# Some Extremal Problems for Strictly Totally Positive Matrices

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ABSTRACT

For a strictly totally positive $M \times N$ matrix $A$ we show that the ratio $\|A\mathbf{x}\|_p / \|\mathbf{x}\|_p$ has exactly $R = \min\{M, N\}$ nonzero critical values for each fixed $p \in (1, \infty)$. Letting $\lambda_i$ denote the $i$th critical value, and $\mathbf{x}^i$ an associated critical vector, we show that $\lambda_1 > \cdots > \lambda_R > 0$ and $\mathbf{x}^i$ (unique up to multiplication by a constant) has exactly $i-1$ sign changes. These critical values are generalizations to $l^p$ of the $s$-numbers of $A$ and satisfy many of the same extremal properties enjoyed by the $s$-numbers, but with respect to the $l^p$ norm.

## 1. INTRODUCTION

In the course of our investigations into $n$-widths of Sobolev spaces in $L^p$, we were led to a consideration of the following matrix problems.

Let $A$ be an $M \times N$ matrix, and $p \in [1, \infty]$. For $\mathbf{x} \in \mathbb{R}^m$, set $\|\mathbf{x}\|_p = (\sum_{i=1}^m |x_i|^p)^{1/p}$, $1 \leqslant p < \infty$, and $\|\mathbf{x}\|_\infty = \max\{|x_i|: i = 1, \ldots, m\}$. Consider the following three quantities:

$$\sigma_n^1(p) = \min_{P_n} \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|(A - P_n)\mathbf{x}\|_p}{\|\mathbf{x}\|_p}, \tag{A}$$

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where $P_n$ ranges over all $M \times N$ matrices of rank $n$;

$$\sigma_n^2(p) = \min_{X_n} \max_{\substack{\mathbf{x} \perp X_n \\ \mathbf{x} \neq 0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}, \tag{B}$$

where $X_n$ is any subspace of $\mathbb{R}^N$ of dimension $n$;

$$\sigma_n^3(p) = \max_{X_{n+1}} \min_{\substack{\mathbf{x} \in X_{n+1} \\ \mathbf{x} \neq 0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}, \tag{C}$$

where $X_{n+1}$ is any subspace of $\mathbb{R}^N$ of dimension $n+1$.

It is easily proven that $\sigma_n^1(p) \geqslant \sigma_n^2(p) \geqslant \sigma_n^3(p)$. For $p = 2$, it is well known that equality holds for any matrix $A$, i.e., $\sigma_n^1(p) = \sigma_n^2(p) = \sigma_n^3(p)$ (by the Rayleigh-Ritz characterization of eigenvalues). The common value is the square root of the $(n+1)$st eigenvalue of $A^TA$ (arranged in nonincreasing order of magnitude), i.e., the singular values or $s$-numbers of $A$. Appropriate optimal $P_n$, $X_n$, and $X_{n+1}$ in (A), (B) and (C), respectively, may be obtained from the singular value decomposition of $A^TA$. For $p \neq 2$ very little is known except when $A$ is a diagonal matrix (see [6]), or when $p = 1, \infty$ and $A$ is strictly totally positive (STP) (see Micchelli and Pinkus [5] for $p = \infty$, and a duality argument gives $p = 1$).

We prove the following result.

THEOREM 1.1. *Let $A$ be an $M \times N$ STP matrix and $p \in [1, \infty]$. Then for each $n$, $0 \leqslant n <$ rank $A = \min\{M, N\} = R$,*

(1) $\sigma_n^1(p) = \sigma_n^2(p) = \sigma_n^3(p) \equiv \sigma_n(p)$

(2) $\sigma_0(p) > \cdots > \sigma_{R-1}(p) > 0$.

We also identify an optimal rank $n$ matrix in (A) and optimal subspaces in (B) and (C).

For $A$ as above it follows by a theorem of Gantmacher and Krein [1] that $A^TA$ has simple, distinct, positive eigenvalues $\lambda_1 > \cdots > \lambda_R > 0$, and if $\mathbf{x}^i$ is the eigenvector of $A^TA$ associated with $\lambda_i$, then $S^+(\mathbf{x}^i) = S^-(\mathbf{x}^i) = i - 1$, $i = 1, \ldots, R$ (see the next section for a definition of $S^+$ and $S^-$). When $p = 2$, Melkman and Micchelli [4] constructed an optimal rank $n$ matrix in (A) and an optimal subspace in (B) which were *not* derived from the singular value decomposition of $A$, but depended on the STP property of $A$ and the sign change property of the eigenvectors. These are the results which are generalized here. The case $p = 1, \infty$ has been solved, and we therefore consider only $p \in (1, \infty)$. Effectively, we set

$$F(\mathbf{x}) = \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}$$

and consider critical vectors of this map, i.e., vectors for which $\partial F(\mathbf{x})/\partial x_k = 0$, $k = 1, \dots, N$. These equations are

$$\sum_{i=1}^{M} a_{ik}|(A\mathbf{x})_i|^{p-1}\operatorname{sgn}((A\mathbf{x})_i) = \lambda^p |x_k|^{p-1}\operatorname{sgn}(x_k),$$

$$k = 1, \dots, N. \quad (1.1)$$

For $p = 2$, this simply reduces to the eigenvalue problem $A^T A\mathbf{x} = \lambda^2 \mathbf{x}$. [Multiplying each side of (1.1) by $x_k$ and summing over $k$, it follows that $\lambda = \|A\mathbf{x}\|_p/\|\mathbf{x}\|_p$.] We say that $(\lambda, \mathbf{x})$ is a *critical value* of our problem, for fixed $p$, if $(\lambda, \mathbf{x})$ satisfies (1.1). We call $\lambda$ a *critical number*, and $\mathbf{x}$ a *critical vector*. (Note that we may multiply $\mathbf{x}$ by any nonzero constant, so that we consider critical vectors up to multiplication by constants.) For $A$ STP we obtain the existence of exactly $R$ critical values $(\lambda_n(p), \mathbf{x}^n(p))$, $n = 1, \dots, R$ [with $\lambda_n(p) > 0$, $n = 1, \dots, R$] where

