Some Remarks on Zero-Increasing Transformations

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Abstract. Let T be a linear operator which maps polynomials of degree n to polynomials of degree n, for every n. We discuss the problem of trying to characterize the set of all such operators which satisfy

$$Z_I(p) \le Z_J(Tp)$$

for every real-valued polynomial p, where Z_I counts the number of zeros on the interval I of $\mathbb{R}$.

§1. Introduction

Let Π denote the space of all real-valued polynomials, and π_n the subspace of all polynomials of degree at most n. In this paper we will consider linear operators $T : \Pi \to \Pi$ for which

$$T : \pi_n \to \pi_n$$

for each $n \in \mathbb{Z}_+$. For each polynomial p, we let $Z_I(p)$ denote the number of zeros of p, counting multiplicity, on the interval $I \subseteq \mathbb{R}$.

The problem we will discuss is that of characterizing those T for which

$$Z_I(p) \le Z_J(Tp)$$

for all $p \in \Pi$. We call such operators *zero-increasing* (ZI). Problems of this type have a long and distinguished history. The interested reader may wish to browse in the books of Marden [18], Obreschkoff [19], and Pólya–Szegő [23].

Let us start with a few simple examples of such operators in order to convince ourselves that they do exist and can be non-trivial. Perhaps the simplest (but trivial) example is

$$(Tp)(x) = (x - a)p'(x)$$

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where $a \in I$. From Rolle's Theorem (the most basic and fundamental of the zero counting methods on $\mathbb{R}$) we have

$$Z_I(p) \le Z_I(Tp)$$

for every polynomial p and any interval I. Similarly it is easy to generalize this example to

$$(Tp)(x) = q_k(x)p^{(k)}(x)$$

where q_k is some fixed polynomial of degree k with k zeros in I. A somewhat more complicated example is the following. Let H_k denote the kth degree Hermite polynomial with leading coefficient 2^k. Then

$$Z_{\mathbb{R}}\left(\sum_{k=0}^{n} a_k x^k\right) \le Z_{\mathbb{R}}\left(\sum_{k=0}^{n} a_k H_k(x)\right)$$

for every choice of $\{a_k\}$ and n. We will "prove" this inequality later. We also have

$$Z_{[-1,1]}\left(\sum_{k=0}^{n} a_k x^k\right) \le Z_{[-1,1]}\left(\sum_{k=0}^{n} a_k T_k(x)\right)$$

and

$$Z_{[-1,1]}\left(\sum_{k=0}^{n} a_k x^k\right) \le Z_{[-1,1]}\left(\sum_{k=0}^{n} a_k U_k(x)\right)$$

where T_k and U_k are the Chebyshev polynomials of the first and second kind, respectively, i.e.,

$$T_k(\cos\theta) = \cos k\theta$$

and

$$U_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta}\,.$$

This and similar results are sometimes stated only for the special case where if

$$\sum_{k=0}^{n} a_k x^k$$

has only real zeros, then

$$\sum_{k=0}^{n} a_k H_k(x)$$

has only real zeros. Or if

$$\sum_{k=0}^{n} a_k x^k$$

has all its zeros in $[-1,1]$ then

$$\sum_{k=0}^{n} a_k T_k(x)$$

(or $\sum_{k=0}^{n} a_k U_k(x)$) has all its zeros in $[-1,1]$. We will later discuss this variation on the general problem. In this form the result for Hermite polynomials and for the two Chebyshev polynomials appears in Iserles and Saff [13], although their proof actually gives the more general result. In fact these results, in the general case, were already in Burbea [4]. Examples of other ZI operators can be found in the above two references and in Iserles, Nørsett [11], and Iserles, Nørsett and Saff [12].

We can list more and more of these transformations. But we will not. Rather we will try to develop the general theory, if at all possible. We will mainly consider the cases where I is the whole real line or one of its rays.

§2. Some Classic Theorems

We start with a result called the Hermite–Poulain Theorem. It is based on questions posed by Hermite [8], and answered by Poulain [24].

Hermite–Poulain Theorem. *Assume*

$$g(x) = \sum_{k=0}^{m} b_k x^k$$

is a polynomial with all real zeros and $b_0 \neq 0$. *Then for any polynomial* p

$$Z_{I\!R}(p) \le Z_{I\!R}\left(\sum_{k=0}^{m} b_k p^{(k)}\right) .$$

Proof. The main idea in the proof of this result consists in noting that

$$\sum_{k=0}^{m} b_k p^{(k)} = (g(D)p)(x)$$

where

$$g(D) = \sum_{k=0}^{m} b_k D^k = \sum_{k=0}^{m} b_k \frac{d^k}{dx^k} .$$

That is, $\sum_{k=0}^{m} b_k p^{(k)}$is obtained from p by the application of a constant coefficient ordinary differential equation. In addition, this ordinary differential equation is what we now call *disconjugate*. In this case, this simply means that g has only real zeros, i.e.,

$$g(D) = b_m \prod_{j=1}^{m} (D - \alpha_j I)$$

and $\alpha_j \in I\!R$. We also impose the condition $b_0 \neq 0$ which implies $\alpha_j \neq 0$. Otherwise the resulting polynomial loses degree and the theorem is not correct in the stated form.

