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Best Approximations by Smooth Functions

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1. INTRODUCTION

Let $W_\infty^{(r)}$ denote the Sobolev space of functions in $C^{(r-1)}[0, 1]$ whose $(r-1)$st derivative is absolutely continuous and whose rth derivative is an element of $L^\infty[0, 1]$, i.e.,

$$W_\infty^{(r)} = W_\infty^{(r)}[0, 1] = \{f: f^{(r-1)} \text{ abs. cont. and } \|f^{(r)}\|_\infty < \infty\}.$$

$B_\infty^{(r)}$ shall denote those functions f in $W_\infty^{(r)}$ for which $\|f^{(r)}\|_\infty \leqslant 1$.

Sattes, in his dissertation [12] (see also [13]), considered the problem of approximating a continuous function, in the uniform norm, by functions of $B_\infty^{(r)}$. He obtained the following result.

THEOREM 1.1 (U. Sattes). *Let $r \geqslant 2$ and $g \in C[0, 1] \backslash B_\infty^{(r)}$. Then $f^* \in B_\infty^{(r)}$ is a best approximation to g, in L^∞ (such a best approximation necessarily exists) if and only if there exists a subinterval $(\alpha, \beta) \subset [0, 1]$ and a positive integer $M \geqslant r+1$ for which the following conditions hold*

(i) *$f^*|_{[\alpha,\beta]}$ is a Perfect spline of degree r with exactly $M-r-1$ knots and $|f^{*(r)}(x)| = 1$ a. e. on $[\alpha, \beta]$, i.e., there exists $\alpha = \xi_0 < \xi_1 < \cdots < \xi_{M-r-1} < \xi_{M-r} = \beta$ for which*

$$f^{*(r)}(x) = \varepsilon(-1)^i, \qquad \xi_{i-1} < x < \xi_i,$$

$i = 1,..., M-r$, where $\varepsilon = +1$ or -1, fixed.

(ii) *$(g - f^*)(x)$ equioscillates on M points $\alpha = x_1 < \cdots < x_M = \beta$ in $[\alpha, \beta]$, i.e.,*

$$(g - f^*)(x_i) = \delta(-1)^i \| g - f^* \|_\infty, \qquad i = 1,..., M,$$

where $\delta = +1$ or -1, fixed.

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(iii) $x_{i+1} < \xi_i < x_{i+r}$, $i = 1,\dots, M-r-1$.

(iv) $\varepsilon = (-1)^r \delta$, *or equivalently* $f^{*(r)}(x) = \operatorname{sgn}((g - f^*)(x_M))$ *for* $x \in (\xi_{M-r-1}, \xi_{M-r})$.

Furthermore every best approximation to $g(x)$ *from* $B_\infty^{(r)}$ *agrees with* $f^*(x)$ *on* $[\alpha, \beta]$.

This theorem evoked our interest for various reasons. Firstly, the class of approximants, unlike those generally considered in problems of approximation theory, is neither a finite dimensional subspace nor is it a varisolvent family of functions (see Rice [11]). Secondly the fact that one obtains an interval of uniqueness, namely $[\alpha, \beta]$, is reminiscent of results obtained in approximating continuous functions by splines of some fixed degree with a fixed number of variable knots (see, e.g., Braess [1]). Thirdly this result is an additional example of the importance of Perfect splines which have been shown to be fundamental in a number of extremal problems in L^∞ (see, e.g., Fisher and Jerome [3], Karlin [7], Pinkus [10], Tichomirov [14] and references therein).

Motivated by Sattes' result we were led to a consideration of best approximations to continuous functions, in the uniform norm, from the class

$$\mathscr{M} = \left\{ f(x) \colon f(x) = \int_0^1 K(x, y)\, h(y)\, dy,\ l(y) \leqslant h(y) \leqslant u(y) \right\},$$

where $u, l \in C[0, 1]$, fixed, $u > l$, and where $K(x, y)$ is a strictly totally positive kernel. For this class we obtain existence, uniqueness and characterization of the best approximant to $g \in C[0, 1]$ from $\mathscr{M}$ (Theorem 3.1).

Stimulated by Theorem 3.1 and very much using the full characterization obtained therein, we then discuss best approximations from sets of the form

$$\mathscr{P}_n^- = \left\{ f(x) \colon f(x) = \sum_{j=0}^{n} (-1)^{j+1} \int_{\xi_j}^{\xi_{j+1}} K(x, y)\, dy,\right.$$
$$\left. 0 = \xi_0 \leqslant \xi_1 \leqslant \cdots \leqslant \xi_n \leqslant \xi_{n+1} = 1 \right\}$$

and

$$\mathscr{P}_n^+ = \left\{ f(x) \colon f(x) = \sum_{j=0}^{n} (-1)^{j} \int_{\xi_j}^{\xi_{j+1}} K(x, y)\, dy,\right.$$
$$\left. 0 = \xi_0 \leqslant \xi_1 \leqslant \cdots \leqslant \xi_n \leqslant \xi_{n+1} = 1 \right\}.$$

For each of the sets $\mathscr{P}_n^-$ and $\mathscr{P}_n^+$ (which are very much analogous to Perfect splines) we prove uniqueness of the best approximation as well as a full characterization result (Theorem 4.1) which is similar to that obtained in Theorem 3.1. In Section 4 we also consider the problem

$$\min\{\| g - cf\|_\infty : f \in \mathscr{P}_n^-, c \geqslant 0\}$$

as well as the analogous problem for $\mathscr{P}_n^+$ and once again, we are able to give a concise description of the unique best approximant (Theorem 4.2).

In Section 5, we replace the set $\mathscr{M}$ by $\mathscr{M}_\infty = \{f(x): f(x) = \int_0^1 K(x, y)\, d\mu(y)\}$, where μ is any nonnegative finite Borel measure and obtain analogues to Theorems 4.1 and 4.2. An interesting problem somewhat similar in structure to the above is given in Theorem 5.6.

2. Preliminaries

In this work we shall consider functions both of the form $f(x) = \int_0^1 K(x, y)\, h(y)\, dy$, where $h \in L^\infty[0, 1]$, and of the form $f(x) = \int_0^1 K(x, y)\, d\mu(y)$, where μ is a finite Borel measure on $[0, 1]$, and where $K(x, y)$ is a strictly totally positive kernel. This section contains preliminary material which shall be used in the subsequent sections.

Definition 2.1. A kernel $K(x, y) \in C([0, 1] \times [0, 1])$ is said to be strictly totally positive, abbreviated STP, if

$$K\begin{pmatrix} x_1,\dots, x_n \\ y_1,\dots, y_n \end{pmatrix} = \det\,(K(x_i, y_j))_{i,j=1}^n > 0$$

for all choices of $0 \leqslant x_1 < \cdots < x_n \leqslant 1$, $0 \leqslant y_1 < \cdots < y_n \leqslant 1$ and all $n \geqslant 1$.

A full exposition of the theory of totally positive kernels may be found in the books of Gantmacher and Krein [4], and Karlin [6]. One particular property of STP kernels which very much interests us in this work, and which is of fundamental importance in problems of L^∞-approximation is that of variation diminishing. To fully explain this property, we shall use the following definitions.

Defintion 2.2. Let $\mathbf{x} = (x_1,\dots, x_m) \in \mathbb{R}^m$. Then $S^-(\mathbf{x})$ denotes the number of sign changes in the sequence obtained from $x_1,\dots, x_m$ by deleting all zero entries. $S^+(\mathbf{x})$ denotes the maximum number of sign changes in the sequence $x_1,\dots, x_m$, where we allow each zero entry to be replaced by 1 or -1.

DEFINITION 2.3. Let $f \in C[0, 1]$. We define $S^+(f) = \sup S^+[f(x_1),..., f(x_m)]$, where the supremum is taken over all sets $0 \leqslant x_1 < \cdots < x_m \leqslant 1$, and m arbitrary.

DEFINITION 2.4. A finite Borel measure μ defined on $[0, 1]$ is said to have m relevant sign changes, denoted by $S^-(\mu) = m$, if

(i) there exist disjoint sets $A_1,..., A_{m+1}$ of $[0, 1]$ with $A_i < A_{i+1}$ (i.e., $x < y$ for all $x \in A_i$, $y \in A_{i+1}$), and $\bigcup_{i=1}^{m+1} A_i = [0, 1]$.

(ii) $\mu(A_i) \neq 0$, $i = 1,..., m+1$, and μ is either a nonnegative or nonpositive measure on A_i.

(iii) $\mu(A_i)\, \mu(A_{i+1}) < 0$, $i = 1,..., m$.

When considering functions of the form $f(x) = \int_0^1 K(x, y)\, h(y)\, dy$ for $h \in L^\infty$, then by $S^-(h) = m$ we mean that $S^-(\mu) = m$, where $h(y)\, dy = d\mu(y)$.

