = \left|\int_0^1 F(x,y)\phi_i(y)\,dy\right| \leq \varepsilon \int_0^1 |\phi_i(y)|dy \leq \varepsilon \left(\int_0^1 |\phi_i(y)|^2 dy\right)^{1/2} \leq \varepsilon\,.$$

Choose c so that $c\phi_0 - \phi_1 \geq 0$ on $[\delta, 1-\delta]$, but vanishes at some point therein. We claim that $c\phi_0 - \phi_1 \geq 0$ on all of $[0,1]$. Since $\phi_0 \geq 0$, we have $c\phi_0 - \phi_1 \geq -\varepsilon$ on $[0,1]$. Now for $x \in [0,\delta] \cup [1-\delta,1]$

$$(c\phi_0 - \phi_1)(x) = \int_0^1 F(x,y)(c\phi_0 - \phi_1)(y)\,dy$$

$$\geq \int_0^\delta F(x,y)(c\phi_0 - \phi_1)(y)\,dy + \int_{1-\delta}^1 F(x,y)(c\phi_0 - \phi_1)(y)\,dy$$

$$\geq \int_0^\delta \varepsilon(-\varepsilon)dy + \int_{1-\delta}^1 \varepsilon(-\varepsilon)dy = -2\delta\varepsilon^2\,.$$

Repeating this k times, each time using the newer lower estimate for $c\phi_0 - \phi_1$ on $[0,\delta] \cup [1-\delta,1]$, we get that $(c\phi_0 - \phi_1)(x) \geq -(2\delta\varepsilon)^k\varepsilon$ on $[0,\delta] \cup [1-\delta,1]$. Since this is true for every k we have that $c\phi_0 - \phi_1 \geq 0$ on all of $[0,1]$, and so we may apply the previous method of proof to obtain $m = 0$. ■

It is important to note that the above argument can easily be adapted to the case of a real continuous, symmetric function of many variables. We will use this fact.

The pth compound kernel $K_{[p]}(\mathbf{x},\mathbf{y})$, defined on $R_p \times R_p$, is continuous, symmetric (in $\mathbf{x}$ and $\mathbf{y}$), and *anti-symmetric* within each of its variables $\mathbf{x}$ and $\mathbf{y}$. That is, interchanging the order of x_i and x_{i+1} is the same as multiplying by -1. Note that by assumption

$$K_{[p]}(\mathbf{x},\mathbf{y}) \geq 0$$

for all $(\mathbf{x},\mathbf{y}) \in S_p \times S_p$, and

$$K_{[p]}(\mathbf{x},\mathbf{x}) > 0$$

for all $\mathbf{x} \in \operatorname{int} S_p$. Furthermore, if ψ is anti-symmetric then

$$(K\psi)(\mathbf{x}) = \int_{R_p} K_{[p]}(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,d\mathbf{y} = p!\int_{S_p} K_{[p]}(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,d\mathbf{y}\,. \tag{3.3}$$

In the next two lemmas we identify all the eigenvalues and eigenfunctions of $K_{[p]}$.

Lemma 3.4. *Let $\{\lambda_i\}_0^\infty$ and $\{\phi_i\}_0^\infty$ be the eigenvalues and eigenfunctions, respectively, of K, as previously defined. Then $p!\lambda_{i_1}\cdots\lambda_{i_p}$ is an eigenvalue of $K_{[p]}$ on $R_p \times R_p$, with associated eigenfunction $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$, for each set of integers $0 \leq i_1 < \cdots < i_p$.*

Proof: From (3.3),

$$\int_{R_p} K_{[p]}(\mathbf{x},\mathbf{y})\left(\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}\right)(\mathbf{y})\,d\mathbf{y} = p!\int_{S_p} K_{[p]}(\mathbf{x},\mathbf{y})\left(\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}\right)(\mathbf{y})\,d\mathbf{y}.$$

From the Basic Composition Formula,

$$\begin{aligned} p!\int_{S_p} K_{[p]}(\mathbf{x},\mathbf{y})\left(\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}\right)(\mathbf{y})\,d\mathbf{y} &= p!\left((K\phi_{i_1}) \wedge \cdots \wedge (K\phi_{i_p})\right)(\mathbf{x}) \\ &= p!\lambda_{i_1}\cdots\lambda_{i_p}\left(\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}\right)(\mathbf{x}). \quad \blacksquare \end{aligned}$$

We only consider distinct $i_1,\ldots,i_p$ since if they are not distinct, then $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p} = 0$.

Lemma 3.5. *The only eigenvalues and eigenfunctions of $K_{[p]}$ on $R_p \times R_p$ are those given in Lemma 3.4.*

Proof: Recall that

$$d_p = \frac{1}{p!}\int_{R_p} K_{[p]}(\mathbf{x},\mathbf{x})\,d\mathbf{x}\,.$$

From the Fredholm theory of integral equations

$$p!\,d_p = \sum_{j=0}^{\infty} \nu_j\,,$$

where the $\{\nu_j\}_0^\infty$ are all the eigenvalues of $K_{[p]}$ listed to their algebraic multiplicities as root of the Fredholm determinant of $K_{[p]}$. $K_{[p]}$ is continuous, symmetric and positive definite. As such all its eigenvalues are positive. The lemma is therefore proven if we can show that

$$d_p = \sum_{0\le i_1<\cdots<i_p} \lambda_{i_1}\cdots\lambda_{i_p}\,.$$

(It is not trivial to see that $K_{[p]}$ is positive definite. However it is true. A simple way of getting around the necessity of showing this fact is to consider $(K_2)_{[p]}$, i.e., replace K by its second iterate K_2. Since $(K_2)_{[p]} = (K_{[p]})_2$, we have for every $\psi \in C(R_p)$

$$\int_{R_p}\int_{R_p} (K_2)_{[p]}(\mathbf{x},\mathbf{y})\psi(\mathbf{x})\psi(\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} = \int_{R_p}\left(\int_{R_p} K_{[p]}(\mathbf{x},\mathbf{z})\psi(\mathbf{x})\,d\mathbf{x}\right)^2 d\mathbf{z} \ge 0\,,$$

and thus $(K_2)_{[p]}$ is positive definite. We then obtain the result for $(K_2)_{[p]}$, which in turn implies the result for $K_{[p]}$. However, for ease of exposition, we will work with $K_{[p]}$.)

Recall that $s_r = \sum_{i=0}^\infty \lambda_i^r$, $r = 1,2,\ldots$. $K_{[p]}(\mathbf{x},\mathbf{x})$ is a determinant which is a sum of factors of the form $K(x_1,x_{j_1})\cdots K(x_p,x_{j_p})$ with appropriate sign. Collecting terms of the form $K(x_1,x_{\ell_1})K(x_{\ell_1},x_{\ell_2})\cdots K(x_{\ell_r},x_1)$ with distinct $1,\ell_1,\ldots,\ell_r$, and all possible r, and integrating, it may be shown that d_p satisfies the recurrence formula

$$pd_p = \sum_{r=1}^{p}(-1)^{r-1}s_r d_{p-r}$$

(see Goursat [16, p. 114]) where $d_0 = 1$. The unique solution to the above set of equations (with $d_0 = 1$) is

$$d_p = \sum_{0\le i_1<\cdots<i_p} \lambda_{i_1}\cdots\lambda_{i_p}\,.$$

This proves the result. ∎

We now have all the tools needed to finish the proof of Theorem 3.2.

Proof of Theorem 3.2: For each p we have from Lemmas 3.4 and 3.5 that all the eigenvalues and eigenfunctions of the integral equation

$$\int_{S_p} K_{[p]}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,d\mathbf{y} = \lambda\phi(\mathbf{x})$$

are $\{\lambda_{i_1}\cdots\lambda_{i_p}\}$ and $\{\phi_{i_1}\wedge\cdots\wedge\phi_{i_p}\}$, respectively, for $0 \le i_1 < \cdots < i_p$.

From the proof of Proposition 3.3 applied to $K_{[p]}$, it may be deduced that the largest eigenvalue of $K_{[p]}$ is simple, and the associated eigenfunction may be chosen to be strictly positive on the interior of its domain of definition.

Since

$$\lambda_0 \geq \lambda_1 \geq \cdots > 0$$

this immediately implies that

$$\lambda_0 \lambda_1 \cdots \lambda_{p-1} > \lambda_0 \lambda_1 \cdots \lambda_{p-2} \lambda_p$$

for each $p = 1, 2, \ldots$, and thus

$$\lambda_0 > \lambda_1 > \cdots > \lambda_p > \cdots$$

i.e., the eigenvalues of K are all simple. The simplicity of the eigenvalue $\lambda_0 \lambda_1 \cdots \lambda_{p-1}$ of $K_{[p]}$ implies that the associated eigenfunction, namely $\phi_0 \wedge \cdots \wedge \phi_{p-1}$, is uniquely determined up to multiplication by a nonzero constant. Multiplying ϕ_{p-1} by an appropriate constant we may therefore choose the ϕ_{p-1} so that $(\phi_0 \wedge \cdots \wedge \phi_{p-1})(\mathbf{x}) > 0$ for all $\mathbf{x} \in \operatorname{int} S_p$, $p = 1, 2, \ldots$. This is exactly Property (D). ■

The above theorem is proven only for symmetric kernels. However it contains the main ideas used in the proof of almost all the analogous results. Firstly, prove that for a suitably positive kernel the largest eigenvalue is positive and simple and the associated eigenfunction may be chosen positive. Secondly, use $K_{[p]}$ to obtain results for all eigenvalues and eigenfunctions. The two general results needed in this context had essentially been proven about a decade before the papers of Kellogg. They are theorems of Jentzsch (a generalization thereof is actually needed) and Schur. These theorems generalize the analogous results for matrices by Perron and Kronecker. We discuss these results in the next sections.