As the $D - \alpha_j I$ *commute*, to prove the theorem it suffices to verify

$$Z_{\mathbb{R}}(p) \le Z_{\mathbb{R}}((D - \alpha I)p)$$

for $\alpha \in \mathbb{R}\backslash\{0\}$.

This may be done in a variety of ways. The most elegant I have seen is that due to Poulain himself. He writes

$$(D - \alpha I)p = e^{\alpha x} D(e^{-\alpha x} p)\,.$$

As $\alpha \neq 0$ and p is a polynomial, $e^{-\alpha x}p(x)$ has an additional "zero" at ∞ or $-\infty$ depending upon the sign of α. We now apply Rolle's Theorem. $\square$

Note, in fact, that if $\alpha > 0$ then $D(e^{-\alpha x}p)$ has a zero interlacing those of p and to the right thereof. Thus, refining a bit the above we also have:

Hermite–Poulain Theorem (II). *Let*

$$g(x) = \sum_{k=0}^{m} b_k x^k$$

be a polynomial with all positive zeros. Then

$$Z_{[A,\infty)}(p) \le Z_{[A,\infty)}\left(\sum_{k=0}^{m} b_k p^{(k)}\right),$$

for every polynomial p, and $A \in \mathbb{R}$.

Disconjugate ordinary differential equations have an inverse given by integrating against a Green's function with certain desirable qualities. We will consider the inverse of $g(D)$, operating as a map from polynomials to polynomials of the same degree. The operator $g(D)$ formally exists since $b_0 \neq 0$ and

$$g(D)x^n = b_0 x^n + \sum_{k=0}^{n-1} b_{n,k} x^k\,.$$

But this is not an especially useful form of g from which to derive its inverse. Let us start with the simplest case $g(D) = D - \alpha I$, i.e., the operator

$$(D - \alpha I)p = q\,.$$

This is equivalent to

$$e^{\alpha x} D(e^{-\alpha x} p) = q$$

which then easily translates into

$$e^{-\alpha x} p(x) = \int e^{-\alpha y} q(y)\, dy$$

for some appropriate antiderivative.

For the above to make sense, in that it takes polynomials to polynomials, it follows that for $\alpha > 0$

$$p(x) = -\int_x^\infty e^{\alpha(x-y)} q(y)\, dy\,,$$

and for $\alpha < 0$

$$p(x) = \int_{-\infty}^x e^{\alpha(x-y)} q(y)\, dy\,.$$

In other words, if

$$(Sq)(x) = \begin{cases} -\int_x^\infty e^{\alpha(x-y)} q(y)\, dy, & \alpha > 0 \\ \int_{-\infty}^x e^{\alpha(x-y)} q(y)\, dy, & \alpha < 0 \end{cases}$$

then $S = T^{-1}$ and

$$Z_{\mathbb{R}}(Sq) \le Z_{\mathbb{R}}(q)$$

for every polynomial q.

This is not a surprising result. S not only takes polynomials of degree n to polynomials of degree n, but it is an integral equation with a *totally positive difference kernel* (called a *Pólya frequency function*). It was proven by Schoenberg [25] (see also Karlin [14]) that an operator given by such an integral equation is *variation diminishing*, i.e., satisfies

$$Z_{\mathbb{R}}(Sq) \le Z_{\mathbb{R}}(q)$$

for all continuous functions for which the above integral makes sense (and not only polynomials) if and only if the kernel is totally positive (or to be more precise *sign regular*). (We might possibly call this property *zero decreasing.* However the term variation diminishing is well established. What is actually meant by the term is that the number of *variations in sign* of the Sq is less than or equal that of q.)

What we just did applies to the simple operator $D - \alpha I$. However multiplying such operators is equivalent (for the inverse) of convolving the difference kernels. Thus the appropriate inverse operator to

$$\prod_{j=1}^m (D - \alpha_j I)$$

is an integral equation whose kernel is obtained by convolving totally positive kernels, and thus is also totally positive. There is one further fact which we wish to highlight using this simple example. The kernel of the above S, i.e., of $(D - \alpha I)^{-1}$, is for $\alpha > 0$

$$K_\alpha(x) = \begin{cases} 0\,, & x > 0 \\ -e^{\alpha x}, & x < 0 \end{cases}$$

and has the two-sided Laplace transform

$$\frac{1}{s-\alpha} = \frac{1}{g(s)} \quad \text{for } s < \alpha\,.$$

For $\alpha < 0$ the kernel is given by

$$K_\alpha(x) = \begin{cases} e^{\alpha x}, & x > 0 \\ 0\,, & x < 0\,, \end{cases}$$

and has the two-sided Laplace transform

$$\frac{1}{s-\alpha} = \frac{1}{g(s)} \quad \text{for } s > \alpha\,.$$

Note that in both cases the two-sided Laplace transform equals $1/g(s)$, i.e., is intimately connected with the ordinary differential equation, and exists in some neighborhood of the origin.