(1) $\lambda_1(p) > \cdots > \lambda_R(p) > 0$,
(2) $S^+(\mathbf{x}^i(p)) = S^-(\mathbf{x}^i(p)) = i - 1$, $i = 1, \dots, R$,

and the critical vector $\mathbf{x}^i(p)$ associated with the critical value $\lambda_i(p)$ is unique (up to multiplication by constants). (There is also an $N - R$ dimensional subspace $Y_{N-R}$ such that $(0, \mathbf{x})$ is a critical value for all $\mathbf{x} \in Y_{N-R}$.) Furthermore,

$$\sigma_n(p) = \lambda_{n+1}(p) = \frac{\|A\mathbf{x}^{n+1}(p)\|_p}{\|\mathbf{x}^{n+1}(p)\|_p}, \qquad n = 0, 1, \dots, R-1.$$

The optimal rank $n$ matrix of (A) and optimal subspaces of (B) and (C) are constructed from $\mathbf{x}^{n+1}(p)$ [and do not depend on the other $\mathbf{x}^i(p)$].

The following questions immediately present themselves. Firstly, does $\sigma_n^1(p) = \sigma_n^2(p) = \sigma_n^3(p)$ for all $p \in [1, \infty]$ and every matrix $A$? Secondly, do there exist at most $R$ critical values of $F(\mathbf{x})$ for arbitrary $A$ of rank $R$ (with nonzero critical numbers)? We do not know the answers to these questions. It is known that when considering $\|A\mathbf{x}\|_p/\|\mathbf{x}\|_q$, $p, q \in [1, \infty]$, $p \neq q$, i.e., different norms on the numerator and denominator, then the answer to both questions is in the negative even for diagonal matrices.

The organization of the paper runs as follows. In Section 2 we define the notation and present some known results which will be subsequently used. In Section 3 we first assume that (1.1) has a critical value $(\lambda, \mathbf{x})$ with $S^{-}(\mathbf{x}) = n$ and show, from this property, how to prove Theorem 1.1. We then prove the existence of the required solutions of (1.1).

We note that in the terminology of approximation theory, $\sigma_n^1(p)$ is known as the linear $n$-width of $\mathscr{A}_p = \{A\mathbf{x} : \|\mathbf{x}\|_p \leqslant 1\}$ in $l_p^M$, $\sigma_n^2(p)$ is the Gel'fand $n$-width of $\mathscr{A}_p$ in $l_p^M$, and $\sigma_n^3(p)$ is the Bernstein $n$-width of $\mathscr{A}_p$ in $l_p^M$. A fourth quantity, called the Kolmogorov $n$-width, is not introduced here, but its value is the same as these others for $\mathscr{A}_p$ in $l_p^M$ (see [6] for a fuller exposition).

## 2. PRELIMINARIES

Let $A = (a_{ij})_{i=1}^{M}{}_{j=1}^{N}$, $a_{ij} \in \mathbb{R}$, and $R = \min\{M, N\}$.

DEFINITION 2.1. The matrix $A$ is said to be strictly totally positive (STP) if

$$A\begin{pmatrix} i_1, \dots, i_k \\ j_1, \dots, j_k \end{pmatrix} = \det\left(a_{i_m j_n}\right)_{m=1}^{k}{}_{n=1}^{k} > 0$$

for all choices of $1 \leqslant i_1 < \cdots < i_k \leqslant M$, $1 \leqslant j_1 < \cdots < j_k \leqslant N$, and all $k = 1, \dots, R$. It is said to be totally positive (TP) if inequality replaces strict inequality. If

$$A\begin{pmatrix} i_1, \dots, i_k \\ j_1, \dots, j_k \end{pmatrix} > 0$$

for ordered $\{i_m\}$, $\{j_n\}$ as above and $k = 1, \dots, r$, then $A$ is said to be strictly totally positive of order $r$ ($\mathrm{STP}_r$).

(The results of this paper actually hold for a class of matrices called strictly sign regular of order $r$, and rank $r$; see e.g. Karlin [2] for the definition. However, it is simpler to assume that $A$ is STP and of full rank.)

An important property of STP matrices is that of variation diminishing. To explain this property we need the following.

DEFINITION 2.2. Let $\mathbf{x} \in \mathbb{R}^m$. Then for $\mathbf{x} \neq \mathbf{0}$:

(i) $S^-(\mathbf{x})$ counts the number of ordered sign changes in the vector $\mathbf{x} = (x_1, \ldots, x_m)$ where zero entries are discarded.
(ii) $S^+(\mathbf{x})$ counts the maximum number of ordered sign changes in the vector $\mathbf{x} = (x_1, \ldots, x_m)$ where zero entries are given arbitrary values.

For convenience, we set $S^+(\mathbf{0}) = m$.
Thus, for example, if $\mathbf{x} = (1,0,1,0,-1)$, then $S^-(\mathbf{x}) = 1$ and $S^+(\mathbf{x}) = 3$. Note that if $S^+(\mathbf{x}) = S^-(\mathbf{x})$, then

(1) $x_1, x_m \neq 0$,
(2) $x_i = 0$ implies $x_{i-1}x_{i+1} < 0$.

These facts will be repeatedly used.

PROPOSITION 2.1. *Let $A$ be an $M \times N$ STP matrix and $\mathbf{x} \neq \mathbf{0}$. Then*

$$S^+(A\mathbf{x}) \leqslant S^-(\mathbf{x}).$$

*In particular, if $A\mathbf{x} = \mathbf{0}$, then $S^-(\mathbf{x}) \geqslant M$.*

A proof of this statement and many other facts concerning STP matrices may be found in Karlin [2].