THEOREM 2.1. *Let $K(x, y)$ be an* STP *kernel and μ a finite Borel measure. Set*

$$f(x) = \int_0^1 K(x, y)\, d\mu(y).$$

Then,

(i) $S^+(f) \leqslant S^-(\mu)$,

(ii) *if $S^+(f) = S^-(\mu) < \infty$, then f and μ exhibit the same arrangement of generalized signs. That is to say that if the first nonzero sign of μ near zero is, say, positive, then $f(0) \geqslant 0$, and if $f(0) = 0$, then $f(\varepsilon) < 0$ for all $\varepsilon > 0$ sufficiently small.*

The proof of the above theorem is essentially to be found in Karlin [6, p. 233]. However, for completeness, and in order that the reader have a familiarity with the techniques, we present the proof here. The proof depends on the following facts to which we shall have frequent recourse.

DEFINITION 2.5. Let $u_i \in C[0, 1]$, $i = 1,..., m+1$. $\{u_1,..., u_{m+1}\}$ is said to be a Tchebycheff (T^+) system if

$$U\begin{pmatrix} x_1,..., x_{m+1} \\ 1,..., m+1 \end{pmatrix} = \det(u_j(x_i))_{i,j=1}^{m+1} > 0$$

for all choices of $0 \leqslant x_1 < \cdots < x_{m+1} \leqslant 1$. We say that $\{u_1,..., u_{m+1}\}$ is a Descartes system if $\{u_{i_1},..., u_{i_k}\}$ is a T^+ system for all choices of $1 \leqslant i_1 < \cdots < i_k \leqslant m+1$, and $k = 1,..., m+1$.

The following proposition may be found in Karlin and Studden [8, p. 25].

PROPOSITION 2.2. *Assume that* $\{u_1,\ldots,u_{m+1}\}$ *is a Descartes system on* $[0,1]$, *and*

$$u(x)=\sum_{i=1}^{m+1} a_i u_i(x).$$

(i) $S^+(u)\leqslant S^-(\mathbf{a})$, *where* $\mathbf{a}=(a_1,\ldots,a_{m+1})$.

(ii) *If* $S^+(u)=S^-(\mathbf{a})$, *then the same arrangement of generalized signs occurs in* $S^+(u)$ *and* $S^-(\mathbf{a})$. *Thus, if* $a_i>0$, $a_j=0$, $j=1,\ldots,i-1$, *then* $u(0)\geqslant 0$ *and if* $u(0)=0$, *then* $u(\varepsilon)<0$ *for all* $\varepsilon>0$ *sufficiently small.*

Proof of Theorem 2.1. Assume that $S^-(\mu)=m$, and let $\{A_i\}_{i=1}^{m+1}$ be as in the definition of $S^-(\mu)$. Without loss of generality, we assume that $(-1)^{i+1}\mu(A_i)>0$, $i=1,\ldots,m+1$. Form

$$u_i(x)=(-1)^{i+1}\int_{A_i} K(x,y)\,d\mu(y),\; i=1,\ldots,m+1.$$

We claim that since $K(x,y)$ is STP, $\{u_1,\ldots,u_{m+1}\}$ is a Descartes system on $[0,1]$. To prove this fact, let $1\leqslant i_1<\cdots<i_k\leqslant m+1$, and $0\leqslant x_1<\cdots<x_k\leqslant 1$. Then, by the basic composition formula, see Karlin [6, p. 17],

$$U\begin{pmatrix}x_1,\ldots,x_k\\ i_1,\ldots,i_k\end{pmatrix}=\int_{A_{i_1}}\cdots\int_{A_{i_k}} K\begin{pmatrix}x_1,\ldots,x_k\\ y_1,\ldots,y_k\end{pmatrix}|d\mu(y_k)|\cdots|d\mu(y_1)|>0,$$

since K is STP and $|\mu(A_i)|>0$ for all $i=1,\ldots,m+1$.

Thus

$$f(x)=\sum_{i=1}^{m+1}(-1)^{i+1}u_i(x)$$

and from Proposition 2.2, the statements of Theorem 2.1 follow.

DEFINITION 2.6. A function $g\in C[0,1]$ is said to equioscillate on k points if there exist $0\leqslant x_1<\cdots<x_k\leqslant 1$ for which

$$\varepsilon g(x_i)(-1)^{i+1}=\|g\|_\infty,\qquad i=1,\ldots,k,$$

where $\varepsilon=+1$ or -1, fixed. If, in addition, $\varepsilon=+1$, then we say that g equioscillates on k points, *starting positively*, while if $\varepsilon=-1$, then we say that g equioscillates on k points, *starting negatively*.

The following easily proven results shall be used repeatedly throughout this work.

PROPOSITION 2.3. *Assume that* $g, h \in C[0, 1]$, $\|g\|_\infty \geqslant \|h\|_\infty$, *and* g *equioscillates on* k *points. Then*

$$S^+(g-h) \geqslant k-1.$$

Furthermore, if $S^+(g-h) = k-1$ *and the equioscillations of* $g(x)$ *start positively, then the generalized sign pattern of* $(g-h)(x)$ *starts positively.*

For each n, set

$$\Lambda_n = \{\boldsymbol{\xi}: \boldsymbol{\xi} = (\xi_1, ..., \xi_n), \xi_0 = 0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = 1\}$$

and for each $\boldsymbol{\xi} \in \Lambda_n$, let

$$h_{\boldsymbol{\xi}}(y) = (-1)^i, \qquad \xi_{i-1} < y < \xi_i, i = 1, ..., n+1.$$

PROPOSITION 2.4. (a) *If* $\boldsymbol{\xi} \in \Lambda_n$, $\boldsymbol{\eta} \in \Lambda_k$, *then*

$$S^-(h_{\boldsymbol{\xi}} \pm h_{\boldsymbol{\eta}}) \leqslant \min\{n, k\}.$$

(b) *If* $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Lambda_n$, *then*

$$S^-(h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}) \leqslant n-1.$$

3. MAIN THEOREM

Let $u, l \in C[0, 1]$ and $u(y) > l(y)$ for all $y \in [0, 1]$. (This condition may be weakened.) Set

$$\begin{aligned} \mathscr{M} &= \mathscr{M}[u, l] \\ &= \left\{ f(x): f(x) = \int_0^1 K(x, y)\, h(y)\, dy, l(y) \leqslant h(y) \leqslant u(y), y \in [0, 1] \right\}. \end{aligned}$$

THEOREM 3.1. *Assume that* $K(x, y)$ *is* STP *and* $g \in C[0, 1] \backslash \mathscr{M}$. *Then there exists a unique best approximant to* g *from* $\mathscr{M}$. *This best approximant*

$$f^*(x) = \int_0^1 K(x, y)\, h^*(y)\, dy$$

is uniquely characterized as follows.

There exists a nonnegative (finite) integer M, *knots* $\xi_0^* = 0 < \xi_1^* < \cdots < \xi_M^* < 1 = \xi_{M+1}^*$, $M+1$ *points of equioscillation* $0 \leqslant x_1^* < \cdots < x_{M+1}^* \leqslant 1$, *and an* $\varepsilon = \pm 1$, *such that*

(i) $\varepsilon(g-f^*)(x_i^*)(-1)^{i+1} = \|g-f^*\|_\infty$, $i = 1,\ldots, M+1$

(ii) *if* $\varepsilon = 1$, *then a.e.*

$$\begin{aligned} h^*(y) &= u(y), \qquad \xi_{i-1}^* < y < \xi_i^*, \ i \text{ odd} \\ &= l(y), \qquad \xi_{i-1}^* < y < \xi_i^*, \ i \text{ even} \end{aligned}$$

while if $\varepsilon = -1$, *then a.e.*

$$\begin{aligned} h^*(y) &= u(y), \qquad \xi_{i-1}^* < y < \xi_i^*, \ i \text{ even} \\ &= l(y), \qquad \xi_{i-1}^* < y < \xi_i^*, \ i \text{ odd}. \end{aligned}$$

Furthermore, if for any $g \in C[0,1]$ *there exists a nonnegative integer* M, *and a function* $f^* \in \mathscr{M}$ *satisfying* (i) *and* (ii), *and if* $\|g-f^*\|_\infty > 0$, *then* $g \notin \mathscr{M}$.

Remark. Upon completion of this research, C. A. Micchelli brought to our attention a preprint of K. Glashoff, which has since appeared as [5]. In this elegant paper, Glashoff proves the above theorem, except for the full characterization. He proves existence, uniqueness, and also that the above h^* is either equal to u or l with a finite number of jumps. Glashoff's proof is somewhat differnt than ours in that we also prove the full characterization which will be used in the next sections. His motivation in considering such a problem stems from a finite bang–bang principle which has applications in optimal control theory.