Gantmacher and Krein, in their book Gantmacher, Krein [13], present an analogous proof of the above result for a continuous, symmetric kernel K (referencing Kellogg). Their proof follows much the same pattern. They prove Proposition 3.3 by considering the maximum of $(K\phi, \phi)$ over all ϕ of norm 1. This maximum is attained by an eigenfunction of the eigenvalue λ_0. They then prove the various properties from this fact (see the next section). They also prove Lemma 3.5 by a different method. They return to the Mercer expansion

$$K(x, y) = \sum_{i=0}^{\infty} \lambda_i \phi_i(x) \phi_i(y)$$

and show directly (essentially by a generalization of the Cauchy-Binet formula) that

$$K_{[p]}(\mathbf{x}, \mathbf{y}) = \sum_{0 \leq i_1 < \cdots < i_p} \lambda_{i_1} \cdots \lambda_{i_p} \left(\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}\right)(\mathbf{x}) \left(\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}\right)(\mathbf{y}) .$$

Before ending this section and Kellogg's contribution to total positivity, let us note that Kellogg also explains, in [24], how he came to consider kernels satisfying Property (K). Since we find this of interest, we will briefly explain what he does. Kellogg was interested in determining necessary and sufficient conditions on a (symmetric) kernel K so that its eigenfunctions satisfy Property (D). To exactly determine such conditions is all but impossible. However he noted a common property of kernels whose eigenfunctions satisfy Property (D), and that is that

$$\int_0^1 K(x,y)\left(\sum_{j=0}^n a_j\phi_j(y)\right)dy$$

has, and I quote verbatim, *no more sign changes in the interior of* $(0,1)$ *than* $\sum_{j=0}^n a_j\phi_j(x)$*, and in many cases, this property holds for any continuous function* f, (see Kellogg [24, p. 146]). Kellogg then assumes that K has the property that

$$S_{(0,1)}\left(\int_0^1 K(\cdot,y)f(y)\,dy\right)\le S_{(0,1)}(f)$$

for any $f\in C[0,1]$, and then argues as follows. Choose $0<y_0<y_1<\cdots<y_n<1$. It is not difficult to see that the above variation-diminishing property implies (via a limiting argument) that

$$\sum_{j=0}^n c_jK(\cdot\,,y_j)$$

has at most n sign changes in $(0,1)$. (Choose f with support in the neighborhoods of the y_j, and approximate the appropriate linear combinations of the point measures at the y_j.) Thus given any $n+2$ points $0<x_0<x_1<\cdots<x_{n+1}<1$, the $(n+2)\times(n+1)$ matrix $(K(x_i,y_j))_{i=0}^{n+1}{}_{j=0}^{n}$, as an operator from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+2}$, does not include in its range any point of the interior of the quadrant containing $\mathbf{a}=(1,-1,1,\ldots,(-1)^{n+1})$. This may be seen to imply that no two of the $n+2$ numbers

$$\det\left(K(x_i,y_j)\right)_{\substack{i=0\\i\neq r}}^{n+1}{}_{j=0}^{n}$$

can have strictly opposite sign. Since K is symmetric, we can interchange the role of the $\{x_i\}$ and the $\{y_j\}$. Assuming some "irreducibility" of the kernel (for example, assuming condition (b) of Property (K)), it then follows that for some $\varepsilon_n\in\{-1,1\}$ we have

$$\varepsilon_nK\begin{pmatrix}x_0,x_1,\ldots,x_n\\y_0,y_1,\ldots,y_n\end{pmatrix}\ge 0$$

for all $0<x_0<x_1<\cdots<x_n<1$ and $0<y_0<y_1<\cdots<y_n<1$.

This is far from a full, correct proof. However Kellogg is, in effect, saying that he was motivated by the fact that if the symmetric kernel K is variation diminishing (VD), then it is sign regular (SR).

§4. The Kernel Case

In this section we present a full proof of the theorem concerning spectral properties of the integral equation with totally positive kernel. (We will demand more than total positivity but less than strict total positivity.) This theorem was announced in Gantmacher [16], with indications as to a proof.

Theorem 4.1. *Let $K \in C([0,1] \times [0,1])$ satisfy*

$$K\begin{pmatrix} x_1, \dots, x_p \\ y_1, \dots, y_p \end{pmatrix} \geq 0, \qquad p = 1, \dots,$$

for any $0 \leq x_1 < \cdots < x_p \leq 1$, and $0 \leq y_1 < \cdots < y_p \leq 1$, and

$$K\begin{pmatrix} x_1, \dots, x_p \\ x_1, \dots, x_p \end{pmatrix} > 0, \qquad p = 1, \dots,$$

for any $0 \leq x_1 < \cdots < x_p \leq 1$. Then every eigenvalue of the integral equation

$$\int_0^1 K(x,y)\phi(y)\,dy = \lambda\,\phi(x)$$

(as a root of the Fredholm determinant of K) is positive and simple. Let

$$\lambda_0 > \lambda_1 > \cdots > \lambda_p > \cdots > 0$$

denote these eigenvalues, and

$$\phi_0, \phi_1, \dots$$

the associated (real) eigenfunctions. Then

1) for every choice of non-trivial $(a_m, \dots, a_n)$

$$m \leq S\left(\sum_{i=m}^{n} a_i \phi_i\right) \leq Z\left(\sum_{i=m}^{n} a_i \phi_i\right) \leq n\,,$$

2) the zeros of ϕ_n and ϕ_{n+1} strictly interlace.

Remark. Gantmacher states this result with the second condition holding only over $0 < x_1 < \cdots < x_p < 1$ and thus the zero counting in (1) is also restricted to $(0,1)$. There is no difference in the proof in these two cases.

We should like to say a few words about Gantmacher before embarking upon the proof. Feliks Ruvimovich Gantmacher (1908–1964) was born and studied in Odessa. In 1934 he moved to Moscow where he resided until his death. Gantmacher is, of course, known for his excellent book *The Theory of Matrices* [7], and his book with Krein [13]. He was also one of the organizers and editors of the journal *Uspekhi Mat. Nauk* (*Russian Math. Surveys*).

Gantmacher was instrumental in the establishment and organization of the well-known Moscow Physico-Technical Institute, where from 1953 until his death he headed the Department of Theoretical Applied Mathematics. An obituary on Gantmacher may be found in [8].

The proof of Theorem 4.1 is rather lengthy. We will develop two main preliminary results. The first is Jentzsch's Theorem, which is a generalization of Perron's Theorem (Theorem 5.3). Jentzsch [17] proved this theorem in the case of a continuous kernel which is strictly positive on its domain. We have written the statement of Theorem 4.2 in the form in which we will use it. Because the kernel is not strictly positive on all of its domain of definition, Jentzsch's original proof must be modified. This can be done. However we have taken here a different approach which takes ideas from Anselone, Lee [2]. Recall that

$$S_p^* = \{\mathbf{x} = (x_1, \ldots, x_p) \,:\, 0 \le x_1 < \cdots < x_p \le 1\}\,,$$

and $S_p = \overline{S_p^*}$ is the closed p-dimensional simplex.

Theorem 4.2. *Suppose $L \in C(S_p \times S_p)$ satisfies*
a) $L \ge 0$ on $S_p \times S_p$
b) $L(\mathbf{x},\mathbf{x}) > 0$ for all $\mathbf{x} \in S_p^*$.

Then L has an eigenvalue λ^* which is positive and simple, as a root of the Fredholm determinant of L, and strictly larger in modulus than all other eigenvalues of L. Furthermore, the eigenfunction ϕ^* associated with λ^* may be chosen to be strictly positive on S_p^*.

Proof: For each positive integer n, set

$$B_n = \{\mathbf{x} : \mathbf{x} \in S_p,\ d(\mathbf{x}, \partial S_p) \ge 1/n\}\,,$$

where $d(\mathbf{x}, \partial S_p)$ is some measure of the distance from $\mathbf{x}$ to the boundary of S_p. We first restrict our attention to the eigenvalue problem on B_n. We assume n is chosen sufficiently large such that $B_n \ne \emptyset$. Since $L(\mathbf{x},\mathbf{x}) > 0$ for $\mathbf{x} \in B_n$, it follows that

$$\int_{B_n} L(\mathbf{x},\mathbf{y})\, d\mathbf{y} \ge c > 0$$

for some constant c and all $\mathbf{x} \in B_n$. Thus there exists for any function $\phi \in C(B_n)$ with $\phi > 0$ on B_n, some $\mu > 0$ for which

$$\int_{B_n} L(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\, d\mathbf{y} \ge \mu\phi(\mathbf{x})$$

for all $\mathbf{x} \in B_n$. Let λ_n^* be the supremum over all such μ as we vary over all $\phi > 0$ on B_n. That is,

$$\lambda_n^* = \sup\{\lambda \,:\, \int_{B_n} L(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\, d\mathbf{y} \ge \lambda\phi(\mathbf{x}),\ \phi \in C(B_n),\ \phi > 0\}\,.$$

Let $\mu_m \uparrow \lambda_n^*$, and $\phi_m \in C(B_n)$ satisfy $\phi_m > 0$ on B_n and

$$\int_{B_n} L(\mathbf{x},\mathbf{y})\phi_m(\mathbf{y})\,d\mathbf{y} \geq \mu_m\phi_m(\mathbf{x})\,.$$

Define

$$\psi_m(\mathbf{x}) = \int_{B_n} L(\mathbf{x},\mathbf{y})\phi_m(\mathbf{y})\,d\mathbf{y}\,,$$

for $\mathbf{x} \in B_n$. Then $\psi_m \in C(B_n)$ satisfies $\psi_m > 0$ on B_n and

$$\int_{B_n} L(\mathbf{x},\mathbf{y})\psi_m(\mathbf{y})\,d\mathbf{y} \geq \mu_m\psi_m(\mathbf{x})\,.$$

Normalize ψ_m to have (uniform) norm equal to one. Since the linear operator L is completely continuous, as an operator from $C(B_n)$ to $C(B_n)$, there exists a subsequence of the ψ_m which converges to some $\zeta^* \in C(B_n)$ satisfying $\zeta^* \geq 0$ ($\zeta^* \neq 0$), and

$$\int_{B_n} L(\mathbf{x},\mathbf{y})\zeta^*(\mathbf{y})\,d\mathbf{y} \geq \lambda_n^*\zeta^*(\mathbf{x})\,.$$