## 3. THE MAIN RESULT

We assume that $A$ is an $M \times N$ STP matrix, $R = \min\{M, N\}$, and $p \in (1, \infty)$, fixed. For $\mathbf{x} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$, set

$$F(\mathbf{x}) = \|A\mathbf{x}\|_p / \|\mathbf{x}\|_p,$$

and $G_k(\mathbf{x}) = \partial F(\mathbf{x}) / \partial x_k$, $k = 1, \ldots, N$, $G(\mathbf{x}) = (G_1(\mathbf{x}), \ldots, G_N(\mathbf{x}))$. An easy calculation shows that $G(\mathbf{x}) = \mathbf{0}$ if and only if

$$\sum_{i=1}^{M} a_{ik} |(A\mathbf{x})_i|^{p-1} \operatorname{sgn}((A\mathbf{x})_i) = \lambda^p |x_k|^{p-1} \operatorname{sgn}(x_k),$$

$$k = 1, \ldots, N, \quad (3.1)$$

where $\lambda$ is some constant. We say that $(\lambda, \mathbf{x})$, $\mathbf{x} \neq \mathbf{0}$, is a *critical value* of our problem (for fixed $p$) if $(\lambda,\mathbf{x})$ satisfies (3.1). We call $\lambda$ a *critical number*, and $\mathbf{x}$ a *critical vector*. (Note that if $\mathbf{x}$ is a critical vector, then so is $\alpha\mathbf{x}$ for all $\alpha\in\mathbb{R}\setminus\{0\}$.)

We first prove some simple facts.

LEMMA 3.1. *If $(\lambda,\mathbf{x})$ is a critical value, then* $\lambda=\|A\mathbf{x}\|_p/\|\mathbf{x}\|_p$.

*Proof.* Multiply each equation in (3.1) by $x_k$ and sum over $k$. ■

LEMMA 3.2. *If $(\lambda,\mathbf{x})$ is a critical value and $S^-(\mathbf{x})\leqslant R-1$, then $\lambda>0$.*

*Proof.* From Lemma 3.1, $\|A\mathbf{x}\|_p=\lambda\|\mathbf{x}\|_p$, $\mathbf{x}\neq\mathbf{0}$. If $\lambda=0$, then $A\mathbf{x}=\mathbf{0}$. Now $R=\min\{M,N\}$ and $A$ is of rank $R$. Since $A\mathbf{x}=\mathbf{0}$, then by Proposition 2.1 $S^-(\mathbf{x})\geqslant M\geqslant R$. By assumption $S^-(\mathbf{x})\leqslant R-1$. ■

PROPOSITION 3.3. *If $(\lambda,\mathbf{x})$ is a critical value and $\lambda>0$, then*

$$S^-(\mathbf{x})=S^+(\mathbf{x})=S^-(A\mathbf{x})=S^+(A\mathbf{x}).$$

*Proof.* This proposition is an application of Proposition 2.1 and the fact that $\alpha$ and $|\alpha|^{p-1}\operatorname{sgn}(\alpha)$ have the same sign for $\alpha\in\mathbb{R}$. To be precise, since $\lambda>0$,

$$S^-(\mathbf{x})\leqslant S^+(\mathbf{x})=S^+\left(\left\{\lambda^p|x_k|^{p-1}\operatorname{sgn}(x_k)\right\}_{k=1}^N\right)$$

$$=S^+\left(\left\{\sum_{i=1}^M a_{ik}|(A\mathbf{x})_i|^{p-1}\operatorname{sgn}((A\mathbf{x})_i)\right\}_{k=1}^N\right)$$

from (3.1). By Proposition 2.1,

$$\leqslant S^-\left(\left\{|(A\mathbf{x})_i|^{p-1}\operatorname{sgn}((A\mathbf{x})_i)\right\}_{i=1}^M\right)$$

$$=S^-(A\mathbf{x})\leqslant S^+(A\mathbf{x})\leqslant S^-(\mathbf{x}).$$

Thus equality holds throughout, and the proposition is proved. ■

Note that since $R=\min\{M,N\}$, and $\mathbf{x}\in\mathbb{R}^N$, $A\mathbf{x}\in\mathbb{R}^M$, it follows that if $(\lambda,\mathbf{x})$ is a critical value, then $\lambda>0$ if and only if $S^-(\mathbf{x})\leqslant R-1$.

PROPOSITION 3.4. *Assume* $(\lambda,\mathbf{x})$ *and* $(\lambda,\mathbf{y})$ *are critical values with the same* $\lambda>0$. *Then* $\mathbf{x}=\alpha\mathbf{y}$ *for some* $\alpha\in\mathbb{R}$.

*Proof.* Assume not, i.e., $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. We use the following simple fact. For any $a,b\in\mathbb{R}$, $\operatorname{sgn}(a+b)=\operatorname{sgn}\big(|a|^{p-1}\operatorname{sgn}(a)+|b|^{p-1}\operatorname{sgn}(b)\big)$. Thus for any $\alpha,\beta\in\mathbb{R}$ (assume $\alpha^2+\beta^2>0$)

$$\operatorname{sgn}(\alpha x_k+\beta y_k)=\operatorname{sgn}\big(|\alpha|^{p-1}\operatorname{sgn}(\alpha)\,|x_k|^{p-1}\operatorname{sgn}(x_k)+|\beta|^{p-1}\operatorname{sgn}(\beta)\,|y_k|^{p-1}\operatorname{sgn}(y_k)\big).$$