We divide the proof of the theorem into a series of lemmas.

LEMMA 3.1. *To each* $g \in C[0,1]$, *there exists a best approximant* $f^* \in \mathscr{M}$.

Proof. The set $\{h\colon l(y) \leqslant h(y) \leqslant u(y)\}$ is a bounded set and $K\colon h(\cdot) \to \int_0^1 K(\cdot, y)\, h(y)\, dy$ is a compact linear operator. Hence $\mathscr{M}$ is compact, and a best approximation necessarily exists.

LEMMA 3.2. *Let* f^* *be a best approximant to* $g \in C[0,1] \setminus \mathscr{M}$ *from* $\mathscr{M}$. *Then* $(g-f^*)(x)$ *equioscillates on at most a finite number of points.*

Proof. Since $g - f^*$ is uniformly continuous on the compact interval $[0,1]$, there is a $\delta > 0$ such that $|(g-f^*)(x) - (g-f^*)(y)| < 2\|g-f^*\|_\infty$, whenever $|x-y| < \delta$. Thus the distance between consecutive points, of a set on which $g-f^*$ equioscillates, is at least δ. This proves the finiteness of the set.

For any Lebesgue measurable set E of $[0,1]$, $m(E)$ shall denote its Lebesgue measure.

LEMMA 3.3. *Let*

$$E = \{y: h^*(y) = u(y) \text{ or } h^*(y) = l(y)\}.$$

Then $m(E) = 1$.

Proof. Assume $m(E) < 1$. Let

$$F_n = \left\{ y: l(y) + \frac{1}{n} \leqslant h^*(y) \leqslant u(y) - \frac{1}{n} \right\}.$$

Since $F_{n+1}^c \subseteq F_n^c$, and $\bigcap_{n=1}^{\infty} F_n^c = E$, it follows that $\lim_{n\to\infty} m(F_n^c) = m(E) < 1$. Thus there exists an n for which $m(F_n) > 0$. Assume that $(g - f^*)(x)$ equioscillates on exactly $N+1$ points ($N \geqslant 0$). Since $m(F_n) > 0$, there exist points $0 = \beta_0 < \beta_1 < \cdots < \beta_N < \beta_{N+1} = 1$, for which

$$m((\beta_{i-1}, \beta_i) \cap F_n) > 0, \qquad i = 1,\dots, N+1.$$

Let $G_i = (\beta_{i-1}, \beta_i) \cap F_n$, and set

$$v_i(x) = \int_{G_i} K(x, y)\, dy, \qquad i = 1,\dots, N+1.$$

Since $K(x, y)$ is an STP kernel, the set $\{v_1,\dots, v_{N+1}\}$ spans a Descartes system on $[0, 1]$. By the well known characterization theorem, the zero function is not the best approximation to $g - f^*$ from this subspace. For some $v = \sum_{i=1}^{N+1} a_i v_i$, we have $\| g - f^* - v \|_\infty < \| g - f^* \|_\infty$. Choose $\lambda \in (0, 1]$ such that $|\lambda a_i| < 1/n$, $i = 1,\dots, N+1$. Then $f^* + \lambda v \in \mathscr{M}$, and $\| g - f^* - \lambda v \|_\infty < \| g - f^* \|_\infty$. This contradicts the optimality of f^*. Thus $m(E) = 1$.

As in Definition 2.4, given measurable sets I and J of $[0, 1]$, we say that $I < J$ if $x < y$ for all $x \in I$ and $y \in J$.

LEMMA 3.4. *Assume that there exist* $M+1$ *sets* $I_1 < \cdots < I_{M+1}$ *with* $m(I_j) > 0$, $j = 1,\dots, M+1$, *for which a.e.*

$$\begin{aligned} h^*(y) &= u(y), && \text{on } I_k,\ k \text{ odd}, \\ &= l(y), && \text{on } I_k,\ k \text{ even}, \end{aligned}$$

or

$$\begin{aligned} h^*(y) &= u(y), && \text{on } I_k,\ k \text{ even}, \\ &= l(y), && \text{on } I_k,\ k \text{ odd}. \end{aligned}$$

Then $g - f^*$ *equioscillates on at least* $M+1$ *points, starting positively, or negatively, respectively.*

Proof. Assume that the former case holds, i.e., $h^*(y) = u(y)$ on I_1, etc.... We must show that $g - f^*$ equioscillates on at least $M + 1$ points, starting positively. Suppose that $g - f^*$ equioscillates on exactly $N + 1$ points. If $N > M$, then we can, by deleting suitable points of equioscillation, obtain our result. Set

$$v_i(x) = \int_{I_i} K(x, y)\, dy, \qquad i = 1, \dots, M + 1.$$

Case I. $g - f^*$equioscillates on $N + 1$ points, $N \leqslant M$, starting negatively.

For some $v(x) = \sum_{i=1}^{N+1} a_i v_i(x)$, we have $\| g - f^* - v \|_\infty < \| g - f^* \|_\infty$ since $\{v_i\}_{i=1}^{N+1}$ spans a Descartes system on $[0, 1]$. Since $g - f^*$ equioscillates on $N + 1$ points, it follows that $v = (g - f^*) - (g - f^* - v)$ has at least N sign changes. Furthermore $g - f^*$ starts negatively and therefore v starts negatively. Thus by Proposition 2.2, $a_i(-1)^i > 0$, $i = 1, \dots, N + 1$. Now for all $\lambda \in (0, 1]$,

$$\| g - f^* - \lambda v \|_\infty < \| g - f^* \|_\infty$$

and for $\lambda > 0$, sufficiently small, $f^* + \lambda v \in \mathscr{M}$. This contradicts the optimality of f^* so that case I cannot occur.

Case II. $g - f^*$ equioscillates on $N + 1$ points, $N \leqslant M - 1$, starting positively.

For some $v(x) = \sum_{i=2}^{N+2} b_i v_i(x)$, we have $\| g - f^* - v \|_\infty < \| g - f^* \|_\infty$. As in case I, it follows that v has exactly N sign changes, and since $g - f^*$ starts positively, $b_i(-1)^i > 0$, $i = 2, \dots, N + 2$. A contradiction now follows as in case I.

Since neither case I nor case II may obtain, the lemma is proved.

Because $g - f^*$ must equioscillate on at least $M + 1$ points and since, by Lemma 3.2, M must be finite, it therefore follows that there exist $(\xi_1^*, \dots, \xi_M^*)$ as in the statement of the theorem. We have thus proven that every best approximation f^* from $\mathscr{M}$ to $g \in C[0, 1] \setminus \mathscr{M}$ must satisfy statements (i) and (ii) of the theorem. From this fact, it is a simple matter to prove the uniqueness of the best approximation. This may be done via a convexity argument. However, we wish to prove more, namely, that any function f^* satisfying statements (i) and (ii) of the theorem is necessarily the unique best approximant.

LEMMA 3.5. *Let* $g \in C[0, 1] \setminus \mathscr{M}$ *and let* $f^* \in \mathscr{M}$ *satisfy statements* (i) *and* (ii) *of Theorem* 3.1. *Then* f^* *is the unique best approximant to* g *from* $\mathscr{M}$.

Proof. Let

$$f^*(x) = \int_0^1 K(x, y)\, h^*(y)\, dy$$

satisfy statements (i) and (ii) of the theorem and assume, without loss of generality, that $\varepsilon = 1$. Let

$$f(x) = \int_0^1 K(x, y)\, h(y)\, dy \in \mathscr{M}$$

satisfy $\| g - f \|_\infty \leqslant \| g - f^* \|_\infty, f \not\equiv f^*$. Since $(g - f^*)(x)$ equioscillates on at least $M + 1$ points, starting positively, then from Proposition 2.3,

$$S^+((g - f^*) - (g - f)) = S^+(f - f^*) \geqslant M.$$

Furthermore, if equality holds, then the orientation of the generalized sign of $(f - f^*)(x)$ starts positively. Now,

$$(f - f^*)(x) = \int_0^1 K(x, y)(h - h^*)(y)\, dy.$$

Thus from Theorem 2.1(i),

$$S^+(f - f^*) \leqslant S^-(h - h^*).$$

Since $l(y) \leqslant h(y) \leqslant u(y)$ for all $y \in [0, 1]$, $S^-(h - h^*) \leqslant M$. Thus,

$$S^+(f - f^*) = S^-(h - h^*) = M.$$

Because $\varepsilon = 1$, the orientation of the generalized sign of $(h - h^*)(y)$ starts negatively. This fact contradicts Theorem 2.1(ii), proving the lemma.