Let $\zeta_0 = \zeta^*$, and set

$$\zeta_r(\mathbf{x}) = \int_{B_n} L(\mathbf{x},\mathbf{y})\zeta_{r-1}(\mathbf{y})\,d\mathbf{y}\,, \qquad r = 1, 2, \ldots\,.$$

A simple calculation shows that

$$\zeta_r(\mathbf{x}) = \int_{B_n} L_r(\mathbf{x},\mathbf{y})\zeta^*(\mathbf{y})\,d\mathbf{y}\,,$$

where L_r is the rth iterate of L (see Section 2) on B_n. For fixed n, there exists a $\delta > 0$ such that for all $\mathbf{x},\mathbf{y} \in B_n$ with $d(\mathbf{x},\mathbf{y}) < \delta$ we have

$$L(\mathbf{x},\mathbf{y}) > 0\,.$$

Thus if $\mathbf{x},\mathbf{y} \in B_n$ and $d(\mathbf{x},\mathbf{y}) < 2\delta$ then we have

$$L_2(\mathbf{x},\mathbf{y}) > 0\,,$$

while if $\mathbf{x},\mathbf{y} \in B_n$ and $d(\mathbf{x},\mathbf{y}) < r\delta$ then

$$L_r(\mathbf{x},\mathbf{y}) > 0\,.$$

As such for some m, depending on n,

$$L_r(\mathbf{x},\mathbf{y}) > 0$$

for all $\mathbf{x}, \mathbf{y} \in B_n$ and all $r \geq m$. Let $r \geq m$. Note that

$$\int_{B_n} L(\mathbf{x}, \mathbf{y})\zeta_r(\mathbf{y})\, d\mathbf{y} \geq \lambda_n^* \zeta_r(\mathbf{x}), \tag{4.1}$$

for all $\mathbf{x} \in B_n$. By the definition of λ_n^* there must be equality in (4.1) for some $\mathbf{x}^* \in B_n$. Therefore

$$0 = \int_{B_n} L_r(\mathbf{x}^*, \mathbf{y}) \left[\int_{B_n} L(\mathbf{y}, \mathbf{z})\zeta^*(\mathbf{z})\, d\mathbf{z} - \lambda_n^* \zeta^*(\mathbf{y}) \right] d\mathbf{y}\,.$$

Since L_r is strictly positive on $B_n \times B_n$ and the value within the brackets is non-negative, we must in fact have

$$\int_{B_n} L(\mathbf{x}, \mathbf{y})\zeta^*(\mathbf{y})\, d\mathbf{y} = \lambda_n^* \zeta^*(\mathbf{x})\,.$$

for all $\mathbf{x} \in B_n$. Thus in addition $\zeta^* > 0$ on B_n.

Set

$$\eta(\mathbf{x}) = \begin{cases} \zeta^*(\mathbf{x}), & \mathbf{x} \in B_n \\ 0, & \mathbf{x} \notin B_n \end{cases}$$

and

$$\phi(\mathbf{x}) = \int_{B_{n+1}} L(\mathbf{x}, \mathbf{y})\eta(\mathbf{y})\, d\mathbf{y}\,.$$

Then $\phi \in C(B_{n+1})$, $\phi \geq 0$ $(\phi \neq 0)$, and

$$\int_{B_{n+1}} L(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\, d\mathbf{y} \geq \lambda_n^* \phi(\mathbf{x}),$$

for all $\mathbf{x} \in B_{n+1}$. This simple fact shows that λ_n^* is a nondecreasing sequence of n. Furthermore λ_n^* tends to

$$\lambda^* = \sup\{\lambda : \int_{S_p} L(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\, d\mathbf{y} \geq \lambda\phi(\mathbf{x}),\ \phi \in C(S_p),\ \phi \geq 0,\ (\phi \neq 0)\}\,.$$

Let ζ_n^* be the ζ^* suitably normalized. Since L is completely continuous there exists a subsequence of the ζ_n^* which converges to a $\phi^* \in C(S_p)$ satisfying $\phi^* \geq 0$ $(\phi^* \neq 0)$ on S_p, and

$$\int_{S_p} L(\mathbf{x}, \mathbf{y})\phi^*(\mathbf{y})\, d\mathbf{y} = \lambda^* \phi^*(\mathbf{x})\,.$$

Since $L(\mathbf{x}, \mathbf{x}) > 0$ for $\mathbf{x} \in S_p^*$, it follows that $\phi^* > 0$ on S_p^*. λ^* is a positive eigenvalue with associated eigenfunction positive on S_p^*. It remains to prove the remaining properties.

If λ is any other eigenvalue of L with associated eigenfunction ψ, then

$$\lambda\psi(\mathbf{x}) = \int_{S_p} L(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,d\mathbf{y}\,.$$

Thus

$$|\lambda|\,|\psi(\mathbf{x})| \le \int_{S_p} L(\mathbf{x},\mathbf{y})|\psi(\mathbf{y})|d\mathbf{y}\,,$$

which implies, by the extremal property of λ^*, that $|\lambda| \le \lambda^*$.

From this fact, and by the same arguments as above, there exists a left eigenfunction φ^* associated with the eigenvalue λ^*. That is, a function $\varphi^* \in C(S_p)$ which is strictly positive on S_p^* and satisfies

$$\lambda^*\varphi^*(\mathbf{y}) = \int_{S_p} \varphi^*(\mathbf{x})L(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,.$$

Assume λ is an eigenvalue with associated eigenfunction ψ and $|\lambda| = \lambda^*$. Then

$$\lambda^* \int_{S_p} \varphi^*(\mathbf{x})|\psi(\mathbf{x})|d\mathbf{x} = \int_{S_p} \varphi^*(\mathbf{x})|\lambda|\,|\psi(\mathbf{x})|d\mathbf{x}$$

$$\le \int_{S_p} \varphi^*(\mathbf{x}) \left[\int_{S_p} L(\mathbf{x},\mathbf{y})|\psi(\mathbf{y})|d\mathbf{y}\right] d\mathbf{x} = \int_{S_p} \left[\int_{S_p} \varphi^*(\mathbf{x})L(\mathbf{x},\mathbf{y})\,d\mathbf{x}\right] |\psi(\mathbf{y})|d\mathbf{y}$$

$$= \lambda^* \int_{S_p} \varphi^*(\mathbf{y})|\psi(\mathbf{y})|d\mathbf{y}\,.$$

Since φ^* is strictly positive on S_p^* we must have

$$\lambda^*|\psi(\mathbf{x})| = \int_{S_p} L(\mathbf{x},\mathbf{y})|\psi(\mathbf{y})|d\mathbf{y}\,,$$

for all $\mathbf{x} \in S_p$. This together with

$$\lambda\psi(\mathbf{x}) = \int_{S_p} L(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,d\mathbf{y}\,.$$

implies that there exists a constant γ, $|\gamma| = 1$, such that

$$|\psi(\mathbf{x})| = \gamma\psi(\mathbf{x})$$

for all $\mathbf{x} \in S_p$. Thus if λ is an eigenvalue with $|\lambda| = \lambda^*$, then $\lambda = \lambda^*$, and the associated eigenfunction may be assumed to be positive on S_p^*. This also implies that the geometric multiplicity of λ^* is one, for otherwise there exists an eigenfunction with at least one change of sign. If the algebraic multiplicity is not one, then there exists a $\phi^{**} \in C(S_p)$ such that

$$\int_{S_p} L(\mathbf{x},\mathbf{y})\phi^{**}(\mathbf{y})\,d\mathbf{y} = \lambda^*\phi^{**}(\mathbf{x}) + \alpha\phi^*(\mathbf{x})\,,$$

for some $\alpha \neq 0$. Now

$$\lambda^* \int_{S_p} \varphi^*(\mathbf{y})\phi^{**}(\mathbf{y})\,d\mathbf{y} = \int_{S_p}\left[\int_{S_p}\varphi^*(\mathbf{x})L(\mathbf{x},\mathbf{y})d\mathbf{x}\right]\phi^{**}(\mathbf{y})\,d\mathbf{y}$$

$$= \int_{S_p}\varphi^*(\mathbf{x})\left[\int_{S_p} L(\mathbf{x},\mathbf{y})\phi^{**}(\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x} = \int_{S_p}\varphi^*(\mathbf{x})\left(\lambda^*\phi^{**}(\mathbf{x}) + \alpha\phi^*(\mathbf{x})\right)d\mathbf{x}$$

$$= \lambda^*\int_{S_p}\varphi^*(\mathbf{x})\phi^{**}(\mathbf{x})\,d\mathbf{x} + \alpha\int_{S_p}\varphi^*(\mathbf{x})\phi^*(\mathbf{x})\,d\mathbf{x}\,.$$

This implies that $\alpha\int_{S_p}\varphi^*(\mathbf{x})\phi^*(\mathbf{x})\,d\mathbf{x} = 0$. However, $\alpha \neq 0$, and $\varphi^*\,\phi^* > 0$ on all of S_p^*, which is impossible. This proves the theorem. ■

The other preliminary result which we will use is due to Schur, generalizes Kronecker's Theorem (Theorem 5.4), and may be found in Schur [32]. It is given in Theorem 4.4. To prove it we will use this next result, which is of independent interest. In both Theorem 4.3 and Theorem 4.4 we are following the proof of Schur.

Let $M \in C\left([0,1]^p \times [0,1]^p\right)$. Associated with each eigenvalue λ^* of M is its algebraic multiplicity m (i.e., the multiplicity of the zero $1/\lambda^*$ of the Fredholm determinant of M). There exist m (and no more than m) linearly independent functions $\psi_1,\ldots,\psi_m$ (generalized eigenfunctions) such that

$$\int_{[0,1]^p} M(\mathbf{x},\mathbf{y})\psi_i(\mathbf{y})\,d\mathbf{y} = \sum_{j=1}^{m} b_{ij}\psi_j(\mathbf{x})\,,\quad i = 1,\ldots,m,$$

and $\det(b_{ij} - \lambda\delta_{ij})_{i,j=1}^m = (\lambda^* - \lambda)^m$. We will call such a system of functions a *complete invariant system* for the eigenvalue λ^*.