In general, if

$$g(D) = b_m \prod_{j=1}^{m} (D - \alpha_j I)\,,$$

then the two-sided Laplace transform of the difference kernel of the inverse operator is given by

$$\frac{1}{b_m \prod_{j=1}^{m}(s-\alpha_j)} = \frac{1}{g(s)}\,,$$

and exists for all s in some neighborhood of the origin. (It actually exists in the strip

$$\{z : \max_{\alpha_j<0} \alpha_j < \operatorname{Re} z < \min_{\alpha_j>0} \alpha_j\}\,.)$$

We will shortly return to the Hermite–Poulain Theorem. However we first pick up another thread. There is an additional class of operators which lead to operators with the ZI property. This is a result due to another outstanding 19th century French analyst, namely Laguerre [16].

Laguerre's Theorem. *Assume g is a polynomial with only real zeros, all of which lie outside the interval $[0, n]$. Then for every polynomial p of degree at most n*

$$p(x) = \sum_{k=0}^{n} a_k x^k$$

we have

$$Z_{\mathbb{R}}\left(\sum_{k=0}^{n} a_k x^k\right) \le Z_{\mathbb{R}}\left(\sum_{k=0}^{n} a_k g(k) x^k\right)\,.$$

These are different types of operator from those in the Hermite–Poulain Theorem. They are "diagonal" in the sense that

$$Tx^k = g(k)x^k$$

for $k = 0, 1, \ldots, n$. Note that there is a restriction here in that these operators only have the ZI property for polynomials of degree at most n, although there is no restriction on the degree of g. Thus these operators in themselves need not be ZI operators on all of Π. However we will use them to construct other ZI operators.

Proof. The idea of this proof is again linked to ordinary differential equations.

If

$$g(x) = b \prod_{j=1}^{m} (x - \alpha_j)$$

then

$$(Tp)(x) = \sum_{k=0}^{n} a_k g(k) x^k = b \prod_{j=1}^{m} (xD - \alpha_j I) p(x)$$

simply because

$$b \prod_{j=1}^{m} (xD - \alpha_j I) x^k = b \prod_{j=1}^{m} (k - \alpha_j) x^k = g(k) x^k \,,$$

for each k. Thus

$$Tp = g(xD)p \,.$$

The ordinary differential equation $g(xD)$ is an Euler equation. Again, as the

$$xD - \alpha_j I$$

commute, it suffices to prove that

$$Z_{I\!R}(p) \le Z_{I\!R}((xD - \alpha I)p)$$

for $\alpha \notin [0, n]$.

Why is this true? We consider separately zeros of p which are at zero, positive and negative. As p and $(xD - \alpha I)p$ have the same order zero at $x = 0$, there is no problem with zeros at zero. For $x > 0$

$$(xD - \alpha I)p = x^{\alpha+1} D(x^{-\alpha} p(x)) \,.$$

Now $x^{-\alpha} p(x)$ has a zero at zero, for $\alpha < 0$, while for $\alpha > n$, $x^{-\alpha} p(x)$ has a "zero" at ∞ (since p is of degree at most n). In both cases, applying Rolle's Theorem we get

$$Z_{(0,\infty)}(p) \le Z_{(0,\infty)}((xD - \alpha I)p) \,.$$

For $x < 0$, a similar argument holds. $\square$

From the above we also have

Laguerre's Theorem (II). *Let g be a polynomial with all negative zeros. Then*

$$Z_{[0,A]}\left(\sum_{k=0}^{n} a_k x^k\right) \le Z_{[0,A]}\left(\sum_{k=0}^{n} a_k g(k) x^k\right),$$

for every polynomial p, and $A>0$ (and similarly on $[-A,0]$).

If there is a zero of g in $[0,n]$, then both the above argument and the results are not valid. However, if we agree to count the number of zeros of the zero polynomial as infinite, then the above result will also hold for polynomials g with all real zeros outside $[r,n]$, r a positive integer, if $g(0)=\cdots=g(r-1)=0$, see Craven and Csordas [6].

Laguerre's Theorem (III). *For any $r\in\mathbb{Z}_+$, let h be a polynomial of the form*

$$h(x)=x(x-1)(x-2)\cdots(x-r)q(x)g(x)$$

where g is as in Laguerre's Theorem, and q is any polynomial with all real zeros in $[0,r+1)$. Then

$$Z_{\mathbb{R}}\left(\sum_{k=0}^{n} a_k x^k\right) \le Z_{\mathbb{R}}\left(\sum_{k=0}^{n} a_k h(k) x^k\right).$$

Proof. To prove this result it suffices to prove it for h of the form

$$h(x)=x(x-1)(x-2)\cdots(x-r)q(x).$$

We then apply Laguerre's Theorem to obtain the full result.