LEMMA 3.6. *Let $g \in C[0, 1]$ and assume that there exists an $f^* \in \mathscr{M}$ satisfying conditions* (i) *and* (ii) *of the statement of the theorem, for which $\| g - f^* \|_\infty > 0$. Then $g \notin \mathscr{M}$.*

Proof. Let

$$f^*(x) = \int_0^1 K(x, y)\, h^*(y)\, dy,$$

where $h^*(y)$ has M knots. Assume $g \in \mathscr{M}$. Thus

$$g(x) = \int_0^1 K(x, y)\, h(y)\, dy$$

for some h satisfying $l(y) \leqslant h(y) \leqslant u(y)$ for all $y \in [0, 1]$. Since $(g-f^*)(x)$ equioscillates on at least $M+1$ points,

$$M \leqslant S^+(g-f^*) \leqslant S^-(h-h^*)$$

by Theorem 2.1(i). From the form of h^*, $S^-(h-h^*) \leqslant M$. Thus $M = S^+(g-f^*) = S^-(h-h^*)$.

A contradiction now ensues from part (ii) of Theorem 2.1 as in the proof of Lemma 3.5.

This lemma completes the proof of Theorem 3.1.

It is generally impossible to determine the number M. In fact as g approaches the set $\mathscr{M}$, this number may well blow up to ∞. If, however,

$$g(x) = \int_0^1 K(x, y)\, \tilde{h}(y)\, dy,$$

where $\tilde{h}(y) \geqslant u(y)$ for all $y \in [0, 1]$, then $M=0$, and $h^*(y) = u(y)$. Similarly if $\tilde{h}(y) \leqslant l(y)$, then $M=0$ and $h^*(y) = l(y)$.

At times, bounds on M are available. For example, if $\tilde{h}(y) \notin [l(y), u(y)]$ for $y \in [0, 1]$, and if $S^-(\tilde{h}-h) = r$ for every h for which $l \leqslant h \leqslant u$, then $M \leqslant r$. The proof of this fact is an immediate consequence of Theorem 2.1.

Since M is in general an unknown quantity, it is natural to ask for some sort of algorithmic method by which we may deduce its value. In this next section we attempt to provide such a method.

4. Approximations by Generalized Perfect Splines

In this section we consider best approximations from functions of the form $\int_0^1 K(x, y)\, h(y)\, dy$, where $|h(y)| = 1$, a.e., the number of sign changes of h is at most n, and h has a fixed orientation. Thus we here assume that $u(y) = 1$ and $l(y) = -1$. (It should, however, be noted that all results obtained in this section apply *mutatis mutandis* to the case of general $u, l \in C[0, 1]$, with $l(y) < u(y)$, all y.) Hence in this section we have

$$\mathscr{M} = \left\{ f(x) : f(x) = \int_0^1 K(x, y)\, h(y)\, dy, \|h\|_\infty \leqslant 1 \right\}.$$

For each nonnegative integer n, let Λ_n be as defined in Section 2. $\bar{\Lambda}_n$ shall denote the closure of the simplex Λ_n, as given by

$$\bar{\Lambda}_n = \{\boldsymbol{\xi} : \boldsymbol{\xi} = (\xi_1, ..., \xi_n), 0 = \xi_0 \leqslant \xi_1 \leqslant \cdots \leqslant \xi_n \leqslant \xi_{n+1} = 1\}.$$

To each $\xi \in \bar{\Lambda}_n$, set $f_n^-(x;\xi) = \sum_{j=0}^{n}(-1)^{j+1}\int_{\xi_j}^{\xi_{j+1}} K(x,y)\,dy$, and

$$f_n^+(x;\xi) = \sum_{j=0}^{n} (-1)^j \int_{\xi_j}^{\xi_{j+1}} K(x,y)\,dy = -f_n^-(x;\xi).$$

(The signs $+$ and $-$ on the function f_n denote the sign of $h(y)$ on the first interval (ξ_0, ξ_1).) Now, let

$$\mathscr{P}_n^{\pm} = \{f_n^{\pm}(x;\xi) : \xi \in \bar{\Lambda}_n\}.$$

The interior of $\mathscr{P}_n^-$ is to be regarded as the set $\{f_n^-(x;\xi) : \xi \in \Lambda_n\}$, with a similar statement holding for $\mathscr{P}_n^+$. The sets $\mathscr{P}_n^-$ and $\mathscr{P}_n^+$ are closed and compact and

$$\mathscr{P}_n^- \cap \mathscr{P}_n^+ = \mathscr{P}_{n-1}^- \cup \mathscr{P}_{n-1}^+.$$

For notational ease, for given $\xi \in \bar{\Lambda}_n$, we shall also let

$$h_\xi(y) = (-1)^i, \qquad \xi_{i-1} < y < \xi_i,\ i = 1,\dots, n+1.$$

We shall study the problem of approximating, in $L^\infty[0,1]$, functions $g \in C[0,1]$ by elements of $\mathscr{P}_n^-$ and of $\mathscr{P}_n^+$.

THEOREM 4.1. *Let $g \in C[0,1]$ and assume that $K(x,y)$ is an* STP *kernel.*

(a) *There exists a unique best approximation $f^*(x;\xi^*) \in \mathscr{P}_n^-$ to g from $\mathscr{P}_n^-$. f^* is either the best approximation to g from $\mathscr{M}$, or $\xi^* \in \Lambda_n$ and f^* is uniquely characterized by the property that $g - f^*$ equioscillates on exactly $n+1$ points, starting positively.*

(b) *There exists a unique best approximation $f^*(x;\xi^*) \in \mathscr{P}_n^+$ to g from $\mathscr{P}_n^+$. f^* is either the best approximation to g from $\mathscr{M}$, or $\xi^* \in \Lambda_n$ and f^* is uniquely characterized by the property that $g - f^*$ equioscillates on exactly $n+1$ points, starting negatively.*

Remark. Note that the characterization of the best approximation given above is very similar to that given in Theorem 3.1. The important difference is that the orientation between the points of equioscillation of $(g - f^*)(x)$ and the sign of $h^*(y)$ is reversed if f^* is not the best approximation from $\mathscr{M}$.

For convenience, the proof of this theorem is also divided into a series of lemmas.

LEMMA 4.1. *To each $g \in C[0,1]$, there exists a best approximant from $\mathscr{P}_n^-$ and from $\mathscr{P}_n^+$.*

Proof. The lemma follows from the fact that $\mathscr{P}_n^-$ and $\mathscr{P}_n^+$ are compact.

Because of the symmetry between $\mathscr{P}_n^-$ and $\mathscr{P}_n^+$, we shall only prove the theorem for $\mathscr{P}_n^-$. The proof of the theorem for $\mathscr{P}_n^+$ is entirely analogous.

Before proving the existence of a function $f^*(x; \xi^*) \in \mathscr{P}_n^-$ with $\xi^* \in \Lambda_n$, for which $g - f^*$ equioscillates on $n+1$ points, starting positively, under the assumption

$$\min\{\| g - f\|_\infty : f \in \mathscr{P}_n^-\} > \min\{\| g - f\|_\infty : f \in \mathscr{M}\}, \tag{1}$$

we shall first prove that any such function is necessarily the unique best approximation.

LEMMA 4.2. *Assume that there exists an $f^* \in \mathscr{P}_n^-$ satisfying the conditions given in Theorem* 4.1(a). *Then f^* is the unique best approximation to g from $\mathscr{P}_n^-$.*

Proof. If (1) does not hold, then the uniqueness of the best approximation from $\mathscr{P}_n^-$ is a consequence of the uniqueness proven in Theorem 3.1. Assume that (1) holds, and $g - f^*$ equioscillates on $n+1$ points. (If the equioscillations start negatively, then (1) does not hold by Theorem 3.1.)

Assume $f_n^-(x; \xi) \in \mathscr{P}_n^-$, and

$$\| g(\cdot) - f_n^-(\cdot\,; \xi)\|_\infty \leqslant \| g(\cdot) - f^*(\cdot\,; \xi^*)\|_\infty .$$

Since $g - f^*$ equioscillates on $n+1$ points

$$S^+((g - f^*) - (g - f_n^-)) \geqslant n.$$

Now

$$S^+((g - f^*) - (g - f_n^-)) = S^+(f_n^- - f^*),$$

and

$$f_n^-(x; \xi) - f^*(x; \xi^*) = \int_0^1 K(x, y)[h_\xi(y) - h_{\xi^*}(y)]\, dy.$$

Thus by Theorem 2.1,

$$n \leqslant S^-(h_\xi - h_{\xi^*}).$$

However, $\xi^* \in \Lambda_n$ and $\xi \in \bar{\Lambda}_n$, and thus by Proposition 2.4,

$$S^-(h_\xi - h_{\xi^*}) \leqslant n - 1.$$

This contradiction proves the lemma.