A countable (or finite) system of functions $\{\phi_i\}_{i=1}$ is said to be a *complete system of principal functions* for M if

1) Each $\phi_i \in C([0,1]^p)$, and the $\phi_1,\ldots,\phi_k$ are linearly independent for each finite k.
2) For each i,

$$\int_{[0,1]^p} M(\mathbf{x},\mathbf{y})\phi_i(\mathbf{y})\,d\mathbf{y} = \sum_{j=1}^{i} a_{ij}\phi_j(\mathbf{x})\,,\quad i = 1,2\ldots\,,$$

with $a_{ii} \neq 0$.
3) If $\psi_1,\ldots,\psi_m$ is a complete invariant system for an eigenvalue λ^*, then each ψ_j is a linear combination of a finite number of the $\{\phi_i\}_{i=1}$.

We should think of a complete system of principal functions for M as a convenient form for dealing with the eigenvalues and (generalized) eigenfunctions of M. The eigenvalue associated with each ϕ_i is simply a_{ii}.

The first result to be proved has to do with tensor products.

Theorem 4.3. *Let* $L_k \in C([0,1]\times[0,1])$, $k=1,\ldots,p$, *and*

$$L(\mathbf{x},\mathbf{y}) = \prod_{k=1}^{p} L_k(x_k,y_k)$$

where $\mathbf{x}=(x_1,\ldots,x_p), \mathbf{y}=(y_1,\ldots,y_p)\in[0,1]^p$. *Assume* $\{\phi_{ik}\}_{i=1}$ *are, for each* $k\in\{1,\ldots,p\}$, *a complete system of principal functions for* L_k. *Then* $\{\phi_{i_1 1}\cdots\phi_{i_p p}\}_{i_1,\ldots,i_p=1}$ *is a complete system of principal functions for* L.

Remark. We use the expression $\phi_{i_1 1}\cdots\phi_{i_p p}$ for the function $\Phi(x_1,\ldots,x_p) = \phi_{i_1 1}(x_1)\cdots\phi_{i_p p}(x_p)$.

Proof: We wish to prove that the $\{\phi_{i_1 1}\cdots\phi_{i_p p}\}$ are a complete system of principal functions for L. We first order them, and then prove that they satisfy the three properties in the definition of such a system.

We order the $\mathbf{i}=(i_1,\ldots,i_p)\in\mathbb{N}^p$ in the usual fashion. We say $\mathbf{i}>\mathbf{j}$ if $|\mathbf{i}|>|\mathbf{j}|$ ($|\mathbf{i}|=i_1+\cdots+i_p$) or if $|\mathbf{i}|=|\mathbf{j}|$ and the first non-zero term in the sequence $i_1-j_1,\ldots,i_p-j_p$ is positive.

Each of the $\phi_{i_1 1}\cdots\phi_{i_p p}\in C([0,1]^p)$ and it is easily seem that any finite number of them is linearly independent. Thus the first property holds. To prove the second property, recall that for each $k\in\{1,\ldots,p\}$ and $i=1,2,\ldots,$

$$\int_{[0,1]} L_k(x,y)\phi_{ik}(y)\,dy = \sum_{j=1}^{i} a_{ij}^{(k)}\phi_{jk}(x)\,,$$

where $a_{ii}^{(k)}=\lambda_{ik}\neq 0$. Thus

$$\begin{aligned}\int_{[0,1]^p} L(\mathbf{x},\mathbf{y})(\phi_{i_1 1}\cdots\phi_{i_p p})(\mathbf{y})\,d\mathbf{y} &= \prod_{k=1}^{p}\left[\int_{[0,1]} L_k(x_k,y_k)\phi_{i_k k}(y_k)\,dy_k\right]\\ &= \prod_{k=1}^{p}\left(\sum_{j=1}^{i_k} a_{i_k j}^{(k)}\phi_{jk}(x_k)\right)\\ &= \sum_{\mathbf{j}\le\mathbf{i}} A_{\mathbf{i},\mathbf{j}}\,(\phi_{j_1 1}\cdots\phi_{j_p p})(\mathbf{x})\,,\end{aligned}$$

where $A_{\mathbf{i},\mathbf{i}}=\prod_{k=1}^{p} a_{i_k k}^{(k)} = \lambda_{i_1 1}\cdots\lambda_{i_p p}$.

The above simply implies that what we thought were eigenvalues and eigenfunctions are, in fact, eigenvalues and eigenfunctions. The third property is the more interesting. It says that there are no others. Let λ^* be an eigenvalue of L and $\psi_1,\ldots,\psi_m$ an associated complete invariant system. Thus

$$\int_{[0,1]^p} L(\mathbf{x},\mathbf{y})\psi_s(\mathbf{y})\,d\mathbf{y} = \sum_{t=1}^{m} b_{st}\psi_t(\mathbf{x})\,,\qquad s=1,\ldots,m\,,$$

and $\det(b_{st} - \lambda\delta_{st})_{s,t=1}^m = (\lambda^* - \lambda)^m$. Set

$$\zeta_s(x_1, \dots, x_p) = \int_{[0,1]^{p-1}} L_2(x_2, y_2) \cdots L_p(x_p, y_p)\psi_s(x_1, y_2, \dots, y_p)\, dy_2 \cdots dy_p\,.$$

Thus for each $s = 1, \dots, m$,

$$\int_{[0,1]} L_1(x_1, y_1)\zeta_s(y_1, x_2, \dots, x_p)\, dy_1 = \sum_{t=1}^m b_{st}\psi_t(x_1, x_2, \dots, x_p)\,. \tag{4.2}$$

Multiplying both sides by $L_2(\cdot, x_2) \cdots L_p(\cdot, x_p)$ and integrating with respect to $x_2, \dots, x_p$, we see that

$$\int_{[0,1]^p} L(\mathbf{x}, \mathbf{y})\zeta_s(\mathbf{y})\, d\mathbf{y} = \sum_{t=1}^m b_{st}\zeta_t(\mathbf{x})\,, \qquad s = 1, \dots, m\,.$$

This simply says that $\zeta_1, \dots, \zeta_m$ is another complete invariant system for the eigenvalue λ^* of L. (It follows from (4.2) that the $\zeta_1, \dots, \zeta_m$ are linearly independent.) Thus each ζ_s is a linear combination of the $\psi_1, \dots, \psi_m$, and substituting in (4.2) we obtain

$$\int_{[0,1]} L_1(x_1, y_1)\psi_s(y_1, x_2, \dots, x_p)\, dy_1 = \sum_{t=1}^m d_{st}\psi_t(x_1, x_2, \dots, x_p), \tag{4.3}$$

for each $s = 1, \dots, m$, for some non-singular matrix $(d_{st})_{s,t=1}^m$.

Since the $\{\phi_{i1}\}_{i=1}$ are a complete system of principal functions for L_1, it follows that for each fixed $x_2, \dots, x_p$, each ψ_s, $s = 1, \dots, m$, is a linear combination of a finite number of the ϕ_{i1}. Thus for each fixed $x_2, \dots, x_p$,

$$\psi_s(x_1, x_2, \dots, x_p) = \sum_{i=1}^N f_{si}^{(1)}\phi_{i1}(x_1)\,, \qquad s = 1, \dots, m\,.$$

We may prove, using (4.3), that the vector $(f_{sN}^{(1)})_{s=1}^m$ is an eigenvector of the matrix $(d_{st})_{s,t=1}^m$ with eigenvalue $a_{NN}^{(1)}$. The $a_{ii}^{(1)}$ (eigenvalues of L_1) tend to zero. Thus we can obtain a bound for N, which is independent of $x_2, \dots, x_p$. Therefore

$$\psi_s(x_1, x_2, \dots, x_p) = \sum_{i=1}^N f_{si}^{(1)}(x_2, \dots, x_p)\phi_{i1}(x_1)\,, \qquad s = 1, \dots, m\,.$$

where the $\{f_{si}^{(1)}\}_{i=1}^N$ are uniquely defined functions of $x_2, \dots, x_p$. What has been done with respect to L_1 and the variable x_1 can also be done for L_k and the variable x_k for each $k \in \{1, \ldots, p\}$. Thus for some N sufficiently large, and each $s = 1, \ldots, m$ and $k = 1, \ldots, p$ we have

$$\psi_s(x_1, \ldots, x_p) = \sum_{i=1}^{N} f_{si}^{(k)}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_p)\phi_{ik}(x_k)\,. \tag{4.4}$$

A simple lemma now shows that since (4.4) holds for $k = 1, \ldots, p$, where the $(\phi_{ik})_{i=1}^{N}$ are linearly independent, then we must have

$$\psi_s(x_1, \ldots, x_p) = \sum_{i_1, \ldots, i_p = 1}^{N} c_{i_1, \ldots, i_p}^{(s)} \phi_{i_1 1}(x_1) \cdots \phi_{i_p p}(x_p)\,,$$

for each $s = 1, \ldots, m$. This proves property three and the theorem. ∎

We can now prove the theorem of Schur which we shall use. Let $K \in C([0,1] \times [0,1])$ and $K_{[p]} \in C(S_p \times S_p)$ where

$$K_{[p]}(\mathbf{x}, \mathbf{y}) = K \begin{pmatrix} x_1, \ldots, x_p \\ y_1, \ldots, y_p \end{pmatrix}$$

for each $\mathbf{x} = (x_1, \ldots, x_p), \mathbf{y} = (y_1, \ldots, y_p) \in S_p$.

Theorem 4.4. *Let $K \in C([0,1] \times [0,1])$. Assume $\{\phi_i\}_{i=1}$ is a complete system of principal functions for K, with associated eigenvalues $\{\lambda_i\}_{i=1}$. Then $\{\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}\}_{1 \le i_1 < \cdots < i_p}$ is a complete system of principal functions for $K_{[p]} \in C(S_p \times S_p)$ with associated eigenvalues $\{\lambda_{i_1} \cdots \lambda_{i_p}\}_{1 \le i_1 < \cdots < i_p}$.*

Proof: To prove that $\{\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}\}_{1 \le i_1 < \cdots < i_p}$ is a complete system of principal functions for $K_{[p]}$, we must first order then and then prove that they satisfy the properties of such a system.