As previously, for $p(x)=\sum_{k=0}^{n} a_k x^k$ we have

$$\sum_{k=0}^{n} a_k h(k) x^k = h(xD)p.$$

Now it is readily checked that

$$xD(xD-I)(xD-2I)\cdots(xD-rI)p = x^{r+1}p^{(r+1)}(x).$$

From Rolle's Theorem we have

$$Z_{\mathbb{R}}(p)\le r+1+Z_{\mathbb{R}}(p^{(r+1)}) = Z_{\mathbb{R}}(x^{r+1}p^{(r+1)}(x)).$$

Let us now consider

$$(xD-\alpha I)xD(xD-I)(xD-2I)\cdots(xD-rI)p.$$

As the $(xD-\alpha I)$ commute, the order here is unimportant. From the previous analysis it therefore follows that this equals

$$(xD-\alpha I)x^{r+1}p^{(r+1)}(x),$$

and a simple calculation shows that this is equivalent to

$$x^{r+1}(xD-(\alpha-r-1)I)p^{(r+1)}(x)\,.$$

Thus we have

$$(xD-\alpha_1 I)\cdots(xD-\alpha_k I)xD(xD-I)(xD-2I)\cdots(xD-rI)p(x)=$$
$$x^{r+1}(xD-(\alpha_1-r-1)I)\cdots(xD-(\alpha_k-r-1)I)p^{(r+1)}(x)\,.$$

As each $\alpha_i\in[0,r+1)$, we have $\alpha_i-r-1<0$ and thus from Laguerre's Theorem and Rolle's Theorem

$$\begin{aligned}
&Z_{I\!R}(p)\le r+1+Z_{I\!R}(p^{(r+1)}(x))\\
&\le r+1+Z_{I\!R}((xD-(\alpha_1-r-1)I)\cdots(xD-(\alpha_k-r-1)I)p^{(r+1)}(x))\\
&=Z_{I\!R}(x^{r+1}(xD-(\alpha_1-r-1)I)\cdots(xD-(\alpha_k-r-1)I)p^{(r+1)}(x))\\
&=Z_{I\!R}((xD-\alpha_1 I)\cdots(xD-\alpha_k I)xD(xD-I)(xD-2I)\cdots(xD-rI)p(x)),
\end{aligned}$$

which is exactly the result we wanted. □

Can we easily invert $g(xD)$? Obviously yes. The inverse is simply given by the diagonal operator taking x^k to $x^k/g(k)$, $k=0,1,\dots,n$. But this form of the inverse is not really useful. Another form of the inverse is more insightful. We first state it for $g(xD)=xD-\alpha I$.

For $\alpha<0$

$$\begin{aligned}
\frac{1}{k-\alpha}&=\int_0^1 t^{k-\alpha-1}dt\\
&=\int_0^1 t^{-\alpha-1}t^k dt
\end{aligned}$$

and thus for $x>0$

$$\frac{x^k}{k-\alpha}=\int_0^x\left(\frac{y}{x}\right)^{-\alpha-1}\frac{1}{x}\,y^k dy\,.$$

Similarly for $\alpha>n$ (and $0\le k\le n$) it follows that

$$\frac{x^k}{k-\alpha}=-\int_x^\infty\left(\frac{y}{x}\right)^{-\alpha-1}\frac{1}{x}\,y^k dy\,.$$

Thus for each p of degree at most n, the operator $S_\alpha=(xD-\alpha I)^{-1}$ is given by

$$(S_\alpha p)(x)=\int_0^x\left(\frac{y}{x}\right)^{-\alpha-1}\frac{1}{x}\,p(y)\,dy$$

for $\alpha<0$, and by

$$(S_\alpha p)(x)=-\int_x^\infty\left(\frac{y}{x}\right)^{-\alpha-1}\frac{1}{x}\,p(y)\,dy$$

for $\alpha > n$, and for such p we have

$$Z_{(0,\infty)}(S_\alpha p) \le Z_{(0,\infty)}(p)\,.$$

(The above integrals do not converge for all $p \in \pi_n$ if $0 \le \alpha \le n$.) A similar result holds on $(-\infty, 0)$. In some sense we have just complicated things. However two things are worth noting. Firstly,

$$\left(\left(\prod_{j=1}^{m} S_{\alpha_j}\right) p\right)(x)$$

also has the form

$$\int_0^\infty L\left(\frac{y}{x}\right)\frac{1}{x}\,p(y)\,dy$$

for some appropriate L. Secondly, if we substitute $y = e^u$ and $x = e^v$, then the "kernel" of the integral operator S_α has the form

$$e^{\alpha(v-u)}e^{-u}\,,$$

where it is not zero. This is "essentially" the same kernel we saw previously, in reference to the inverse of $(D - \alpha I)$.

§3. Consequences of the Hermite–Poulain Theorem

The question we now pose is the following. Are the two sets of operators we have so far discussed, i.e., those obtained from the Hermite–Poulain Theorem and those from Laguerre's Theorem, in some sense the essential building blocks of all linear ZI operators taking polynomials of degree n to polynomials of degree n, all n?

Assuming we are interested in "all linear operators" then we must consider not only the specific operators given by the Hermite–Poulain Theorem and Laguerre's Theorem, but also those operators in their "closure". (And we must understand which operators are in their closure.) Note, for example, that neither the translation nor the dilation operator in contained in what we have so far discussed, and these operators obviously must be considered.