Now, from (3.1),

$$\begin{aligned}\sum_{i=1}^{M}a_{ik}\Big[&|\alpha|^{p-1}\operatorname{sgn}(\alpha)|(A\mathbf{x})_i|^{p-1}\operatorname{sgn}((A\mathbf{x})_i)\\&+|\beta|^{p-1}\operatorname{sgn}(\beta)|(A\mathbf{y})_i|^{p-1}\operatorname{sgn}((A\mathbf{y})_i)\Big]\\=\lambda^p\Big[&|\alpha|^{p-1}\operatorname{sgn}(\alpha)|x_k|^{p-1}\operatorname{sgn}(x_k)\\&+|\beta|^{p-1}\operatorname{sgn}(\beta)|y_k|^{p-1}\operatorname{sgn}(y_k)\Big],\qquad k=1,\dots,N.\end{aligned}$$

Following the method of proof of Proposition 3.3, we see that

$$S^-(\alpha\mathbf{x}+\beta\mathbf{y})=S^+(\alpha\mathbf{x}+\beta\mathbf{y})=S^-(A(\alpha\mathbf{x}+\beta\mathbf{y}))=S^+(A(\alpha\mathbf{x}+\beta\mathbf{y}))$$

for all $\alpha,\beta\in\mathbb{R}$, $\alpha^2+\beta^2>0$. This cannot hold for all $\alpha,\beta$. For example choose $\alpha,\beta$ such that $\alpha x_1+\beta y_1=0$. Then $S^-(\alpha\mathbf{x}+\beta\mathbf{y})<S^+(\alpha\mathbf{x}+\beta\mathbf{y})$. ■

Proposition 3.4 will not be directly used in this section, but is important in the next section together with the corollary to the following main result.

THEOREM 3.5. *Let* $(\lambda,\mathbf{x})$ *be a critical value with* $S^-(\mathbf{x})=n\leqslant R-1$. *Then*

$$\sigma_n^1(p)=\sigma_n^2(p)=\sigma_n^3(p)=\lambda=\frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}.$$

REMARK. The full statement concerning the form of an optimal rank $n$ matrix for $\sigma_n^1(p)$ and optimal subspaces for $\sigma_n^2(p)$ and $\sigma_n^3(p)$ will be given after the proof of the theorem.

Assuming the theorem to be true, we have the following corollary.

COROLLARY 3.6. *If* $(\lambda, \mathbf{x})$ *and* $(\mu, \mathbf{y})$ *are critical values with* $S^-(\mathbf{x}) = S^-(\mathbf{y}) \leqslant R-1$, *then* $\lambda = \mu$ *and* $\mathbf{x} = \alpha \mathbf{y}$ *for some* $\alpha \in \mathbb{R}$.

*Proof.* From Theorem 3.5, $\lambda = \mu$, and thus from Proposition 3.4, $\mathbf{x} = \alpha \mathbf{y}$. ■

It therefore follows, on the basis of Theorem 3.5, that there exist at most $R$ critical values $(\lambda, \mathbf{x})$ with $\lambda > 0$. In the next section we prove the existence of $R$ such critical values.

*Proof of Theorem 3.5.* Recall that we have fixed $p \in (1, \infty)$. Since $\sigma_n^1(p) \geqslant \sigma_n^2(p) \geqslant \sigma_n^3(p)$, we shall prove that $\sigma_n^1(p) \leqslant \lambda \leqslant \sigma_n^3(p)$. For ease of notation we normalize $\mathbf{x}$ so that $\|\mathbf{x}\|_p = 1$ and $x_1 > 0$. [From Proposition 3.3, $S^-(\mathbf{x}) = S^+(\mathbf{x})$, so that $x_1 \neq 0$.]

*Claim 1.* $\lambda \leqslant \sigma_n^3(p)$. Since $S^-(\mathbf{x}) = S^+(\mathbf{x}) = n$, there exist indices $1 \leqslant k_1 < \cdots < k_n < N$ such that

$$x_j(-1)^{r+1} \geqslant 0, \qquad k_{r-1} + 1 \leqslant j \leqslant k_r,$$

$r = 1, \ldots, n+1$, where $k_0 = 0$, $k_{n+1} = N$, and for each $r$ there exists a $j$, $k_{r-1} + 1 \leqslant j \leqslant k_r$, for which $x_j \neq 0$. Let $B$ denote the $M \times (n+1)$ matrix whose $r$th column $\mathbf{b}^r$ is given by

$$\mathbf{b}^r = \sum_{j=k_{r-1}+1}^{k_r} \mathbf{a}^j |x_j|, \qquad r = 1, \ldots, n+1,$$

where $\mathbf{a}^j$ is the $j$th column vector of $A$. A multilinear expansion easily shows that $B$ is STP, and since $n+1 \leqslant M$, rank $B = n+1$. Let $\mathbf{z} = (1, -1, \ldots, (-1)^n) \in \mathbb{R}^{n+1}$, and note that $B\mathbf{z} = A\mathbf{x}$.

We now define $n+1$ vectors in $\mathbb{R}^N$ as follows: Set

$$(\mathbf{y}^r)_j = \begin{cases} |x_j|, & k_{r-1} + 1 \leqslant j \leqslant k_r, \\ 0, & \text{otherwise}, \end{cases}$$

$r = 1, \ldots, n+1$. Define $X_{n+1}^* = \operatorname{span}\{\mathbf{y}^1, \ldots, \mathbf{y}^{n+1}\}$. Since $\dim X_{n+1}^* = n+1$, we have

$$\sigma_n^3(p) \geqslant \min\left\{\frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p} : \mathbf{x} \in X_{n+1}^* \setminus \{0\}\right\}$$

$$= \min\left\{\frac{\left\|A\left(\sum_{r=1}^{n+1} \alpha_r \mathbf{y}^r\right)\right\|_p}{\left\|\sum_{r=1}^{n+1} \alpha_r \mathbf{y}^r\right\|_p} : \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{n+1}) \neq \mathbf{0}\right\}.$$