It thus remains to prove the existence of an $f^* \in \mathscr{P}_n^-$ satisfying the conditions of the theorem under assumption (1). Our proof is based on an induction argument and repeated use of Theorem 3.1. It is therefore necessary that we first prove the case $n = 0$.

LEMMA 4.3. *Theorem* 4.1 *holds for* $n = 0$.

Proof. We assume that (1) holds for $n = 0$. The set $\mathscr{P}_0^-$ is simply the function $f_0^-(x) = -\int_0^1 K(x, y)\, dy$. Since (1) holds, $\| g - f_0^- \|_\infty > 0$, and there therefore exists a point x^* such that $| g(x^*) - f_0^-(x^*)| = \| g - f_0^- \|_\infty$. If for some $x^* \in [0, 1]$, $(g - f_0^-)(x^*) = -\| g - f_0^- \|_\infty$, then from Theorem 3.1, we contradict (1). Thus $(g - f_0^-)(x^*) = \| g - f_0^- \|_\infty$ and at no point does $g - f_0^-$ take on the value $-\| g - f_0^- \|_\infty$.

LEMMA 4.4. *Theorem* 4.1 *holds for all* n.

Proof. Let $n \geqslant 1$, and assume that the results of the theorem hold for $k \leqslant n - 1$. Let $f^* \in \mathscr{P}_n^-$ denote a best approximation to g from $\mathscr{P}_n^-$ which is not the best approximation from $\mathscr{M}$. We distinguish three cases.

Case I. $\boldsymbol{\xi}^* \in A_n$. Assume that $g - f^*$ does not equioscillate on a set of $n + 1$ points. Then there exists a

$$v(x) = \sum_{i=1}^{n} a_i K(x, \xi_i^*)$$

such that $\| g - f^* - v\|_\infty < \| g - f^* \|_\infty$. Thus for $\lambda \in (0, 1)$,

$$\| g - f^* - \lambda v\|_\infty \leqslant \| g - f^* \|_\infty - \lambda c,$$

where

$$c = \| g - f^* \|_\infty - \| g - f^* - v \|_\infty > 0.$$

Let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$. Assume that the δ_i are sufficiently small so that $\max\{\xi_i^*, \xi_i^* + \delta_i\} < \min\{\xi_{i+1}^*, \xi_{i+1}^* + \delta_{i+1}\}$, $i = 0, 1, \ldots, n$, where $\xi_0^* = \delta_0 = \delta_{n+1} = 0$, and $\xi_{n+1}^* = 1$. Then,

$$\begin{aligned} f_n(x; \boldsymbol{\xi}^* + \boldsymbol{\delta}) - f^*(x; \boldsymbol{\xi}^*) &= \sum_{j=1}^{n} 2(-1)^j \int_{\xi_j^*}^{\xi_j^* + \delta_j} K(x, y)\, dy \\ &= \sum_{j=1}^{n} 2(-1)^j\, \delta_j K(x, \xi_j^*) + o(\boldsymbol{\delta}). \end{aligned}$$

Let $\delta_j = (\frac{1}{2})(-1)^j a_j \lambda$ for $\lambda > 0$, small. Then,

$$f_n(x; \boldsymbol{\xi}^* + \boldsymbol{\delta}) - f^*(x; \boldsymbol{\xi}^*) - \lambda v = o(\lambda).$$

Thus,

$$\|g - f_n^-(\cdot\,;\boldsymbol{\xi}^* + \boldsymbol{\delta})\|_\infty \leqslant \|g - f^* - \lambda v\|_\infty + \|f_n^-(\cdot\,;\xi^* + \boldsymbol{\delta}) - f^* - \lambda v\|_\infty$$
$$\leqslant \|g - f^*\|_\infty - \lambda c + o(\lambda),$$

which, since $c > 0$, contradicts the optimality of f^*.

If there are more than $n+1$ points of equioscillation or if the equioscillations start negatively, then f^* is optimal in $\mathscr{M}$ by Theorem 3.1. So the result holds.

Case II. $f^* \in \mathscr{P}_{n-1}^-$. Since f^* is the best approximation to g from $\mathscr{P}_{n-1}^-$ (and not from $\mathscr{M}$) it follows from the induction hypothesis that h^* has exactly $n-1$ sign changes at $\xi_1^* < \cdots < \xi_{n-1}^*$ and that $g - f^*$ equioscillates at exactly n points, starting positively. Put $\xi_n^* = 1$. Then there exists a function

$$v(x) = \sum_{i=1}^{n} a_i K(x, \xi_i^*)$$

such that $\|g - f^* - v\|_\infty < \|g - f^*\|_\infty$. We wish to apply the analysis of case I. To do this we must choose $\delta_n < 0$, so that $\xi_n^* + \delta_n \in [0, 1]$. It therefore suffices to show that $a_n(-1)^{n-1} > 0$. Since $g - f^*$ has n points of equioscillation, starting positively, it follows, by the reasoning given in the proof of Lemma 3.4, that $v = (g - f^*) - (g - f^* - v)$ has exactly $n-1$ sign changes, starting positively. Thus, by Proposition 2.2, $a_i(-1)^{i+1} > 0$, $i = 1, \ldots, n$, and the result follows.

Case III. $f^* \in \mathscr{P}_{n-1}^+$. This case is totally analogous to case II except that we add the point $\xi_0^* = 0$.

Since the boundary of $\mathscr{P}_n^-$ is contained in $\mathscr{P}_{n-1}^- \cup \mathscr{P}_{n-1}^+$, the proof of Theorem 4.1 is complete.

Let $\mathscr{P}_n^-$ and $\mathscr{P}_n^+$ be as defined above. We are now interested in the problem

$$\min\{\|g - cf\|_\infty : f \in \mathscr{P}_n^-,\ c \geqslant 0\}$$

for $g \in C[0, 1]$. Our approximating class is now the set of functions f of the form

$$f(x) = c \sum_{j=0}^{n} (-1)^{j+1} \int_{\xi_j}^{\xi_{j+1}} K(x, y)\, dy,$$

where $c \geqslant 0$, arbitrary, and $\xi_0 = 0 \leqslant \xi_1 \leqslant \cdots \leqslant \xi_n \leqslant \xi_{n+1} = 1$. The difference between this problem and the previous one is that we allow c to vary over $\mathbb{R}^+$. We, of course, also ask this same question with $\mathscr{P}_n^-$ replaced by $\mathscr{P}_n^+$.

THEOREM 4.2. *Let* $g \in C[0, 1]$ *and* $K(x, y)$ *be an* STP *kernel. Set*

$$c^* = \sup\{c\colon c \geqslant 0, \min\{\| g - cf \|_\infty : f \in \mathcal{P}_n\} = \min\{\| g - cf \|_\infty : f \in \mathcal{M}\}\}.$$

Then,

(i) $0 \leqslant c^* < \infty$,

(ii) $\min\{\| g - cf \|_\infty, f \in \mathcal{P}_n^-, c \geqslant 0\} = \min\{\| g - c^*f \|_\infty : f \in \mathcal{P}_n\}$,

(iii) *For* $c \leqslant c^*$, $\min\{\| g - cf \|_\infty : f \in \mathcal{P}_n\}$ *is a strictly decreasing function of* c, *and* $\min\{\| g - cf \|_\infty : f \in \mathcal{P}_n\} = \min\{\| g - cf \|_\infty : f \in \mathcal{M}\}$.

(iv) *For* $c > c^*$, $\min\{\| g - cf \|_\infty : f \in \mathcal{P}_n^-\}$ *is a strictly increasing function of* c, *and* $\min\{\| g - cf \|_\infty : f \in \mathcal{P}_n^-\} > \min\{\| g - cf \|_\infty : f \in \mathcal{M}\}$.

(v) *Let* $f^*(x) = \int_0^1 K(x, y)\, h^*(y)\, dy$ *be the unique function in* $\mathcal{P}_n^-$ *for which*

$$\min\{\| g - c^*f \|_\infty : f \in \mathcal{P}_n\} = \| g - c^*f^* \|_\infty .$$

Then, if $g \not\equiv c^*f^*$,

(a) $S^-(h^*) = k \leqslant n$,

(b) $(g - c^*f^*)(x)$ *exhibits at least* $l, l \geqslant n + 1$, *points of equioscillation,*

(c) $l \geqslant k + 2$.