The ordering will be the usual lexicographic ordering of $(i_1, \ldots, i_p)$, i.e., $(i_1, \ldots, i_p) > (j_1, \ldots, j_p)$ if the first non-zero term of the sequence $i_1 - j_1, \ldots, i_p - j_p$ is positive. The first property holds since the $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$ are continuous and linearly independent (see Section 2). Assume

$$\int_0^1 K(x, y)\phi_i(y)\, dy = \sum_{j=1}^{i} a_{ij}\phi_j(x)$$

for each $i = 1, \ldots,$ where $a_{ii} = \lambda_i \ne 0$. Then from the Basic Composition Formula

$$\int_{S_p} K_{[p]}(\mathbf{x}, \mathbf{y}) \left(\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}\right)(\mathbf{y})\, d\mathbf{y}$$

$$
\begin{aligned}
&= \big((K\phi_{i_1}) \wedge \cdots \wedge (K\phi_{i_p})\big)(\mathbf{x}) \\
&= \left(\left(\sum_{j_1=1}^{i_1} a_{i_1 j_1}\phi_{j_1}\right) \wedge \cdots \wedge \left(\sum_{j_p=1}^{i_p} a_{i_p j_p}\phi_{j_p}\right)\right)(\mathbf{x}) \\
&= \sum_{\substack{j_1<\cdots<j_p \\ j_\ell \le i_\ell,\, \ell=1,\ldots,p}} A\begin{pmatrix} i_1,\ldots,i_p \\ j_1,\ldots,j_p \end{pmatrix} \big(\phi_{j_1} \wedge \cdots \wedge \phi_{j_p}\big)(\mathbf{x})\,.
\end{aligned}
$$

Now

$$
A\begin{pmatrix} i_1,\ldots,i_p \\ i_1,\ldots,i_p \end{pmatrix} = \prod_{\ell=1}^{p} a_{i_\ell i_\ell} = \lambda_{i_1}\cdots\lambda_{i_p} \neq 0\,.
$$

Thus property two holds (and $\lambda_{i_1}\cdots\lambda_{i_p}$ for $1 \le i_1 < \cdots < i_p$ will be all the eigenvalues of $K_{[p]}$ listed to their algebraic multiplicity as roots of the Fredholm determinant of $K_{[p]}$).

It remains to prove property three. Let λ^* be an eigenvalue of $K_{[p]}$ and $\psi_1,\ldots,\psi_m$ an associated complete invariant system. Thus

$$
\int_{S_p} K_{[p]}(\mathbf{x},\mathbf{y})\psi_s(\mathbf{y})\,d\mathbf{y} = \sum_{t=1}^{m} b_{st}\psi_t(\mathbf{x})\,, \qquad s=1,\ldots,m\,,
$$

with $\det(b_{st} - \lambda\delta_{st})_{s,t=1}^{m} = (\lambda^* - \lambda)^m$. Consider $K_{[p]}$ as being defined on $R_p \times R_p$ ($R_p = [0,1]^p$) and anti-symmetric thereon. Extend ψ_s from S_p to R_p to also be anti-symmetric thereon. It is easily checked that

$$
\int_{R_p} K_{[p]}(\mathbf{x},\mathbf{y})\psi_s(\mathbf{y})\,d\mathbf{y} = p!\sum_{t=1}^{m} b_{st}\psi_t(\mathbf{x})\,, \qquad s=1,\ldots,m\,.
$$

For any $f \in C(R_p)$ which is anti-symmetric, let $\varepsilon_\pi \in \{-1,1\}$ satisfy

$$
f(y_{\pi(1)},\ldots,y_{\pi(p)}) = \varepsilon_\pi f(y_1,\ldots,y_p)
$$

where $\pi = (\pi(1),\ldots,\pi(p)) \in \Pi$, the group of permutations of $\{1,\ldots,p\}$. Then

$$
\begin{aligned}
&\int_{R_p} K_{[p]}(\mathbf{x},\mathbf{y})f(\mathbf{y})\,d\mathbf{y} \\
&= \sum_{\pi\in\Pi} \varepsilon_\pi \int_{R_p} K(x_1,y_{\pi(1)})\cdots K(x_p,y_{\pi(p)})f(y_1,\ldots,y_p)\,dy_1\cdots dy_p \\
&= \sum_{\pi\in\Pi} \int_{R_p} K(x_1,y_{\pi(1)})\cdots K(x_p,y_{\pi(p)})f(y_{\pi(1)},\ldots,y_{\pi(p)})\,dy_1\cdots dy_p \\
&= \sum_{\pi\in\Pi} \int_{R_p} K(x_1,y_{\pi(1)})\cdots K(x_p,y_{\pi(p)})f(y_{\pi(1)},\ldots,y_{\pi(p)})\,dy_{\pi(1)}\cdots dy_{\pi(p)} \\
&= \sum_{\pi\in\Pi} \int_{R_p} K(x_1,y_1)\cdots K(x_p,y_p)f(y_1,\ldots,y_p)\,dy_1\cdots dy_p \\
&= p!\int_{R_p} K(x_1,y_1)\cdots K(x_p,y_p)f(y_1,\ldots,y_p)\,dy_1\cdots dy_p\,.
\end{aligned}
$$

Thus for each $s = 1, \ldots, m$

$$\begin{aligned}\int_{R_p} K_{[p]}(\mathbf{x},\mathbf{y})\psi_s(\mathbf{y})\,d\mathbf{y}& \\ =&p!\int_{R_p} K(x_1,y_1)\cdots K(x_p,y_p)\psi_s(y_1,\ldots,y_p)\,dy_1\cdots dy_p \\ =&p!\sum_{t=1}^{m} b_{st}\psi_t(\mathbf{x})\,,\end{aligned}$$

which implies that $\psi_1, \ldots, \psi_m$ is a complete invariant system for the kernel $\prod_{\ell=1}^{p} K(x_\ell, y_\ell)$. From Theorem 4.3, this implies that each $\psi_s(x_1, \ldots, x_p)$ may be written as a finite linear combination of $\phi_{i_1}(x_1)\cdots\phi_{i_p}(x_p)$ (for arbitrary positive i_j, $j = 1, \ldots, p$).

Assume

$$\psi_s(x_1,\ldots,x_p) = \sum_{i_1,\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p}\phi_{i_1}(x_1)\cdots\phi_{i_p}(x_p)\,.$$

Then

$$\psi_s(x_{\pi(1)},\ldots,x_{\pi(p)}) = \sum_{i_1,\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p}\phi_{i_1}(x_{\pi(1)})\cdots\phi_{i_p}(x_{\pi(p)})\,.$$

The function ψ_s is anti-symmetric. Multiplying this last term by ε_π and summing over all possible $p!$ terms $\pi \in \Pi$, we obtain

$$\begin{aligned}p!\psi_s(x_1,\ldots,x_p) &= \sum_{i_1,\ldots,i_p=1}^{N} a_{i_1,\ldots,i_p}\left(\phi_{i_1}\wedge\cdots\wedge\phi_{i_p}\right)(x_1,\ldots,x_p) \\ &= \sum_{1\le i_1<\cdots i_p\le N} b_{i_1,\ldots,i_p}\left(\phi_{i_1}\wedge\cdots\wedge\phi_{i_p}\right)(x_1,\ldots,x_p)\,.\end{aligned}$$

This proves the third property and the theorem. ■

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1: We start by repeating the proof of Theorem 3.2. Let $\lambda_0, \lambda_1, \ldots,$ denote the eigenvalues of K listed to their algebraic multiplicity as roots of the Fredholm determinant of K, and such that

$$|\lambda_0| \ge |\lambda_1| \ge \cdots\,.$$

Let ϕ_i denote associated (generalized) eigenfunctions. For each p we have from Theorem 4.4 that all the eigenvalues and (generalized) eigenfunctions of the integral equation

$$\int_{S_p} K_{[p]}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,d\mathbf{y} = \lambda\phi(\mathbf{x})$$

are $\{\lambda_{i_1} \cdots \lambda_{i_p}\}$ and $\{\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}\}$, respectively, for $0 \le i_1 < \cdots < i_p$. From Theorem 4.2 applied to $K_{[p]}$, it follows that the largest eigenvalue of $K_{[p]}$ is positive and simple, and the associated eigenfunction may be chosen to be strictly positive on S_p^*. This immediately implies that

$$\lambda_0 \lambda_1 \cdots \lambda_{p-1} > |\lambda_0 \lambda_1 \cdots \lambda_{p-2} \lambda_p|$$

for each $p = 1, 2, \ldots$, and thus

$$\lambda_0 > \lambda_1 > \cdots > \lambda_p > \cdots$$

i.e., the eigenvalues of K are all positive and simple. The simplicity of the eigenvalue $\lambda_0 \lambda_1 \cdots \lambda_{p-1}$ of $K_{[p]}$ implies that the associated eigenfunction, namely $\phi_0 \wedge \cdots \wedge \phi_{p-1}$, is uniquely determined up to multiplication by a nonzero constant. Multiplying ϕ_{p-1} by an appropriate constant we may therefore choose the ϕ_{p-1} so that $(\phi_0 \wedge \cdots \wedge \phi_{p-1})(\mathbf{x}) > 0$ for all $\mathbf{x} \in S_p^*$, $p = 1, 2, \ldots$. It remains to prove properties (1) and (2) of the eigenfunctions.