The Hermite–Poulain Theorem tells us that an operator

$$T = g(D)$$

determined by any polynomial g with all real zeros has the desired property. From continuity considerations the same will be true for all operators which are obtained as limits of $g(D)$'s of the above form. So what is the appropriate limit of the set of polynomials with all real zeros?

This and related questions were examined by Laguerre [15] who was the first to characterize those functions which can be uniformly approximated by polynomials whose zeros are all real, or whose zeros are all positive, and also by Pólya [20], [21] who proved and generalized some of these results. These facts may be found in a variety of texts such as Hille [9], Hirschman and Widder [10], and Levin [17]. The set of functions

$$\mathcal{A}_1 = \left\{ Cx^m e^{-c^2x^2+ax} \prod_{k=1}^{\infty} (1+\alpha_k x) e^{-\alpha_k x} \right\}$$

where $C, c, a, \alpha_k \in \mathbb{R}$, $m \in \mathbb{Z}_+$, and $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$, are those functions which may be obtained as the uniform limit (on $[-A, A]$ for any $A > 0$) of polynomials having only real zeros. Defined on $\mathbb{C}$, these are entire functions. (Note that this is a rather restricted set, and it should be contrasted with what the Weierstrass Theorem tells us about approximation by polynomials.) We sometimes call this set of functions the first Pólya–Laguerre (or Laguerre–Pólya) class.

In the Hermite–Poulain Theorem we also need the fact that $g(0) \neq 0$. This simply implies that $m = 0$ (and $C \neq 0$) in the above. Thus we have that if

$$g(D) = Ce^{-c^2D^2+aD} \prod_{k=1}^{\infty} (I + \alpha_k D) e^{-\alpha_k D}$$

then

$$Z_{\mathbb{R}}(p) \leq Z_{\mathbb{R}}(g(D)p)$$

for every polynomial p. We understand the $e^{-c^2D^2}$ and e^{aD} as given by their power series expansion and as such $g(D)$ is well-defined on the set of polynomials.

What are each of the operators $e^{-c^2D^2}$, e^{aD}, and $(I + \alpha_k D)$? The third operator we have already discussed. It is also readily checked that

$$(e^{aD}p)(x) = p(x+a)\,,$$

i.e., e^{aD} is just the shift operator, and obviously

$$Z_{\mathbb{R}}(p) = Z_{\mathbb{R}}(e^{aD}p)\,.$$

The operator $e^{-c^2D^2}$ is more interesting. It may be shown, see Carnicer, Peña, Pinkus [5], that

$$e^{-c^2D^2}x^k = c^k H_k\left(\frac{x}{2c}\right)$$

where H_k is the kth degree Hermite polynomial (with leading coefficient 2^k). Thus

$$Z_{\mathbb{R}}\left(\sum_{k=0}^{n} a_k x^k\right) \leq Z_{\mathbb{R}}\left(\sum_{k=0}^{n} a_k H_k(x)\right)\,,$$

as claimed in the introduction of this paper.

What about the converse? That is, if T is a linear operator taking π_n to π_n for all n and satisfying

$$Z_{\mathbb{R}}(p) \le Z_{\mathbb{R}}(Tp)$$

for every polynomial p, does this imply that

$$T = g(D)$$

for some $g \in \mathcal{A}_1$?

The answer is no!! For $g \in \mathcal{A}_1$ $(m = 0)$ we have $g(0) = C(\neq 0)$ which immediately implies that

$$g(D)x^n = Cx^n + \sum_{k=0}^{n-1} b_{n,k}x^k\,.$$

Thus $g(D)$ is an operator on the set of all polynomials which acts, on the monomials, as a triangular matrix with "constant diagonal entries".

What if we therefore ask the same question, but restrict ourselves to operators of this particular form? That is, assume T is a linear operator of the form

$$Tx^n = Cx^n + \sum_{k=0}^{n-1} b_{n,k}x^k\,,$$

which satisfies

$$Z_{\mathbb{R}}(p) \le Z_{\mathbb{R}}(Tp)$$

for every polynomial p. Does this imply that $T = g(D)$ for some $g \in \mathcal{A}_1$? The answer now is yes. This result may be found in Carnicer, Peña, Pinkus [5].

What can we say about the inverse of such $T = g(D)$? As $g \in \mathcal{A}_1$ $(g(0) \neq 0)$ is the limit of the previously considered polynomials with all real roots, we can identify the inverse operator $S = T^{-1} = (g(D))^{-1}$. There are two possibilities. If $T = g(D) = Ce^{aD}$ then, up to the constant C, T is just a shift and thus S is the reverse shift and is given by $(1/C)e^{-aD}$. If, however, $g \in \mathcal{A}_1$, $g(0) \neq 0$, and $g(x) \neq Ce^{ax}$, then there exists a kernel K whose two-sided Laplace transform is $1/g$ in some neighborhood of the origin, i.e.,

$$\frac{1}{g(s)} = \int_{-\infty}^{\infty} e^{-sx}K(x)\,dx\,,$$

in some neighborhood of the origin, and

$$g(D)p = q$$

if and only if

$$p(x) = \int_{-\infty}^{\infty} K(x-y)q(y)\,dy\,.$$

This result is actually due to Pólya [21] who was motivated to consider this question for the same reasons which we have so far described. That is, he started with the Hermite–Poulain Theorem, considered the limiting operators obtained from functions in $\mathcal{A}_1$, and then considered the inverse of these operators. The fact that K, as a difference kernel, is totally positive and other related facts were proven by Schoenberg [25], [26] (also in Schoenberg [27]). It is for exactly these reasons that Schoenberg called such kernels K Pólya frequency functions. These results are also discussed in Hirschman and Widder [10], Karlin [14], and Widder [28].