By definition, $\mathbf{b}^r = A\mathbf{y}^r$, $r = 1, \dots, n+1$, and

$$\left\|\sum_{r=1}^{n+1} \alpha_r \mathbf{y}^r\right\|_p = \left(\sum_{r=1}^{n+1} |\alpha_r|^p c_r\right)^{1/p},$$

where $c_r = \|\mathbf{y}^r\|_p^p$ (since the $\mathbf{y}^r$ have distinct support), $c_r > 0$, $r = 1, \dots, n+1$, and $\sum_{r=1}^{n+1} c_r = \|\mathbf{x}\|_p^p = 1$. Thus

$$\sigma_n^3(p) \geqslant \min\left\{\frac{\|B\boldsymbol{\alpha}\|_p}{\left(\sum_{r=1}^{n+1} |\alpha_r|^p c_r\right)^{1/p}} : \boldsymbol{\alpha} \neq \mathbf{0}\right\}.$$

It is easily seen that this minimum is necessarily attained at a critical value $(\mu, \boldsymbol{\alpha}^*)$ for $B$, i.e., where the derivatives, with respect to each $\alpha_r$, of the above ratio vanish. (Note that $\mu > 0$, since $M \geqslant n+1$, and $B$ has rank $n+1$.) Thus if

$$\mu = \frac{\|B\boldsymbol{\alpha}^*\|_p}{\left(\sum_{r=1}^{n+1} |\alpha_r^*|^p c_r\right)^{1/p}} = \min\left\{\frac{\|B\boldsymbol{\alpha}\|_p}{\left(\sum_{r=1}^{n+1} |\alpha_r|^p c_r\right)^{1/p}} : \boldsymbol{\alpha} \neq \mathbf{0}\right\},$$

then

$$\sum_{i=1}^{M} b_{ir} |(B\boldsymbol{\alpha}^*)_i|^{p-1} \operatorname{sgn}((B\boldsymbol{\alpha}^*)_i) = \mu^p c_r |\alpha_r^*|^{p-1} \operatorname{sgn}(\alpha_r^*),$$
$$r = 1, \dots, n+1. \quad (3.2)$$

We claim that $(\lambda, \mathbf{z})$ also satisfies (3.2) as a result of (3.1). To see this multiply each side of (3.1) by $x_k$, sum on $k$ between $k_{r-1}+1$ and $k_r$, and recall that $Bz = Ax$.

We wish to prove that $\lambda = \mu$. We use the STP property of $B$ and the fact that $\mathbf{z}$ strictly alternates in sign to prove this fact.

Assume that $\boldsymbol{\alpha}^*$ and $\mathbf{z}$ are linearly independent vectors (otherwise it follows that $\lambda = \mu$). An application of the method of proof of Proposition 3.4 (and 3.3) shows that for all $\gamma, \delta \in \mathbb{R}$, $\gamma^2 + \delta^2 > 0$,

$$S^+\left(\left\{\lambda^{p/(p-1)}c_r^{1/(p-1)}\gamma z_r + \mu^{p/(p-1)}c_r^{1/(p-1)}\delta\alpha^*_r\right\}_{r=1}^{n+1}\right) \leqslant S^-\left(\left\{\gamma z_r + \delta\alpha_r^*\right\}_{r=1}^{n+1}\right)$$

Since $c_r > 0$, $r = 1,\dots,n+1$, and $\mu, \lambda > 0$, this reduces (setting $\gamma = 1$) to

$$S^+\left(\mathbf{z} + \left(\frac{\mu}{\lambda}\right)^{p/(p-1)}\delta\boldsymbol{\alpha}^*\right) \leqslant S^-(\mathbf{z} + \delta\boldsymbol{\alpha}^*)$$

for every $\delta \in \mathbb{R}$.

We may assume $\alpha_1^* < 0$. [Since $S^+(\boldsymbol{\alpha}^*) = S^-(\boldsymbol{\alpha}^*)$, we have $\alpha_1^* \neq 0$.] Set

$$\delta_0 = \max\left\{\delta : \delta > 0,\ 1 + \delta\alpha_r^*(-1)^{r+1} \geqslant 0,\ r = 1,\dots,n+1\right\}.$$

Recall that $z_r = (-1)^{r+1}$, $r = 1,\dots,n+1$. Now, $0 < \delta_0 < \infty$, and for $0 < \delta < \delta_0$, $S^-(\mathbf{z} + \delta\boldsymbol{\alpha}^*) = n$, while $S^-(\mathbf{z} + \delta_0\boldsymbol{\alpha}^*) \leqslant n-1$. Thus $S^+(\mathbf{z} + (\mu/\lambda)^{p/(p-1)}\delta_0\boldsymbol{\alpha}^*) \leqslant n-1$, which implies that $(\mu/\lambda)^{p/(p-1)}\delta_0 > \delta_0$, i.e., $\mu > \lambda$. This contradicts the definition of $\mu$. Therefore $\boldsymbol{\alpha}^* = \gamma\mathbf{z}$ for some $\gamma \in \mathbb{R}$ and $\lambda = \mu$. This proves Claim 1.

*Claim* 2. $\sigma_n^1(p) \leqslant \lambda$. Let $0 = k_0 < k_1 < \cdots < k_n < N = k_{n+1}$ be as previously defined, i.e., $x_j(-1)^{r+1} \geqslant 0$, $k_{r-1}+1 \leqslant j \leqslant k_r$, $r = 1,\dots,n+1$, with the additional proviso that if $x_i = 0$, then $i = k_r$ for some $r$. [Recall that since $S^+(\mathbf{x}) = S^-(\mathbf{x})$, then $x_i = 0$ implies $x_{i-1}x_{i+1} < 0$.] Thus for $r = 1,\dots,n$, either

(a) $x_{k_r}x_{k_r+1} < 0$, or
(b) $x_{k_r} = 0$, $x_{k_r-1}x_{k_r+1} > 0$.
We define vectors $\{\mathbf{e}^r\}_{r=1}^n$ as follows:

(a′) If $x_{k_r}x_{k_r+1} < 0$, set

$$(\mathbf{e}^r)_i = \begin{cases} |x_i|^{-(p-1)}, & i = k_r, k_r+1, \\ 0, & \text{otherwise.} \end{cases}$$

(b′) If $x_{k_r} = 0$, set $(\mathbf{e}^r)_i = \delta_{i,k_r}$.