Since for all $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in S_{n+1}^*$

$$(\phi_0 \wedge \cdots \wedge \phi_n)(\mathbf{x}) = \det(\phi_i(x_j))_{i=0}^{n} {}_{j=1}^{n+1} > 0 \,,$$

(this is the same as saying that $\{\phi_0, \ldots, \phi_n\}$ is a T-system on $[0, 1]$) then a contradiction immediately ensues if some non-trivial linear combination of $\phi_0, \ldots, \phi_n$ vanishes at $n+1$ distinct points $0 \le x_1 < \cdots < x_{n+1} \le 1$. That is, in this case

$$\det(\phi_i(x_j))_{i=0}^{n} {}_{j=1}^{n+1} = 0 \,.$$

Thus for $0 \le m \le n$,

$$Z\left(\sum_{i=m}^{n} a_i \phi_i\right) \le n$$

for every non-trivial choice of $(a_m, \ldots, a_n)$.

Let ψ_i denote the left eigenvector of K with eigenvalue λ_i. The $\{\psi_i\}$ and $\{\phi_i\}$ are biorthogonal. Since K^T and K share the same properties, we may assume that for all $\mathbf{x} \in S_m^*$

$$(\psi_0 \wedge \cdots \wedge \psi_{m-1})(\mathbf{x}) > 0 \,.$$

If $f \in C[0, 1]$, and

$$\int_0^1 f(x) \psi_i(x) \, dx = 0 \,, \qquad i = 0, \ldots, m-1 \,,$$

then $S(f) \ge m$. This follows from the fact that given any k points $(0 \le k \le m-1)$ $0 < z_1 < \cdots < z_k < 1$, the function

$$\psi(x) = \det(\psi_i(y_j))_{i,j=0}^{k}$$

where $y_0 = x$, $y_j = z_j$, $j = 1, \ldots, k$, is a linear combination of the $\psi_0, \ldots, \psi_k$ which strictly changes sign at the z_j, and has no other zeros in $[0,1]$. Thus

$$S\left(\sum_{i=m}^{n} a_i \phi_i\right) \geq m\,.$$

This proves (1).

According to (1), ϕ_n has exactly n zeros, each of which is a sign change. Let us denote these zeros by $0 < w_1 < \cdots < w_n < 1$. Set $w_0 = 0$ and $w_{n+1} = 1$. We will prove that ϕ_{n+1} has a zero in each of (w_i, w_{i+1}), $i = 0, \ldots, n$. There are various proofs of this fact. We present a proof partially based on an argument in Kellogg [23], and partially on an argument in Gantmacher, Krein [13].

We first prove that ϕ_n and ϕ_{n+1} have no common zero. Assume to the contrary that $\phi_n(z) = \phi_{n+1}(z) = 0$ for some $z \in (0,1)$. (None of the ϕ_m vanish at the endpoints.) Let $F_1, \ldots, F_{n-1}$ be $n-1$ linearly independent functions in $\mathrm{span}\{\phi_0, \ldots, \phi_{n-1}\}$ which all vanish at z. Let $G \in \mathrm{span}\{\psi_0, \ldots, \psi_{n-1}\}$ be non-trivial and orthogonal to each of the $F_1, \ldots, F_{n-1}$. Such a G exists. Now $Z(G) \leq n-1$ from (1) applied to the left eigenvectors ψ_i. Thus we may choose $d_0 = 0 < d_1 < \cdots < d_n < 1 = d_{n+1}$ such that $z \in \{d_1, \ldots, d_n\}$, and G is of fixed sign on each of (d_i, d_{i+1}), $i = 0, \ldots, n$. Set $F_n = \phi_n$ and $F_{n+1} = \phi_{n+1}$. Since the $F_1, \ldots, F_{n+1}$ are linearly independent, there exists a non-trivial linear combination thereof, F^*, satisfying

$$\int_{d_i}^{d_{i+1}} F^*(x) G(x)\, dx = 0\,, \qquad i = 0, \ldots, n-1\,.$$

Furthermore

$$\int_0^1 F^*(x) G(x)\, dx = 0$$

since G is orthogonal to each of the $F_1, \ldots, F_{n+1}$. Thus

$$\int_{d_i}^{d_{i+1}} F^*(x) G(x)\, dx = 0\,, \qquad i = 0, \ldots, n\,.$$

Because G is of fixed sign on (d_i, d_{i+1}), the function F^* must change sign at least once on each (d_i, d_{i+1}), $i = 0, 1, \ldots, n$. Furthermore $F^*(z) = 0$, and z is one of the d_i. Thus

$$F^* = \sum_{i=0}^{n+1} a_i \phi_i$$

has at least $n+2$ distinct zeros, contradicting (1). Therefore ϕ_n and ϕ_{n+1} have no common zero.

Set

$$f(x) = \frac{\phi_{n+1}(x)}{\phi_n(x)}\,.$$

Obviously f is continuous in (w_i, w_{i+1}) for each $i = 0, \ldots, n$. Assume f is strictly monotone thereon as well. Then ϕ_{n+1} cannot possess two zeros in any of the (w_i, w_{i+1}). However, from (1), ϕ_{n+1} has exactly $n+1$ zeros, and none of these are at the $w_0, w_1, \ldots, w_{n+1}$. Thus ϕ_{n+1} has exactly one zero in each of the (w_i, w_{i+1}), proving (2). Thus it suffices to prove the strict monotonicity of f in each (w_i, w_{i+1}).

If f is not strictly monotone in some (w_i, w_{i+1}), then there exists a $w^* \in (w_i, w_{i+1})$ and $\alpha_0 \in \mathbb{R}$ such that

$$f(w^*) = \alpha_0$$

and $f - \alpha_0$ is either non-negative or non-positive in some neighborhood of w^*. Depending also on the sign of ϕ_n in (w_i, w_{i+1}) it follows that

$$\phi_{n+1} - \alpha_0 \phi_n$$

vanishes at w^* and is either non-negative or non-positive in some neighborhood of w^*. Assume, for the sake of exposition, that $\phi_{n+1} - \alpha_0 \phi_n$ is non-negative in some neighborhood of w^* and $\phi_n > 0$ on (w_i, w_{i+1}). The function $\phi_{n+1} - \alpha_0 \phi_n$ has, by (1), at least n sign changes and at most $n+1$ zeros in $[0, 1]$. Since this function does not change sign at w^*, it has n sign changes at points other than w^*. For ε sufficiently small

$$\phi_{n+1} - (\alpha_0 + \varepsilon)\phi_n$$

has n sign changes close to the n sign changes of $\phi_{n+1} - \alpha_0 \phi_n$. For $\varepsilon > 0$, small, $\phi_{n+1} - (\alpha_0 + \varepsilon)\phi_n$ has two additional sign changes in the neighborhood of w^*, i.e., at least $n+2$ sign changes. This contradicts (1), and proves the theorem. ■

For another approach, in a somewhat more restrictive case, see Karlin [18]. In Lee, Pinkus [26] may be found a generalization which has applications in the theory of ordinary differential equations.

In Pinkus [29] (see also Buslaev [4]) there is to be found a generalization to a non-linear eigenvalue problem. The motivation for this problem came from the study of n-widths. We do not go into details here. The linear case, Theorem 4.1, is the L^2 version of the theory which has been generalized to L^p $(1 \le p \le \infty)$, where a result analogous to Theorem 4.1 is obtained. A matrix version of this theory may be found in Pinkus [28] and Buslaev [5].

§5. The Matrix Case

We will discuss in this section the matrix case. As we previously mentioned, this was actually considered after the kernel case. This fact is well worth recalling. The two main actors in this section are F. R. Gantmacher and M. G. Krein. We start with a few words about Krein.

Mark Grigorievich Krein was born in 1907 in Kiev, and died in 1989 in Odessa. He was a truly eminent and exceedingly prolific mathematician who contributed significantly to and had a tremendous impact in many different areas of mathematics (see Gohberg [14] and [15]). The story of M. G. Krein, and the mathematical schools he built, is fundamentally marred by the tyranny and anti-semitism which were constant factors in his career. Krein was dismissed from his position at the University of Odessa in 1944, and from his part-time position at the Mathematical Institute of the Ukrainian Academy of Sciences in Kiev in 1952. We can only speculate on what might have been if he had been treated with the respect and dignity which was his due. In 1939 he was elected a corresponding member of the Ukrainian Academy of Sciences. He was never elected a full member. This fact prompted the following famous mathematical joke. *Ques:* How do you know that the Ukrainian Academy of Sciences is the best academy in the world? *Ans:* Because Krein is only a corresponding member. From 1944 to 1954 Krein held the chair in theoretical mechanics at the Odessa Naval Engineering Institute, and from 1954 until his retirement he held the chair in theoretical mechanics at the Odessa Civil Engineering Institute.

Altogether Gantmacher and Krein wrote six joint papers and the book we earlier mentioned (in two editions). In Section 1 we referenced two of these papers (one was just the announcement of the other). In another of their papers they proved the Hadamard inequality for TP matrices. That is,

$$A\begin{pmatrix}1,\ldots,n\\1,\ldots,n\end{pmatrix} \le A\begin{pmatrix}1,\ldots,r\\1,\ldots,r\end{pmatrix} A\begin{pmatrix}r+1,\ldots,n\\r+1,\ldots,n\end{pmatrix}$$

for $r = 1,\ldots,n-1$. The proof of this is repeated in the fundamental 1937 paper. The other three joint papers are unrelated to total positivity and so do not interest us here.

In Section 2 we gave the definition of totally positive and strictly totally positive matrices. There are also a set of inbetween matrices called *oscillation* matrices. A is an oscillation matrix if A is TP and some power of A is STP. The following result was first proved in Gantmacher, Krein [11, p. 454-455]. We shall not prove it here as it is not needed in what follows.

Proposition 5.1. *An $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ is an oscillation matrix if and only if A is TP, non-singular, and $a_{i,i+1}, a_{i+1,i} > 0$, $i = 1,\ldots,n-1$. Furthermore, if A is an oscillation matrix, then A^{n-1} is STP.*

Remark. The term "oscillation" matrix was introduced by Gantmacher, Krein [11]. (Note the use of the term also in Kellogg [23].) They called totally positive and strictly totally positive matrices "completely non-negative" and "completely positive", respectively, (and in French). The term "total positiv" (in German) was introduced by Schoenberg [30] in 1930, in a paper where he considers some variation diminishing properties of such matrices, and their generalizations.