What about the second version of the Hermite–Poulain Theorem? That is, we noted that if g has all positive zeros, then

$$Z_{[A,\infty)}(p) \le Z_{[A,\infty)}(g(D)p)$$

for every polynomial p and $A \in \mathbb{R}$. Again we should ask for the appropriate closure of this set of polynomials. It is

$$\mathcal{A}_2^+ = \left\{ Cx^m e^{-ax} \prod_{k=1}^{\infty}(1-\alpha_k x) \right\}$$

where $C \in \mathbb{R}$, $a, \alpha_k \ge 0$, $m \in \mathbb{Z}_+$, and $\sum_{k=1}^{\infty} \alpha_k < \infty$. The difference between this and $\mathcal{A}_1$ is that there is no $e^{-c^2x^2}$ and there is a more restrictive summability condition on the $\{\alpha_k\}$, in addition to the sign conditions. That is, this is the class obtained as the uniform limit, on some interval, of polynomials having only positive zeros. We let $\mathcal{A}_2^-$ be the uniform limit, on some interval, of polynomials having only negative zeros. Note that $g(x) \in \mathcal{A}_2^-$ if and only if $g(-x) \in \mathcal{A}_2^+$.

Assuming $g(0) \ne 0$ ($m = 0$ and $C \ne 0$ in the above) then if

$$g(D) = Ce^{-aD} \prod_{k=1}^{\infty}(I - \alpha_k D)$$

for $a, \alpha_k \ge 0$ and $\sum_{k=1}^{\infty} \alpha_k < \infty$, then

$$Z_{[A,\infty)}(p) \le Z_{[A,\infty)}(g(D)p)$$

for every polynomial p and $A \in \mathbb{R}$. The converse also holds. If T has the form

$$Tx^n = Cx^n + \sum_{k=0}^{n-1} b_{n,k}x^k \,,$$

and

$$Z_{[A,\infty)}(p) \le Z_{[A,\infty)}(Tp)$$

for every polynomial p and $A \in \mathbb{R}$, then $T = g(D)$ for some g as above. It actually suffices to consider only $A \ge 0$ or, $A = -\infty$ and $A = 0$. This result, however, is not true if we only demand the above ZI property to hold for one value of A, see Carnicer, Peña, Pinkus [5].

§4. The Pólya–Schur Property

Before relating to you the parallel generalization of Laguerre's Theorem, let us recall the following. Pólya, Schur [22] in 1914 characterized those linear operators Γ which are "diagonal" on the monomials, i.e., operators of the form

$$\Gamma x^k = \gamma_k x^k\,, \qquad k = 0, 1, \dots\,,$$

and have the property that if p is a polynomial with all its zeros real, then Γp has all its zeros real. As we mentioned in the introduction, this problem is related to the one we have been considering. If T is a ZI operator, then for p with only real zeros it follows that Tp has only real zeros. That is, ZI operators have this Pólya–Schur property. However it transpires that an operator Γ having this Pólya–Schur property does not necessarily have the ZI property. If $\gamma_k = 1+4k^2$, then the associated Γ has the Pólya–Schur property, but not the ZI property. A different example appears in Bakan, Golub [3].

What Pólya and Schur proved is the following:

Pólya–Schur Theorem. *The operator Γ, as above, has the Pólya–Schur property if and only if*

$$\sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \; (= \Gamma\left((1+x)^n\right))$$

has n real zeros all of one sign for all n or, equivalently, the function

$$\phi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$$

is either in $\mathcal{A}_2^+$ or in $\mathcal{A}_2^-$.

These are the same classes $\mathcal{A}_2^\pm$ which we considered previously, but the associated operators are totally different. Pólya and Schur also characterized those "diagonal" operators Γ with the property that if p is a polynomial with all its zeros real and of one sign, then Γp has all its zeros real. Such operators are characterized by the fact that the above ϕ is then in $\mathcal{A}_1$.

What if we now ask to characterize all operators of the form

$$\Gamma x^n = C x^n + \sum_{k=0}^{n-1} b_{n,k} x^k\,,$$

with this Pólya–Schur property. The ZI operators of this form have the Pólya–Schur property and we know how to characterize them. But are they all the operators with this property? Here the answer is in the affirmative. That is, if Γ has the above form of a "triangular operator with constant diagonal entries" and satisfies the Pólya–Schur property, then $\Gamma = g(D)$ for some $g \in \mathcal{A}_1$.