The proof of this theorem we again divide into a series of lemmas. For ease of notation, let $f_c \in \mathcal{P}_n^-$ satisfy

$$\min\{\| g - cf \|_\infty : f \in \mathcal{P}_n^-\} = \| g - cf_c \|_\infty .$$

LEMMA 4.5. *Let* $c > 0$ *and*

$$\min\{\| g - cf \|_\infty : f \in \mathcal{M}\} = \| g - cf_c \|_\infty .$$

Then for all $d \in [0, c)$,

$$\min\{\| g - df \|_\infty : f \in \mathcal{M}\} = \| g - df_d \|_\infty .$$

Proof. The lemma is trivial for $d = 0$. We therefore take $d \in (0, c)$. First assume that $g \neq cf_c$. Thus $g \notin c\mathcal{M}$ and since $d < c$, $g \notin d\mathcal{M}$. From the uniqueness and characterization of Theorem 3.1, we have that

$$\min\{\| g - df \|_\infty : f \in \mathcal{M}\} > \min\{\| g - cf \|_\infty : f \in \mathcal{M}\}.$$

Let $f_c(x) = \int_0^1 K(x, y)\, h_c(y)\, dy$. Thus $S^-(h_c) = k(c) \leqslant n$, and if $k(c) = n$, then the sign pattern of $h_c(y)$ starts negatively. Let $f_d^* \in \mathcal{M}$, $f_d^*(x) =$

$\int_0^1 K(x, y)\, h_d^*(y)\, dy$ denote the unique function in $\mathscr{M}$ for which $\min\{\| g - df\|_\infty : f \in \mathscr{M}\} = \| g - df_d^*\|_\infty$. We shall prove that $f_d^* = f_d$.

Set $S^-(h_d^*) = k(d)$. Thus $(g - df_d^*)(x)$ exhibits, by Theorem 3.1, at least $k(d)+1$ points of equioscillation. Since $\| g - df_d^*\|_\infty > \| g - cf_c\|_\infty$, the function $(g - df_d^*)(x) - (g - cf_c)(x) = (cf_c - df_d^*)(x)$ exhibits at least $k(d)$ sign changes on $[0, 1]$ (Proposition 2.3). Because,

$$(cf_c - df_d^*)(x) = \int_0^1 K(x, y)(ch_c - dh_d^*)(y)\, dy,$$

it follows from Theorem 2.1 that

$$k(d) \leqslant S^+(cf_c - df_d^*) \leqslant S^-(ch_c - dh_d^*).$$

Since $c > d$, $S^-(ch_c - dh_d^*) \leqslant k(c) \leqslant n$. Thus $k(d) \leqslant k(c) \leqslant n$, which implies that $f_d^* \in \mathscr{P}_n^- \cup \mathscr{P}_n^+$. If $k(d)(=k(c)) = n$, then again by Theorem 2.1, the orientation of the sign patterns of $(g - df_d^*)(x) - (g - cf_c)(x)$ and $ch_c(y) - dh_d^*(y)$ must agree. Since the latter function has n sign changes, starting negatively, $(g - df_d^*)(x)$ has exactly $n+1$ points of equioscillation, starting negatively. Theorem 3.1 now implies that the sign pattern of $h_d^*(y)$ must start negatively since df_d^* is the unique best approximation to g from $d\mathscr{M}$ and $S^-(h_d^*) = n$. Thus $f_d^* \in \mathscr{P}_n^-$ and the lemma is proved in this case.

Assume now that $g \equiv cf_c$. Since $d < c$, $g \notin d\mathscr{M}$, and $\min\{\| g - df\|_\infty : f \in \mathscr{M}\} = \| g - df_d^*\|_\infty > 0$, where $f_d^*(x)$ is as defined above. Once again $k(d) \leqslant S^+(cf_c - df_d^*) \leqslant S^-(ch_c - dh_d^*) \leqslant k(c) \leqslant n$. We prove that $f_d^* \in \mathscr{P}_n^-$ by the same reasoning as that given above. This proves the lemma.

The above lemma implies that the set of $c \geqslant 0$ for which

$$\min\{\| g - cf\|_\infty : f \in \mathscr{P}_n^-\} = \min\{\| g - cf\|_\infty : f \in \mathscr{M}\},$$

in an interval (closed by continuity considerations), whose left hand endpoint is zero. Note that in the above lemma we proved that if $0 \leqslant d < c \leqslant c^*$, then $k(d) \leqslant k(c)$, an interesting fact in and of itself. We also proved

LEMMA 4.6. *For $c \leqslant c^*$, $c < \infty$, $\min\{\| g - cf\|_\infty : f \in \mathscr{P}_n^-\}$ is a strictly decreasing function of c, and*

$$\min\{\| g - cf\|_\infty : f \in \mathscr{P}_n^-\} = \min\{\| g - cf\|_\infty : f \in \mathscr{M}\}.$$

LEMMA 4.7. $c^* < \infty$.

Proof. $\min\{\| g - cf\|_\infty : f \in \mathscr{M}\} < \| g\|_\infty < \infty$ for all $c \geqslant 0$, while it is easily shown that $\min\{\| f\|_\infty : f \in \mathscr{P}_n^-\} > 0$. This latter fact assures us that

$$\lim_{c \uparrow \infty} \min\{\| g - cf \|_\infty : f \in \mathscr{P}_n\} = \infty.$$

Thus we cannot have

$$\min\{\| g - cf \|_\infty : f \in \mathscr{P}_n\} = \min\{\| g - cf \|_\infty : f \in \mathscr{M}\}$$

for all $c \geqslant 0$.

LEMMA 4.8. *Let $g \in C[0, 1]$ and $c^* < c < d$. Then*

$$\| g - cf_c \|_\infty < \| g - df_d \|_\infty .$$

Proof. Set $f_c(x) = \int_0^1 K(x, y)\, h_c(y)\, dy$ and $f_d(x) = \int_0^1 K(x, y)\, h_d(y)\, dy$. From Theorem 4.1, $S^-(h_c) = S^-(h_d) = n$, both $h_c(y)$ and $h_d(y)$ start negatively, while $(g - cf_c)(x)$ and $(g - df_d)(x)$ each exhibit exactly $n + 1$ points of equioscillation, starting positively.

Thus, by Theorem 2.1,

$$n \leqslant S^+((g - cf_c) - (g - df_d)) = S^+(df_d - cf_c) \leqslant S^-(dh_d - ch_c).$$

Since $d > c$, $S^-(dh_d - ch_c) = n$, and the sign pattern starts negatively. Hence $n = S^+((g - cf_c) - (g - df_d))$ and the sign pattern thereof must also start negatively. This is only possible (Proposition 2.3) if

$$\| g - cf_c \|_\infty < \| g - df_d \|_\infty .$$

It remains to prove statement (v) of the theorem. Let $f^*(x) = \int_0^1 K(x, y)\, h^*(y)\, dy$ denote the unique function in $\mathscr{P}_n$ for which $\min\{\| g - c^*f \|_\infty : f \in \mathscr{P}_n^-\} = \| g - c^* f^* \|_\infty$. Assume $g \not\equiv c^* f^*$. Since $f^* \in \mathscr{P}_n$, it is certainly true that $S^-(h^*) = k \leqslant n$, and if $S^-(h^*) = n$, then since $\min\{\|g - c^*f\|_\infty : f \in \mathscr{P}_n^-\} = \min\{\|g - c^*f\|_\infty : f \in \mathscr{M}\}$, $(g - c^*f^*)(x)$ exhibits at least $n + 1$ points of equioscillation, starting negatively. Let l denote the number of points of equioscillation of $(g - c^*f^*)(x)$.

LEMMA 4.9. $l \geqslant \max\{n + 1, k + 2\}$.

Proof. For $c > c^*$, $(g - cf_c)(x)$ exhibits exactly $n + 1$ points of equioscillation, starting positively. Because $\lim_{c \downarrow c^*} f_c(x) = f^*(x)$, and $g \not\equiv c^* f^*$, $(g - c^* f^*)(x)$ exhibits at least $n + 1$ points of equioscillation. Thus $l \geqslant n + 1$. Furthermore if $l = n + 1$, then these points of equioscillation start positively. However, if $k = n$ (the only case we must yet consider) then $l \geqslant n + 2$. Otherwise we contradict the orientation of the sign pattern of the equioscillations of $(g - c^* f^*)(x)$.

This proves the theorem.

Let $c^- \geqslant 0$, $f^- \in \mathscr{P}_n^-$ be such that $\min\{\| g - cf \|_\infty : c \geqslant 0,\ f \in \mathscr{P}_n^-\} = \| g - c^- f^- \|_\infty$ and $c^+ \geqslant 0$, $f^+ \in \mathscr{P}_n^+$ be such that $\min\{\| g - cf \|_\infty : c \geqslant 0,\ f \in \mathscr{P}_n^+\} = \| g - c^+ f^+ \|_\infty$. To obtain additional information on the number of equioscillations of $(g - c^- f^-)(x)$ and $(g - c^+ f^+)(x)$, it is necessary to discuss a more general minimum problem.