The main theorem, which we now state, is to be found in Gantmacher, Krein [11] (Theorems 10 and 14, p. 460–467). A proof also appears in Ando [1], Gantmacher [8], and Gantmacher, Krein [13].

Theorem 5.2. *The n eigenvalues of an $n \times n$ oscillation matrix are positive and simple. In addition, if we denote by $\mathbf{u}^k$ a real eigenvector (unique up to multiplication by a non-zero constant) associated with the eigenvalue λ_k, where $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$, then*

$$q - 1 \le S^-\left(\sum_{i=q}^{p} c_i \mathbf{u}^i\right) \le S^+\left(\sum_{i=q}^{p} c_i \mathbf{u}^i\right) \le p - 1$$

for each $1 \le p \le q \le n$ (and c_i not all zero). In particular $S^-(\mathbf{u}^k) = S^+(\mathbf{u}^k) = k - 1$ for $k = 1, \ldots, n$.

We use two known results in the proof of Theorem 5.2. The first of these is Perron's Theorem (Perron [27]).

Theorem 5.3. *Let A be an $n \times n$ matrix, all of whose elements are strictly positive. Then A has a simple, positive eigenvalue which is strictly greater in modulus than all other eigenvalues of A. Furthermore the unique (up to multiplication by a non-zero constant) associated eigenvector may be chosen so that all its components are strictly positive.*

Proof: There are numerous proofs of this result in the literature, and almost countless generalizations. For completeness we present here a proof which seems to be one of the simpler and more transparent (similar, in parts, of course, to the proof of Theorem 4.2).

For vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, we write $\mathbf{x} \ge \mathbf{y}$ if $x_i \ge y_i$, $i = 1, \ldots, n$, and $\mathbf{x} > \mathbf{y}$ if $x_i > y_i$, $i = 1, \ldots, n$. Set

$$\lambda^* = \sup\{\lambda \,:\, A\mathbf{x} \ge \lambda\mathbf{x}\,,\ \text{some } \mathbf{x} \ge \mathbf{0}\}\,.$$

Since all elements of A are strictly positive, we must have $\lambda^* > 0$. A convergence (compactness) argument implies the existence of an $\mathbf{x}^* \ge \mathbf{0}$ ($\mathbf{x}^* \ne \mathbf{0}$) such that $A\mathbf{x}^* \ge \lambda^*\mathbf{x}^*$. If $A\mathbf{x}^* \ne \lambda^*\mathbf{x}^*$, then $A(A\mathbf{x}^*) > \lambda^*(A\mathbf{x}^*)$ since all entries of A are strictly positive. Setting $\mathbf{y}^* = A\mathbf{x}^*$ it follows that $A\mathbf{y}^* > \lambda^*\mathbf{y}^*$, which contradicts the definition of λ^*. Thus

$$A\mathbf{x}^* = \lambda^*\mathbf{x}^*\,.$$

Now $\mathbf{x}^* \ge \mathbf{0}$ ($\mathbf{x}^* \ne \mathbf{0}$) and thus $A\mathbf{x}^* > \mathbf{0}$, whence $\mathbf{x}^* > \mathbf{0}$. We have found a positive eigenvalue with a strictly positive eigenvector.

Let λ be any other eigenvector of A. Thus

$$A\mathbf{y} = \lambda\mathbf{y}$$

for some $\mathbf{y} \in \mathbb{C}^n \backslash \{\mathbf{0}\}$. Now

$$|\lambda|\,|\mathbf{y}| = |\lambda \mathbf{y}| = |A\mathbf{y}| \le A|\mathbf{y}|\,,$$

where $|\mathbf{y}| = (|y_1|, \ldots, |y_n|)$. From the definition of λ^*, it follows that $|\lambda| \le \lambda^*$. If $|\lambda| = \lambda^*$, then we must have $\lambda^*|\mathbf{y}| = A|\mathbf{y}|$ (since otherwise $\lambda^*(A|\mathbf{y}|) < A(A|\mathbf{y}|)$, contradicting the definition of λ^*). Thus for each $i = 1, \ldots, n$,

$$|\lambda|\,|y_i| = \left|\sum_{j=1}^{n} a_{ij} y_j\right| = \sum_{j=1}^{n} a_{ij} |y_j|\,.$$

This implies the existence of a $\gamma \in \mathbb{C}$, $|\gamma| = 1$, such that $\gamma y_j = |y_j|$ for each $j = 1, \ldots, n$. Thus we may in fact assume that if $|\lambda| = \lambda^*$, then $\lambda = |\lambda| = \lambda^*$, and for every associated eigenvector $\mathbf{y}$, we have $\mathbf{y} \ge \mathbf{0}$. This implies two consequences. Firstly, for all eigenvalues $\lambda \ne \lambda^*$ we have $|\lambda| < \lambda^*$. Secondly, the geometric multiplicity of the eigenvalue λ^* is exactly one. For if not, then we can easily construct a real eigenvector which is not of one sign.

It remains to prove that the eigenvalue λ^* is of algebraic multiplicity one. Assume not. Then there exists a vector $\mathbf{y}^*$ (linearly independent of $\mathbf{x}^*$) such that

$$A\mathbf{y}^* = \lambda^* \mathbf{y}^* + \alpha \mathbf{x}^*$$

with some $\alpha \ne 0$. A^T (the transpose of A) has the same eigenvalues as A (and is obviously strictly positive). As such there exists an eigenvector $\mathbf{w}^* > \mathbf{0}$ associated with the eigenvalue λ^*. Now

$$\lambda^*(\mathbf{w}^*, \mathbf{y}^*) = (A^T\mathbf{w}^*, \mathbf{y}^*) = (\mathbf{w}^*, A\mathbf{y}^*) = \lambda^*(\mathbf{w}^*, \mathbf{y}^*) + \alpha(\mathbf{w}^*, \mathbf{x}^*)\,.$$

Since $\mathbf{w}^*, \mathbf{x}^* > \mathbf{0}$, and $\alpha \ne 0$, we have $\alpha(\mathbf{w}^*, \mathbf{x}^*) \ne 0$, a contradiction. Thus the algebraic multiplicity of λ^* is one. ■

The second result we need is called Kronecker's Theorem.

Theorem 5.4. *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ listed to their algebraic multiplicity. Then the $\binom{n}{p}$ eigenvalues of $A_{[p]}$, listed to their algebraic multiplicity, are $\lambda_{i_1} \cdots \lambda_{i_p}$ for $1 \le i_1 < \cdots < i_p \le n$.*

Proof: We will present two proofs. The simpler is the following. Every $n \times n$ matrix A may be written in the form

$$A = P^{-1}TP$$

where T is an upper triangular matrix. As such the diagonal entries of T are simply the eigenvalues of A. Now, from the Cauchy-Binet formula

$$A_{[p]} = (P^{-1})_{[p]} T_{[p]} P_{[p]} = (P_{[p]})^{-1} T_{[p]} P_{[p]}\,.$$

The matrix $T_{[p]}$ is upper triangular and its diagonal entries are simply the products of p distinct diagonal entries of T. This proves the result.

A simple second proof is the following. Associated with each eigenvalue λ_i is a "generalized eigenvector" $\mathbf{u}^i$, $i = 1, \ldots, n$, (see Section 2), such that $\mathbf{u}^1, \ldots, \mathbf{u}^n$ span $\mathbb{C}^n$. Now

$$A_{[p]} \left(\mathbf{u}^{i_1} \wedge \cdots \wedge \mathbf{u}^{i_p} \right) = A\mathbf{u}^{i_1} \wedge \cdots \wedge A\mathbf{u}^{i_p}$$

from which it follows that the $\mathbf{u}^{i_1} \wedge \cdots \wedge \mathbf{u}^{i_p}$ are generalized eigenvectors of $A_{[p]}$ with associated eigenvalues $\lambda_{i_1} \cdots \lambda_{i_p}$. Since these vectors are linearly independent, we have determined all the $\binom{n}{p}$ eigenvalues of $A_{[p]}$. ■

We can now prove Theorem 5.2.

Proof of Theorem 5.2: It suffices to prove the theorem for A STP. For if A has eigenvalues $\lambda_1, \ldots, \lambda_n$, listed to their algebraic multiplicity, then A^m has eigenvalues $\lambda_1^m, \ldots, \lambda_n^m$. If we show that $\lambda_1^m > \cdots > \lambda_n^m > 0$ for all m sufficiently large, then obviously we must have $\lambda_1 > \cdots > \lambda_n > 0$. In addition, if $A\mathbf{u} = \lambda \mathbf{u}$, then $A^m \mathbf{u} = \lambda^m \mathbf{u}$, so that if A has n distinct eigenvalues, then the eigenvectors of A and A^m are exactly the same for all m.

We again follow the reasoning in previous proofs in this paper. Let $|\lambda_1| \geq \cdots \geq |\lambda_n| > 0$. From Theorems 5.3 and 5.4 applied to $A_{[p]}$ we have that

$$\lambda_1 \cdots \lambda_p > |\lambda_1 \cdots \lambda_{p-1} \lambda_{p+1}|$$

for all $p = 1, \ldots, n$, whence

$$\lambda_1 > \cdots > \lambda_n > 0\,.$$

By a suitable normalization of the associated eigenvectors $\mathbf{u}^1, \ldots, \mathbf{u}^n$, it also follows that

$$\mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^p > \mathbf{0}$$

for $p = 1, 2, \ldots, n$.