Thus with respect to the Pólya–Schur property we know how to characterize all "diagonal operators" and also all "triangular operators with constant diagonal entries". This begs the question of how to characterize all such "triangular operators". Unfortunately these two types of operators do not commute. The problem remains open and difficult. Are all such operators simply (possibly infinite) products of operators of the above form?

§5. Consequences of Laguerre's Theorem

We now return to our original problem and to the operators given by Laguerre's Theorem, namely

$$Tx^k = g(k)x^k \,, \qquad k = 0, 1, \ldots, n,$$

where g is a polynomial all of whose zeros are real and lie outside $[0, n]$. These operators have the ZI property on polynomials of degree at most n.

What is the appropriate closure of these polynomials as $n \to \infty$? It is neither $\mathcal{A}_1$ nor $\mathcal{A}_2^-$, but somewhere inbetween. It is given by

$$\mathcal{A}_3 = \left\{ Cx^m e^{-c^2x^2+ax} \prod_{k=1}^{\infty} (1 + \alpha_k x) e^{-\alpha_k x} \right\}$$

where $C, c, a \in \mathbb{R}$, $\alpha_k \geq 0$, $m \in \mathbb{Z}_+$, and $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. That is, it is exactly $\mathcal{A}_1$ except for the restriction $\alpha_k \geq 0$. It is definitely not $\mathcal{A}_2^-$. If $g \in \mathcal{A}_3$, and $g(0) \neq 0$, then

$$Z_{\mathbb{R}} \left(\sum_{k=0}^{n} a_k x^k \right) \leq Z_{\mathbb{R}} \left(\sum_{k=0}^{n} a_k g(k) x^k \right) ,$$

for every set of $\{a_k\}$ and now also for every n.

For example, if $g(x) = e^{ax}$ then $g(k) = b^k$ where $b = e^a$ is any positive value. This gives us the dilation map

$$Tx^k = (bx)^k \,,$$

which exactly preserves the number of real (and positive) zeros. If $g(x) = e^{-c^2x^2}$, then $g(k) = q^{k^2}$ for some $q \in (0, 1)$. Thus

$$Z_{\mathbb{R}} \left(\sum_{k=0}^{n} a_k x^k \right) \leq Z_{\mathbb{R}} \left(\sum_{k=0}^{n} a_k q^{k^2} x^k \right) ,$$

for any $q \in (0, 1)$. (The same holds on $(0, \infty)$.) This g is in $\mathcal{A}_3$, but is not in $\mathcal{A}_2^{\pm}$. We also have the classic example of

$$Z_{\mathbb{R}} \left(\sum_{k=0}^{n} a_k x^k \right) \leq Z_{\mathbb{R}} \left(\sum_{k=0}^{n} a_k \binom{n}{k} x^k \right) ,$$

which comes from the fact that the operator

$$Tx^k = \frac{x^k}{k!}$$

has the desired ZI property since

$$\frac{1}{\Gamma(x+1)} = e^{\gamma x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right) e^{-x/k} ,$$

where $\Gamma(\cdot)$ is the Gamma function, and γ is the Euler constant. This function is in $\mathcal{A}_3$, but not in $\mathcal{A}_2^-$ (even discounting the $e^{\gamma x}$) because $\sum_{k=1}^{\infty} 1/k = \infty$. Both the above examples are due to Laguerre.

The question or conjecture which immediately surfaces is whether every diagonal operator

$$Tx^k = c_k x^k , \qquad k = 0, 1, \ldots$$

(assume $c_k > 0$ for all k) with the ZI property necessarily satisfies

$$c_k = g(k) , \qquad k = 0, 1, \ldots$$

for some $g \in \mathcal{A}_3$. This is still an open conjecture. The result is not known in general. However various cases are known based on work of Bakan, Craven, Csordas and Golub [2], see also [1], [6] and [7].

They proved, for example, that if T, as above, is such that

$$Z_{[0,A]}(p) \le Z_{[0,bA]}(Tp)$$

for some $b > 0$ and all $A > 0$, then

$$c_k = g(k) , \qquad k = 0, 1, \ldots$$

for some g with $e^{-ax} g(x) \in \mathcal{A}_2^-$ for some fixed $a \in \mathbb{R}$. They also proved that if

$$\overline{\lim_{k\to\infty}}\, c_k^{1/k} > 0$$

then again

$$c_k = g(k) , \qquad k = 0, 1, \ldots$$

for some g for which $e^{-ax} g(x) \in \mathcal{A}_2^-$ for some fixed $a \in \mathbb{R}$. (Of course, neither q^{k^2} nor $1/k!$ satisfy these conditions.)

If this conjecture is valid, then one of its consequences is the following. Consider $g \in \mathcal{A}_3$ satisfying $g(0) \neq 0$. Then one possibility is that g is of the form $g(x) = Ce^{ax}$ in which case g generates the dilation operator, up to multiplication by the constant C, which exactly preserves the number of real (and positive) zeros. If g does not have this form then $S = T^{-1}$, which corresponds to the operator with diagonal entries $\{1/g(k)\}$, is also given on $(0, \infty)$ by

$$(Sp)(x) = \int_0^{\infty} L\left(\frac{y}{x}\right) \frac{1}{x}\, p(y)\, dy$$

for some appropriate totally positive L. The cases considered by Bakan, Craven, Csordas and Golub correspond to the case where L has finite support.