The vectors $\{\mathbf{e}^r\}_{r=1}^n$ form a weak Descartes system of degree $n$, i.e., the $N \times n$ matrix formed by the columns $\mathbf{e}^r$ is TP and of rank $n$. Furthermore, each $\mathbf{e}^r$ is orthogonal to the vector

$$\left(|x_1|^{p-1}\operatorname{sgn}(x_1),\ldots,|x_N|^{p-1}\operatorname{sgn}(x_N)\right).$$

We construct $\{\mathbf{f}^r\}_{r=1}^n$ in a totally analogous manner, but with respect to $A\mathbf{x}$ [recall that from Proposition 3.3, $S^+(A\mathbf{x})=S^-(A\mathbf{x})=n$], except that here we set $p=2$, i.e., $(A\mathbf{x},\mathbf{f}^r)=0$, $r=1,\ldots,n$. We now construct the $M\times N$ matrix $D=(d_{ij})_{i=1}^{M}{}_{j=1}^{N}$, where

$$d_{ij}=\frac{\begin{vmatrix}(\mathbf{f}^1,A\mathbf{e}^1) & \cdots & (\mathbf{f}^1,A\mathbf{e}^n) & (\mathbf{f}^1,\mathbf{a}^j)\\ \vdots & & \vdots & \vdots\\ (\mathbf{f}^n,A\mathbf{e}^1) & \cdots & (\mathbf{f}^n,A\mathbf{e}^n) & (\mathbf{f}^n,\mathbf{a}^j)\\ (A\mathbf{e}^1)_i & \cdots & (A\mathbf{e}^n)_i & a_{ij}\end{vmatrix}}{\det\left((\mathbf{f}^s,A\mathbf{e}^t)\right)_{s,t=1}^n}.$$

Thus

$$D\mathbf{y}=\frac{\begin{vmatrix}(\mathbf{f}^1,A\mathbf{e}^1) & \cdots & (\mathbf{f}^1,A\mathbf{e}^n) & (\mathbf{f}^1,A\mathbf{y})\\ \vdots & & \vdots & \vdots\\ (\mathbf{f}^n,A\mathbf{e}^1) & \cdots & (\mathbf{f}^n,A\mathbf{e}^n) & (\mathbf{f}^n,A\mathbf{y})\\ A\mathbf{e}^1 & \cdots & A\mathbf{e}^n & A\mathbf{y}\end{vmatrix}}{\det\left((\mathbf{f}^s,A\mathbf{e}^t)\right)_{s,t=1}^n},$$

where the numerator is to be understood via an expansion by the last row. Since the $\{\mathbf{e}^t\}_{t=1}^n$ and $\{\mathbf{f}^s\}_{s=1}^n$ form TP matrices of rank $n$ and $A$ is STP, the denominator is positive.

Note that

$$D\mathbf{y}=A\mathbf{y}-P_n\mathbf{y},$$

where $P_n$ is an $M\times N$ matrix of rank $n$. From the form of $D$ we see that the range of $P_n$ is spanned by the vectors $A\mathbf{e}^1,\ldots,A\mathbf{e}^n$. Thus

$$\begin{aligned}\sigma_n^1(p)&\leqslant\max_{\mathbf{y}\neq 0}\frac{\|(A-P_n)\mathbf{y}\|_p}{\|\mathbf{y}\|_p}\\ &=\max_{\mathbf{y}\neq 0}\frac{\|D\mathbf{y}\|_p}{\|\mathbf{y}\|_p}.\end{aligned}$$

We list and prove some properties of $D$:

(i) $D\mathbf{x} = A\mathbf{x}$, since $(\mathbf{f}^r, A\mathbf{x}) = 0$, $r = 1, \dots, n$.

(ii) $\operatorname{sgn}(A\mathbf{x})_i d_{ij} \operatorname{sgn}(x_j) \geq 0$ for all $i$, $j$, because of the particular form of the $\{\mathbf{e}^r\}_{r=1}^n$ and $\{\mathbf{f}^r\}_{r=1}^n$.

(iii) If $x_k = 0$, then $d_{ik} = 0$, $i = 1, \dots, M$. If $x_k = 0$, then $(\mathbf{e}^r)_i = \delta_{ik}$ for some $r$, and therefore $(\mathbf{f}^s, A\mathbf{e}^r) = (\mathbf{f}^s, \mathbf{a}^k)$ and $(A\mathbf{e}^r)_i = a_{ik}$, so that $d_{ik} = 0$.

(iv) Set $C = (c_{ij})_{i=1}^{M}{}_{j=1}^{N}$, $c_{ij} = |d_{ij}|$. Then $C$ is TP by an application of Sylvester's determinant identity; see e.g. Karlin [2, p. 3].

(v) $\sum_{i=1}^M d_{ir} |(D\mathbf{x})_i|^{p-1} \operatorname{sgn}((D\mathbf{x})_i) = \lambda^p |x_r|^{p-1} \operatorname{sgn}(x_r)$, $r = 1, \dots, N$.