Set $\mathscr{P}_n = \mathscr{P}_n^- \cup \mathscr{P}_n^+$. We shall consider the problem

$$\min\{\| g - cf \|_\infty : f \in \mathscr{P}_n,\ c \geqslant 0\}.$$

(Note that from Theorem 4.2, the functions $c^+ f^+$ and $c^- f^-$ are the only possible solutions to this problem.)

THEOREM 4.3. *There exists a unique solution* $\hat{c}\hat{f}$, $\hat{c} \geqslant 0$, $\hat{f} \in \mathscr{P}_n$, *to the problem*

$$\min\{\| g - cf \|_\infty : f \in \mathscr{P}_n,\ c \geqslant 0\}.$$

If $g \not\equiv \hat{c}\hat{f}$, *then* $\hat{c}\hat{f}$ *is uniquely characterized by the fact that* $(g - \hat{c}\hat{f})(x)$ *equioscillates on at least* $n + 2$ *points.*

An immediate consequence of the theorem is the following corollary.

COROLLARY. $\| g - c^+ f^+ \|_\infty \neq \| g - c^- f^- \|_\infty$ *unless* $c^+ = c^- = 0$. *Furthermore if, say,* $\| g - c^+ f^+ \|_\infty < \| g - c^- f^- \|_\infty$, *then* $(g - c^+ f^+)(x)$ *equioscillates on at least* $n + 2$ *points and* $(g - c^- f^-)(x)$ *equioscillates on exactly* $n + 1$ *points. (If* $\| g - c^- f^- \|_\infty < \| g - c^+ f^+ \|_\infty$, *then the reverse holds.)*

Various methods exist for proving Theorem 4.3. We shall not present a proof here. Suffice it to say that the set $\{c\mathscr{P}_n : c \geqslant 0\}$ is a class of varisolvent functions all of degree $n + 1$. We can therefore apply Theorem 7.3 of Rice [11].

For fixed $g \in C[0, 1]$, the values c^+ and c^-, as defined above, depended on the choice of n. Let us indicate the dependence on $\mathscr{P}_n^+$ and $\mathscr{P}_n^-$ by setting $c^+(n) = c^+$ and $c^-(n) = c^-$. We have no way of determining which of $c^+(n)$ and $c^-(n)$ is smaller except by calculating $\| g - c^+(n) f^+ \|_\infty$ and $\| g - c^-(n) f^- \|_\infty$. However, an immediate application of the definitions and the fact that $\mathscr{P}_n^- \subseteq \mathscr{P}_{n+1}^{\pm} \subseteq \mathscr{M}$ (and the analogous statement for $\mathscr{P}_n^+$) does give this next result.

PROPOSITION 4.4. *For fixed* $g \in C[0, 1]$,

$$c^+(n), c^-(n) \leqslant c^+(n + 1), c^-(n + 1).$$

5. Generalizations to Nonnegative Measures

Let $\mathscr{B}$ denote the set of finite Borel measures on $[0, 1]$ and

$$\mathscr{M}_c = \left\{ f : f(x) = \int_0^1 K(x, y)\, d\mu(y), \mu \in \mathscr{B}, \mu \geqslant 0 \right\},$$

where by $\mu \geqslant 0$ we mean that μ is a nonnegative measure. We are interested in the problem of approximating a function $g \in C[0, 1]$ by functions in $\mathscr{M}_c$. The following result and its proof parallel those of Theorem 3.1.

THEOREM 5.1. *Let $g \in C[0, 1] \setminus \mathscr{M}_c$ and $K(x, y)$ be an* STP *kernel. Then*

$$\min\{\| g - f \|_c : f \in \mathscr{M}_c\}$$

is uniquely attained by $f^* \in \mathscr{M}_c$ of the form

$$f^*(x) = \sum_{i=1}^{M} a_i^* K(x, \xi_i^*), \tag{2}$$

where M is a nonnegative integer, $a_i^* > 0$, $i = 1,..., M$, and $0 \leqslant \xi_1^* < \cdots < \xi_M^* \leqslant 1$. Furthermore, $f^*(x)$ is uniquely characterized as follows.

(I) *If $M = 0$, i.e., $f^*(x) \equiv 0$, then there exists an $x^* \in [0, 1]$ for which $g(x^*) = -\| g \|_c$.*

(II) *If $M \geqslant 1$, and*

(i) *$0 < \xi_1^* < \cdots < \xi_M^* < 1$, then $(g - f^*)(x)$ equioscillates on at least $2M + 1$ points, starting negatively.*

(ii) *$0 = \xi_1^* < \cdots < \xi_M^* < 1$, then $(g - f^*)(x)$ equioscillates on at least $2M$ points, starting positively.*

(iii) *$0 < \xi_1^* < \cdots < \xi_M^* = 1$, then $(g - f^*)(x)$ equioscillates on at least $2M$ points, starting negatively.*

(iv) *$0 = \xi_1^* < \cdots < \xi_M^* = 1$, then $(g - f^*)(x)$ equioscillates on at least $2M - 1$ points, starting positively.*

Theorem 5.1 is similar to Theorem 3.1. However, before entering into the analysis thereof, let us note that such a theorem holds essentially as a limiting case of Theorem 3.1. What we shall now provide is a sketch, rather than a proof.

In Theorem 3.1, let $l(y)=0$ and $u(y)=A$, where A is some large constant which we shall eventually send to infinity. Set

$$\mathscr{M}_A=\left\{f(x):f(x)=\int_0^1 K(x,y)\,h(y)\,dy,\ 0\leqslant h(y)\leqslant A\right\}.$$

Then $\mathscr{M}_A\subset\mathscr{M}_\infty$ and $\lim_{A\uparrow\infty}\mathscr{M}_A\cap C[0,1]=\mathscr{M}_\infty\cap C[0,1]$. Theorem 3.1 tells us that for each $g\in C[0,1]\setminus\mathscr{M}_\infty$ ($\Rightarrow g\in C[0,1]\setminus\mathscr{M}_A$), there exists a unique approximant $f_A^*\in\mathscr{M}_A$. This unique approximant is characterized as follows.

There exists a nonnegative integer $N(A)$, and knots

$$\xi_0(A)=0<\xi_1(A)<\cdots<\xi_{N(A)}<1=\xi_{N(A)+1}$$

such that either

$$\text{(a)}\qquad f_A^*(x)=A\sum_{j\text{ even}}\int_{\xi_j(A)}^{\xi_{j+1}(A)}K(x,y)\,dy$$

and $(g-f_A^*)(x)$ equioscillates on at least $N(A)+1$ points, starting positively, or

$$\text{(b)}\qquad f_A^*(x)=A\sum_{j\text{ odd}}\int_{\xi_j(A)}^{\xi_{j+1}(A)}K(x,y)\,dy$$

and $(g-f_A^*)(x)$ equioscillates on at least $N(A)+1$ points, starting negatively.

Since $0<\min\{\|g-f\|_\infty:f\in\mathscr{M}_A\}\leqslant\|g\|_\infty<\infty$, and $\|f_A^*\|_\infty\leqslant 2\,\|g\|_\infty$ for all A, it follows that both $N(A)$ and $f_A(x)$ remain bounded as $A\uparrow\infty$. This is possible only if $\lim_{A\to\infty}(\xi_{j+1}(A)-\xi_j(A))=0$ for j even in case (a) and j odd in case (b). In fact we must have

$$\lim_{A\to\infty}A\int_{\xi_j(A)}^{\xi_{j+1}(A)}K(x,y)\,dy=a_iK(x,\xi_i),$$

where $a_i\geqslant 0$, and $\xi_j(A),\ \xi_{j+1}(A)\to\xi_i$, where again this holds for j even in case (a) and j odd in case (b). In other words, knots must either come together in pairs or run off to one of the endpoints. For this reason, the optimal $f^*(x)$ is of the form (2), and the number of equioscillations is essentially double the number of knots, with the orientation as given. As was stated, this is not a proof, although it might be made rigorous. We prefer, mainly for technical reasons, to give a more direct proof.

We first prove that if $f^*(x)$ exists, satisfying condition (I) or (II), then it is the unique solution to our problem.

LEMMA 5.1. *Assume that there exists an f^* of the form* (2) *satisfying condition* (I) *or* (II) *of Theorem* 5.1. *Then f^* is the unique best approximant to g from $\mathcal{M}_\infty$.*

Proof. Assume $M=0$, i.e., $f^* \equiv 0$ and $g(x^*) = -\|g\|_\infty$ for some $x^* \in [0,1]$. Let $f \in \mathcal{M}_\infty$ satisfy $\|g-f\|_\infty \leqslant \|g\|_\infty$. Then $-g(x^*) + (g-f)(x^*) \geqslant 0$, and thus $f(x^*) \leqslant 0$. Since $f(x^*) = \int_0^1 K(x^*, y)\, d\mu(y)$, where $\mu \in \mathscr{B}$, $\mu \geqslant 0$, and $\min_{0 \leqslant y \leqslant 1} K(x^*, y) \geqslant \alpha > 0$, it follows that $d\mu \equiv 0$, i.e., $f \equiv 0$.