It remains for us to prove the sign change properties of the eigenvectors $\mathbf{u}^1, \ldots, \mathbf{u}^n$. Assume $S^+ \left(\sum_{i=q}^p c_i \mathbf{u}^i \right) \geq p$. There then exist $j_0 < \cdots < j_p$ and an $\varepsilon \in \{-1, 1\}$ such that

$$\varepsilon (-1)^k \left(\sum_{i=q}^p c_i \mathbf{u}^i \right)_{j_k} \geq 0\,, \qquad k = 0, \ldots, p\,.$$

Set $\mathbf{u}^0 = \sum_{i=q}^p c_i \mathbf{u}^i$. Thus

$$U \begin{pmatrix} j_0, j_1, \ldots, j_p \\ 0, 1, \ldots, p \end{pmatrix} = \det \left(\mathbf{u}^i_{j_k} \right)_{i,k=0}^p = 0$$

since $\mathbf{u}^0$ is a linear combination of the $\mathbf{u}^1, \ldots, \mathbf{u}^p$. On the other hand, when expanding this matrix by the first column we obtain

$$U\begin{pmatrix} j_0, j_1, \ldots, j_p \\ 0, 1, \ldots, p \end{pmatrix} = \sum_{k=0}^{p} (-1)^k \mathbf{u}^0_{j_k} U \begin{pmatrix} j_0, \ldots, j_{k-1}, j_{k+1}, \ldots, j_p \\ 1, \ldots, p \end{pmatrix}.$$

Since $\mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^p > \mathbf{0}$, we have

$$U\begin{pmatrix} j_0, \ldots, j_{k-1}, j_{k+1}, \ldots, j_p \\ 1, \ldots, p \end{pmatrix} > 0\,,$$

for each $k = 0, \ldots, p$. Thus we must have $\mathbf{u}^0_{j_k} = 0$, $k = 0, \ldots, p$. This is impossible since $\mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^p > \mathbf{0}$ implies that all $p \times p$ minors of the $n \times p$ matrix with columns $\mathbf{u}^1, \ldots, \mathbf{u}^p$ are non-singular. Thus $S^+\left(\sum_{i=q}^{p} c_i \mathbf{u}^i\right) \le p - 1$.

Let $\mathbf{v}^1, \ldots, \mathbf{v}^n$ be left eigenvectors with assocated eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively, of A. We assume, by what we have proven, that $\mathbf{v}^1 \wedge \cdots \wedge \mathbf{v}^{q-1} > \mathbf{0}$ for $q = 2, \ldots, n+1$, and thus

$$S^+\left(\sum_{j=1}^{q-1} b_j \mathbf{v}^j\right) \le q - 2$$

for any choice of non-trivial $(b_1, \ldots, b_{q-1})$. Let $\mathbf{u} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ be such that

$$(\mathbf{u}, \mathbf{v}^j) = 0\,, \qquad j = 1, \ldots, q-1\,.$$

We claim that $S^-(\mathbf{u}) \ge q - 1$. If $S^-(\mathbf{u}) = r \le q - 2$, there exist $1 \le i_1 < \cdots i_r < n$ and an $\varepsilon \in \{-1, 1\}$ such that

$$\varepsilon(-1)^k \mathbf{u}_j \ge 0\,, \qquad i_{k-1} + 1 \le j \le i_k\,,$$

and $\mathbf{u}_j \ne 0$ for some $i_{k-1} + 1 \le j \le i_k$, for $k = 1, \ldots, r+1$, where $i_0 = 0$ and $i_{r+1} = n$. Let

$$\mathbf{v} = \sum_{j=1}^{r+1} b_j \mathbf{v}^j$$

satisfy $\mathbf{v} \ne \mathbf{0}$ and $\mathbf{v}_{i_k} = 0$, $k = 1, \ldots, r$. Since $S^+\left(\sum_{j=1}^{r+1} b_j \mathbf{v}^j\right) \le r$, we must have

$$\delta(-1)^k \mathbf{v}_j > 0\,, \qquad i_{k-1} + 1 \le j < i_k\,,$$

for all $k = 1, \ldots, r+1$, and also $\delta(-1)^{r+1}\mathbf{v}_n > 0$, where $\delta \in \{-1, 1\}$. Thus

$$(\mathbf{u}, \mathbf{v}) \ne 0\,,$$

a contradiction, implying that $S^-(\mathbf{u}) \ge q - 1$. Now

$$\left(\sum_{i=q}^{p} c_i \mathbf{u}^i, \mathbf{v}^j\right) = 0\,, \qquad j = 1, \ldots, q-1\,.$$

Thus

$$S^{-}\left(\sum_{i=q}^{p} c_i \mathbf{u}^i\right) \geq q-1\,,$$

which proves the theorem. ■

Further information regarding the eigenvalues and eigenvectors may be found Gantmacher, Krein [13], Karlin [19], [21], and Karlin, Pinkus [22].

References

1. Ando, T., Totally positive matrices, *Lin. Alg. and Appl.* **90** (1987), 165–219.
2. Anselone, P. M., and J. W. Lee, Spectral properties of integral operators with nonnegative kernels, *Lin. Alg. and Appl.* **9** (1974), 67–87.
3. Birkhoff, G. D., In Memoriam: The Mathematical work of Oliver Dimon Kellogg, *Bull. Amer. Math. Soc.* **39** (1933), 171–177.
4. Buslaev, A. P., Extremal problems of approximation theory, and nonlinear oscillations, *Dokl. Akad. Nauk SSSR* **305** (1989), 1289–1294; English transl. in *Soviet Math. Dokl.* **39** (1989), 379–384.
5. Buslaev, A. P., A variational description of the spectra of totally positive matrices, and extremal problems of approximation theory, *Mat. Zametki* **47** (1990), 39–46; English transl. in *Math. Notes* **47** (1990), 26–31.
6. Gantmacher, F., Sur les noyaux de Kellogg non symétriques, *Comptes Rendus (Doklady) de l'Academie des Sciences de l'URSS* **1 (10)** (1936), 3–5.
7. Gantmacher, F. R., *The Theory of Matrices*, Gostekhizdat, Moscow-Leningrad, 1953; English transl. as *Matrix Theory*, Chelsea, New York, 2 vols., 1959.
8. Gantmacher, F. R., Obituary, in *Uspekhi Mat. Nauk* **20** (1965), 149–158; English transl. as *Russian Math. Surveys*, **20** (1965), 143–151.
9. Gantmacher, F. R., and M. G. Krein, Sur une classe spéciale de déterminants ayant le rapport aux noyaux de Kellog, *Recueil Mat. (Mat. Sbornik)* **42** (1935), 501–508.
10. Gantmacher, F. R., and M. G. Krein, Sur les matrices oscillatoires, *C. R. Acad. Sci. (Paris)* **201** (1935), 577–579.
11. Gantmacher, F. R., and M. G. Krein, Sur les matrices complètement non négatives et oscillatoires, *Composito Math.* **4** (1937), 445–476.
12. Gantmacher, F. R., and M. G. Krein, *Oscillation Matrices and Small Oscillations of Mechanical Systems* (Russian), Gostekhizdat, Moscow-Leningrad, 1941.
13. Gantmacher, F. R., and M. G. Krein, *Ostsillyatsionye Matritsy i Yadra i Malye Kolebaniya Mekhanicheskikh Sistem*, Gosudarstvenoe Izdatel'stvo, Moskva-Leningrad, 1950; German transl. as *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, Akademie

Verlag, Berlin, 1960; English transl. as *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, USAEC, 1961.
14. Gohberg, I., Mathematical Tales, in *The Gohberg Anniversary Collection*, Eds. H. Dym, S. Goldberg, M. A. Kaashoek, P. Lancaster, pp. 17–56, Operator Theory: Advances and Applications, Vol. 40, Birkhäuser Verlag, Basel, 1989.
15. Gohberg, I., Mark Grigorievich Krein 1907–1989, *Notices Amer. Math. Soc.* **37** (1990), 284–285.
16. Goursat, E., *A Course in Mathematical Analysis: Integral Equations, Calculus of Variations, Vol. III, Part 2*, Dover Publ. Inc., New York, 1964.
17. Jentzsch, R., Über Integralgleichungen mit positivem Kern, *J. Reine und Angewandte Mathematik (Crelle)*, **141** (1912), 235–244.
18. Karlin, S., The existence of eigenvalues for integral operators, *Trans. Amer. Math. Soc.* **113** (1964), 1–17.
19. Karlin, S., Oscillation properties of eigenvectors of strictly totally positive matrices, *J. D'Analyse Math.* **14** (1965), 247–266.
20. Karlin, S., *Total Positivity. Volume 1*, Stanford University Press, Stanford, CA, 1968.
21. Karlin, S., Some extremal problems for eigenvalues of certain matrix and integral operators, *Adv. in Math.* **9**, (1972), 93–136.
22. Karlin, S., and A. Pinkus, Oscillation Properties of Generalized Characteristic Polynomials for Totally Positive and Positive Definite Matrices, *Lin. Alg. and Appl.* **8** (1974), 281–312.
23. Kellogg, O. D., The oscillation of functions of an orthogonal set, *Amer. J. Math.* **38** (1916), 1–5.
24. Kellogg, O. D., Orthogonal function sets arising from integral equations, *Amer. J. Math.* **40** (1918), 145–154.
25. Kellogg, O. D., Interpolation properties of orthogonal sets of solutions of differential equations, *Amer. J. Math.* **40** (1918), 225–234.
26. Lee, J. W., and A. Pinkus, Spectral Properties and Oscillation Theorems for Mixed Boundary-Value Problems of Sturm-Liouville Type, *J. Differential Equations* **27** (1978), 190–213.
27. Perron, O., Zur Theorie der Matrices, *Math. Annalen* **64** (1907), 248–263.
28. Pinkus, A., Some Extremal Problems for Strictly Totally Positive Matrices, *Lin. Alg. and Appl.* **64** (1985), 141–156.
29. Pinkus, A., n-Widths of Sobolev Spaces in L^p, *Constr. Approx.* **1** (1985), 15–62.
30. Schoenberg, I. J., Über variationsvermindernde lineare Transformationen, *Math. Z.* **32** (1930), 321–328.
31. Schoenberg, I. J., *I. J. Schoenberg: Selected Papers*, Ed. C. de Boor, 2 Volumes, Birkhäuser, Basel, 1988.
32. Schur, I., Zur Theorie der linearen homogenen Integralgleichungen, *Math. Ann.* **67** (1909), 306–339.
33. Smithies, F., *Integral Equations*, Cambridge University Press, Cambridge, 1970.

Allan Pinkus
Department of Mathematics
Technion
Haifa, 32000
Israel