From (i), $D\mathbf{x} = A\mathbf{x}$, and furthermore, $d_{ij} = a_{ij} - \sum_{k=1}^n (A\mathbf{e}^k)_i b_{kj}$ for some constants $b_{kj}$. Recall that $\mathbf{e}^k$ is orthogonal to $(|x_1|^{p-1}\operatorname{sgn}(x_1), \dots, |x_N|^{p-1}\operatorname{sgn}(x_N))$. Thus

$$\begin{aligned}
\sum_{i=1}^M d_{ir} |(D\mathbf{x})_i|^{p-1} \operatorname{sgn}((D\mathbf{x})_i) &= \sum_{i=1}^M a_{ir} |(A\mathbf{x})_i|^{p-1} \operatorname{sgn}((A\mathbf{x})_i) \\
&\quad - \sum_{k=1}^n b_{kr} \sum_{i=1}^M (A\mathbf{e}^k)_i |(A\mathbf{x})_i|^{p-1} \operatorname{sgn}((A\mathbf{x})_i) \\
&= \lambda^p |x_r|^{p-1} \operatorname{sgn}(x_r) \\
&\quad - \sum_{k=1}^n \sum_{l=1}^N b_{kr} (\mathbf{e}^k)_l \sum_{i=1}^M a_{il} |(A\mathbf{x})_i|^{p-1} \operatorname{sgn}((A\mathbf{x})_i) \\
&= \lambda^p |x_r|^{p-1} \operatorname{sgn}(x_r) \\
&\quad - \lambda^p \sum_{k=1}^n b_{kr} \sum_{l=1}^N (\mathbf{e}^k)_l |x_l|^{p-1} \operatorname{sgn}(x_l) \\
&= \lambda^p |x_r|^{p-1} \operatorname{sgn}(x_r).
\end{aligned}$$

With these properties we now proceed with the proof. Because of the checkerboard nature of the sign of $D$ [by (ii)]

$$\max_{\mathbf{y} \neq \mathbf{0}} \frac{\|D\mathbf{y}\|_p}{\|\mathbf{y}\|_p} = \max_{\mathbf{y} \neq \mathbf{0}} \frac{\|C\mathbf{y}\|_p}{\|\mathbf{y}\|_p}.$$

This maximum is attained for some $(\mu, \mathbf{y}^*)$ satisfying

$$\mu = \frac{\|C\mathbf{y}^*\|_p}{\|\mathbf{y}^*\|_p} = \max_{\mathbf{y} \neq \mathbf{0}} \frac{\|C\mathbf{y}\|_p}{\|\mathbf{y}\|_p}.$$

Since $C \geqslant 0$, The maximum is attained by a $\mathbf{y}^*$ which does not change sign, and we may therefore assume $\mathbf{y}^* \geqslant 0$. The maximum is attained at a critical value, i.e., for $(\mu, \mathbf{y}^*)$ satisfying

$$\sum_{i=1}^{M} c_{ir}\left[(C\mathbf{y}^*)_i\right]^{p-1} = \mu^p (y^*{}_r)^{p-1}, \qquad r = 1,\ldots,N. \tag{3.3}$$

From (ii) and (v) we see that $(\lambda, \mathbf{w})$ also satisfies (3.3), where $\mathbf{w} = (|x_1|,\ldots,|x_N|)$. We claim that $\mu = \lambda$. The proof of this fact is via a Perron-Frobenius type argument. Assume $\mathbf{w}$ and $\mathbf{y}^*$ are linearly independent. (Otherwise the result follows immediately.) From (iii), if $x_r = 0$, then $|d_{ir}| = c_{ir} = 0$, $i = 1,\ldots,M$, which implies by (3.3) that $y_r^* = 0$. By the above, there exists a smallest number $\gamma$ ($\gamma$ finite) such that $\gamma|x_r| \geqslant y_r^*$ for all $r$. Thus

$$\gamma^{p-1}\left[(C\mathbf{w})_i\right]^{p-1} \geqslant \left[(C\mathbf{y}^*)_i\right]^{p-1}, \qquad i = 1,\ldots,M,$$

and therefore

$$\gamma^{p-1}\sum_{i=1}^{M} c_{ir}\left[(C\mathbf{w})_i\right]^{p-1} \geqslant \sum_{i=1}^{M} c_{ir}\left[(C\mathbf{y}^*)_i\right]^{p-1}$$

for $r = 1,\ldots,N$. From (3.3) we obtain

$$\gamma^{p-1}\lambda^p|x_r|^{p-1} \geqslant \mu^p (y_r^*)^{p-1}, \qquad r = 1,\ldots,N.$$

Thus

$$\gamma\left(\frac{\lambda}{\mu}\right)^{p/(p-1)}|x_r| \geqslant y_r^*, \qquad r = 1,\ldots,N.$$

(Strict inequality in fact holds, but this is immaterial here.) From the definition of $\gamma$, it follows that $\lambda \geqslant \mu$. Thus $\lambda = \mu$, proving the claim and completing the proof of the theorem. ∎

Remark. Using the notation of the proof of Theorem 3.5 we may now delineate an optimal rank $n$ matrix and optimal subspaces. $P_n$ as defined above is an optimal rank $n$ matrix for $\sigma_n^1(p)$. The $n$-dimensional subspace $X_n^* = \operatorname{span}\{A^T\mathbf{f}^1,\ldots,A^T\mathbf{f}^n\}$ is optimal for $\sigma_n^2(p)$. The $(n+1)$-dimensional subspace $X_{n+1}^* = \mathrm{span}\{\mathbf{y}^1,\dots,\mathbf{y}^{n+1}\}$ is optimal for $\sigma_n^3(p)$, where the $\{\mathbf{y}^i\}_{i=1}^{n+1}$ are as given in the proof of Claim 1.

REMARK. To better understand $P_n$, one might think of the case where the sign changes of both $\mathbf{x}$ and $A\mathbf{x}$ all occur with zero coefficients. (This more closely corresponds to the continuous version of this problem.) In this case $P_n$ is the rank $n$ matrix which interpolates to $A\mathbf{y}$ at the zero coefficients of $A\mathbf{x}$ from the subspace spanned by the $n$ columns of $A$ whose indices are given by the $n$ zeros of $\mathbf{x}$. In general, these "zeros" are "spread out," so that we must use appropriate linear combinations of consecutive coefficients.