Even assuming that this conjecture is true, there still remains the intriguing problem (as with the Pólya–Schur property) of characterizing all T with the ZI property on $\mathbb{R}$. We seem far from a complete characterization. Even more difficult are these same problems on intervals other than $\mathbb{R}$, and/or with respect to polynomials of a fixed degree.

References

1. Bakan, A., Craven, T., and G. Csordas, Interpolation and the Laguerre-Pólya class, preprint.
2. Bakan, A., Craven, T., Csordas, G., and A. Golub, Weakly increasing zero-diminishing sequences, Serdica Math. J. **22** (1996), 1001–1024.
3. Bakan, A. G., and A. P. Golub, Some negative results on multiplier sequences of the first kind, Ukrainskii Mat. Z. **44** (1992), 305–309 = Ukrainian Math. J. **44** (1992), 264–268.
4. Burbea, J., Total positivity of certain reproducing kernels, Pacific J. Math. **67** (1976), 101–130.
5. Carnicer, J. M., Peña, J. M., and A. Pinkus, On some zero-increasing operators, to appear in Acta Mathematica Hung.
6. Craven, T., and G. Csordas, Complex zero decreasing sequences, Methods Appl. Anal. **2** (1995), 420–441.
7. Craven, T., and G. Csordas, Problems and theorems in the theory of multiplier sequences, Serdica Math. J. **22** (1996), 515–524.
8. Hermite, Ch., Questions 777, 778, 779, Nouv. Ann. Math. **5** (1866), 432 and 479.
9. Hille, E., *Analytic Function Theory*, Volume II, Ginn, Boston, 1962.
10. Hirschman, I. I., and D. V. Widder, *The Convolution Transform*, Princeton University Press, Princeton, 1955.
11. Iserles, A., and S. P. Nørsett, Zeros of transformed polynomials, SIAM J. Math. Anal. **21** (1990), 483–509.
12. Iserles, A., Nørsett, S. P., and E. B. Saff, On transformations and zeros of polynomials, Rocky Mountain J. Math. **21** (1991), 331–357.
13. Iserles, A., and E. B. Saff, Zeros of expansions in orthogonal polynomials, Math. Proc. Camb. Phil. Soc. **105** (1989), 559–573.
14. Karlin, S., *Total Positivity*, Stanford University Press, Stanford, 1968.
15. Laguerre, E., Sur les fonctions du genre zéro et du genre un, C. R. Acad. Sci. **95** (1882), 828–831. Also to be found in *Oeuvres de Laguerre*, Vol I, Ch. Hermite, H. Poincaré, and E. Rouché (eds.), Gauthier-Villars et fils, Paris, 1898, 174–177.
16. Laguerre, E., Sur quelques points de la théorie des équations numériques, Acta Math. **4** (1884), 97–120. Also to be found in *Oeuvres de Laguerre*, Vol I, Ch. Hermite, H. Poincaré, and E. Rouché (eds.), Gauthier-Villars et fils, Paris, 1898, 184–206.
17. Levin, B. Ia., *Distribution of Zeros of Entire Functions*, AMS Transl. Math. Monographs, Vol. 5, Providence, 1964.
18. Marden, M., *Geometry of Polynomials*, AMS Math. Surveys, Volume 3, Providence, 1966.

19. Obreschkoff, N., *Verteilung und berechnung der Nullstellen reeller Polynome*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.
20. Pólya, G., Über Annäherung durch Polynome mit lauter reelen Wurlzeln, Rend. Circ. Mat. Palermo **36** (1913), 279–295.
21. Pólya, G., Algebraische Untersuchungen über ganze Funktionen vom Geschlechte Null und Eins, J. Reine Angewandte Math. **145** (1915), 224–249.
22. Pólya, G., and J. Schur, Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen, J. Reine Angew. Math. **144** (1914), 89–113.
23. Pólya, G., and G. Szegő, *Problems and Theorems in Analysis*, Volume II, Springer, Berlin, 1976.
24. Poulain, A., Théorèmes généraux sur les équations algébriques, Nouv. Ann. Math. **6** (1867), 21–33.
25. Schoenberg, I. J., On Pólya frequency functions. II: Variation-diminishing integral operators of the convolution type, Acta Sci. Math. (Szeged) **12** (1950), 97–106.
26. Schoenberg, I. J., On Pólya frequency functions. I. The totally positive functions and their Laplace transform, J. Analyse Math. **1** (1951), 331–374.
27. Schoenberg, I. J., *I. J. Schoenberg: Selected Papers*, Ed. C. de Boor, Volume 2, Birkhäuser, Basel, 1988.
28. Widder, D. V., *An Introduction to Transform Theory*, Academic Press, NY, 1971.

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