REMARK. If $A$ is an $N \times N$ matrix of full rank, then the above proof may be simplified. Let $\sigma_n^i(p) = \sigma_n^i(p; A)$ to denote the dependence on $A$. It is easily shown that $\sigma_n^2(p; A)\sigma_{N-n-1}^3(p; A^{-1}) = 1$. Furthermore, if $A$ is STP, then $A^{-1}$ is STP after left and right multiplication by the diagonal matrix with diagonal entries alternately one and minus one. In addition, if $(\lambda, \mathbf{x})$ is a critical value for $A$, then $(1/\lambda,\ A\mathbf{x})$ is a critical value for $A^{-1}$ (with the same $p$). It therefore suffices to prove $\sigma_n^1(p; A) \leqslant \lambda$ to obtain equality throughout.

To verify Theorem 1.1, it remains to prove this next result.

THEOREM 3.7. *Let $A$ be an $M \times N$ STP matrix, $p \in (1, \infty)$. Then for each $n$, $0 \leqslant n \leqslant R-1$, $R = \min\{M, N\}$, there exists an $\mathbf{x} \in \mathbb{R}^N$ for which $S^-(\mathbf{x}) = n$ and $(\lambda, \mathbf{x})$ is a critical value, i.e., satisfies* (3.1).

From Corollary 3.6 and the statement thereafter we see that there is at most one vector $\mathbf{x} \in \mathbb{R}^N$ (up to multiplication by a constant) such that $S^-(\mathbf{x}) = n$, $0 \leqslant n \leqslant R-1$, and $\mathbf{x}$ is a critical vector. It therefore suffices to prove the existence of at least $R$ distinct (pairwise linearly independent) critical vectors with nonzero critical numbers. The following proposition is used.

PROPOSITION 3.8. *Let $(\lambda, \mathbf{x})$ be a critical value for $A$ and $p \in (1, \infty)$ with $S^-(\mathbf{x}) = n$, $0 \leqslant n \leqslant R-1$. Set $\mathbf{y} = (y_1, \dots, y_M)$, where*

$$y_k = |(A\mathbf{x})_k|^{p-1}\,\mathrm{sgn}((A\mathbf{x})_k), \qquad k = 1, \dots, M.$$

*Then $(\lambda, \mathbf{y})$ is a critical value for $A^T$ and $q$, where $1/p + 1/q = 1$, and $S^-(\mathbf{y}) = n$.*

*Proof.* By Proposition 3.3, $S^-(\mathbf{x}) = S^-(A\mathbf{x})$. Since $S^-(\mathbf{y}) = S^-(A\mathbf{x})$, it follows that $S^-(\mathbf{y}) = n$. We wish to prove

$$\sum_{i=1}^{N} a_{ki}\left|\left(A^T\mathbf{y}\right)_i\right|^{q-1}\operatorname{sgn}\left(\left(A^T\mathbf{y}\right)_i\right) = \lambda^q |y_k|^{q-1}\operatorname{sgn}(y_k), \qquad k = 1,\dots,M.$$

From the definition of $y_k$,

$$|y_k|^{q-1}\operatorname{sgn}(y_k) = (A\mathbf{x})_k, \qquad k = 1,\dots,M.$$

Since $(\lambda, \mathbf{x})$ is a critical value for $A$ and $p$, we have from (3.1)

$$\sum_{i=1}^{M} a_{ik} y_i = \lambda^p |x_k|^{p-1}\operatorname{sgn}(x_k), \qquad k = 1,\dots,N,$$

i.e., $(A^T\mathbf{y})_i = \lambda^p |x_i|^{p-1}\operatorname{sgn}(x_i)$, $i = 1,\dots,N$, which implies $|(A^T\mathbf{y})_i|^{q-1}\operatorname{sgn}((A^T\mathbf{y})_i) = \lambda^q x_i$, $i = 1,\dots,N$. Thus

$$\begin{aligned}\sum_{i=1}^{N} a_{ki}\left|\left(A^T\mathbf{y}\right)_i\right|^{q-1}\operatorname{sgn}\left(\left(A^T\mathbf{y}\right)_i\right) &= \sum_{i=1}^{N} a_{ki}\lambda^q x_i \\ &= \lambda^q (A\mathbf{x})_k \\ &= \lambda^q |y_k|^{q-1}\operatorname{sgn}(y_k), \qquad k = 1,\dots,M. \ \blacksquare\end{aligned}$$

The proof we present is an application of the Ljusternik-Schnirelman theorem (see [3], [8]), one version of which is:

THEOREM 3.9. *Suppose $f, g \in C^1(\mathbb{R}^k, \mathbb{R})$ with $f, g$ even functions and $g$ strictly convex. Then $f|_{\{\mathbf{x}:\, g(\mathbf{x})=1\}}$ possesses at least $k$ distinct pairs of critical vectors.*

*Proof of Theorem 3.7.* Assume $M \geqslant N$, i.e., $R = N$. Set $f(\mathbf{x}) = \|A\mathbf{x}\|_p$ and $g(\mathbf{x}) = \|\mathbf{x}\|_p$. Then $f$ and $g$ satisfy the hypotheses of Theorem 3.9. The vector $\mathbf{x}$ is a critical vector if the gradient of $f$ is a multiple, $\lambda$, of the gradient of $g$. This is explicitly (3.1). Thus (3.1) holds for at least $N$ distinct pairs (i.e., $\mathbf{x}$ and $-\mathbf{x}$). If $M < N$, consider $A^T$ and $q$, $1/p + 1/q = 1$, and then apply Proposition 3.8. $\blacksquare$

REMARK. We could actually apply a variant of Theorem 3.9 to the case $M < N$ without going over to the transpose problem. However to do this we must introduce the concept of category (see [3]) or genus (see [8]) in order to prove that of the $N$ critical values, at least $M = R$ have nonzero critical numbers.

REMARK. A different, more "elementary," but much lengthier proof may be found in [7], where a generalization of the above result is proved. This other proof uses the strict total positivity of $A$ and the implicit function theorem to vary from $p = 2$, where we know the result to be valid.

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