We shall now assume that $M \geqslant 1$. We prove the result only under the assumption (II)(i). The other cases are proven in a similar manner. Assume that

$$f^*(x) = \sum_{i=1}^{M} a_i^* K(x, \xi_i^*) \left(= \int_0^1 K(x, y)\, d\mu^*(y)\right),$$

where $a_i^* > 0$, $i = 1,..., M$ and $0 < \xi_1^* < \cdots < \xi_M^* < 1$, and $(g - f^*)(x)$ equioscillates on at least $2M+1$ points, starting positively. Let $f \in \mathcal{M}_\infty$, where $f(x) = \int_0^1 K(x, y)\, d\mu(y)$, $\mu \in \mathscr{B}, \mu \geqslant 0$, and assume that $\|g-f\|_\infty \leqslant \|g - f^*\|_\infty$. The equioscillations of $(g-f^*)(x)$ imply that

$$S^+((g-f^*)-(g-f)) \geqslant 2M$$

and equality holds only if the sign pattern begins negatively. By Theorem 2.1,

$$S^+((g-f^*)-(g-f)) = S^+(f-f^*) \leqslant S^-(\mu - \mu^*).$$

From the form of μ and μ^*, $S^-(\mu-\mu^*) \leqslant 2M$, and equality implies that the sign pattern begins positively. Thus we contradict Theorem 2.1(ii) and uniqueness is proven.

LEMMA 5.2. *There exists an $f^* \in \mathcal{M}_\infty$ for which*

$$\inf\{\|g-f\|_\infty : f \in \mathcal{M}_\infty\} = \|g - f^*\|_\infty .$$

Proof. Since the zero function is in $\mathcal{M}_\infty$, we have $\inf\{\|g-f\|_\infty : f \in \mathcal{M}_\infty\} \leqslant \|g\|_\infty$ which easily implies that it suffices to consider only $f \in \mathcal{M}_\infty$ satisfying $\|f\|_\infty \leqslant 2\|g\|_\infty$. $K(x, y)$ is STP and therefore $\inf\{K(x, y): x, y \in [0,1]\} \geqslant \alpha > 0$. These facts together imply that the set

$$\mathcal{M}_\infty \cap \{f : \|f\|_\infty \leqslant 2\|g\|_\infty\}$$

is compact for each fixed $g \in C[0,1]$. Thus the infimum is attained.

Let $f^*(x) = \int_0^1 K(x, y)\, d\mu^*(y)$ denote a best approximant to $g \in C[0, 1] \setminus \mathscr{M}_\gamma$ from $\mathscr{M}_\gamma$.

LEMMA 5.3. *$(g - f^*)(x)$ equioscillates on at most a finite number of points.*

Proof. The proof is exactly that given in Lemma 3.2.

To every nonnegative finite Borel measure μ there exists the decomposition

$$\mu = \mu_A + \mu_c,$$

where μ_A and μ_c are also nonnegative finite Borel measures, μ_A is purely atomic and μ_c is continuous. Let $\mu^* = \mu_A^* + \mu_c^*$.

LEMMA 5.4. $d\mu_c^* \equiv 0$.

Proof. Assume that $d\mu_c^* \not\equiv 0$. Let $N + 1$ denote the number of equioscillations of $(g - f^*)(x)$. For a given $\alpha < \beta$, let $\chi_{(\alpha,\beta)}(y)$ denote the characteristic function of the interval (α, β). Since $d\mu_c^* \not\equiv 0$, there exist points $0 = \beta_0 < \beta_1 < \cdots < \beta_N < \beta_{N+1} = 1$, for which

$$\int_0^1 \chi_{(\beta_{i-1},\beta_i)}(y)\, d\mu_c^*(y) > 0, \qquad i = 1,..., N + 1.$$

Set $u_i(x) = \int_0^1 K(x, y)\, \chi_{(\beta_{i+1},\beta_i)}(y)\, d\mu_c^*(y)$, $i = 1,..., N + 1$. Since $K(x, y)$ is an STP kernel, the set of functions $\{u_1,..., u_{N+1}\}$ forms a Descartes system on $[0, 1]$. We now apply the proof of Lemma 3.3 to obtain a contradiction to the optimality of $f^*(x)$. Thus $d\mu_c^* = 0$.

We have therefore so far proven that

$$f^*(x) = \sum_{i=1}^{M} a_i^* K(x, \xi_i^*),$$

where $a_i^* > 0$ all i, the ξ_i^* are distinct points of $[0, 1]$, and M is either a nonnegative integer or infinity. We first show that M is not infinite. Let N be as given above. Then,

LEMMA 5.5. $M \leqslant N$.

Proof. If $M \geqslant N + 1$, set $u_i(x) = K(x, \xi_i^*)$, for some $i = 1,..., N + 1$, where $0 \leqslant \xi_1^* < \cdots < \xi_{N+1}^* \leqslant 1$. Then $\{u_1,..., u_{N+1}\}$ is a Descarte system and we again obtain a contradiction as in Lemma 5.4.

Since we know M to be finite, we can now show that f^* must be of the desired form.

LEMMA 5.6. *Let $f^* \equiv 0$. Then there exists an $x^* \in [0, 1]$ for which $g(x^*) = -\|g\|_\infty$.*

Proof. Assume not. Thus $g(x) \geqslant \alpha - \|g\|_\infty$ for some $\alpha > 0$ and all $x \in [0, 1]$. Form $f(x) = \int_0^1 K(x, y)\, dy$. Since $K(x, y)$ is STP, $f(x) > 0$ on $[0, 1]$. For $\varepsilon > 0$, sufficiently small $\|g - \varepsilon f\|_\infty < \|g\|_\infty$, while $\varepsilon f \in \mathscr{M}_\infty$. This contradicts the optimality of $f^* \equiv 0$.

We now assume that $M \geqslant 1$, and $f^*(x) = \sum_{i=1}^M a_i^* K(x, \xi_i^*)$, where $a_i^* > 0$, $i = 1,..., M$, and $0 \leqslant \xi_1^* < \cdots < \xi_M^* \leqslant 1$. There are now four cases to consider depending on whether $\xi_1^* = 0$ and/or $\xi_M^* = 1$. The reasoning in all these cases is the same except for technical details. As such we shall only deal with one of the cases, namely,

LEMMA 5.7. *Assume $f^*(x) = \sum_{i=1}^M a_i^* K(x, \xi_i^*)$, where $a_i^* > 0$, $i = 1,..., M$, and $0 < \xi_1^* < \cdots < \xi_M^* < 1$. Then $(g - f^*)(x)$ equioscillates on at least $2M + 1$ points, starting negatively.*

Proof. Set $v_{2i}(x) = K(x, \xi_i^*)$, $i = 1,..., M$, and

$$v_{2i-1}(x) = \int_{\xi_{i-1}^*}^{\xi_i^*} K(x, y)\, dy, \quad i = 1,..., M+1,$$

where $\xi_0^* = 0$, $\xi_{M+1}^* = 1$. Then $\{v_1, v_2,..., v_{2M+1}\}$ is a Descartes system on $[0, 1]$. If $v(x) = \sum_{i=1}^{2M+1} b_i v_i(x)$, where $b_i(-1)^i \geqslant 0$, $i = 1,..., 2M+1$, and $b_{2i} \leqslant a_i^*$, $i = 1,..., M$, then $f^*(x) - \varepsilon v(x) \in \mathscr{M}_\infty$ for all $\varepsilon \in [0, 1]$. Utilizing perturbations of this form and the method of proof of Lemmas 3.3, 3.4, it can be shown that Lemma 5.7 is valid.

This completes the proof of Theorem 5.1.

We wish to obtain an analogue to Theorem 4.1 for $\mathscr{M}_\infty$. The analogues of $\mathscr{P}_n$ and $\mathscr{P}_n^1$ differ depending on the parity of n. We therefore define the following four sets. For $n \geqslant 1$, let

$$Q_{2n} = \left\{ \sum_{i=1}^{n} a_i K(x, \xi_i) : a_i > 0, 0 < \xi_1 < \cdots < \xi_n < 1 \right\},$$

$$Q_{2n}^1 = \left\{ \sum_{i=1}^{n+1} a_i K(x, \xi_i) : a_i > 0, 0 = \xi_1 < \xi_2 < \cdots < \xi_{n+1} = 1 \right\},$$

$$Q_{2n-1} = \left\{ \sum_{i=1}^{n} a_i K(x, \xi_i) : a_i > 0, 0 < \xi_1 < \xi_2 < \cdots < \xi_n = 1 \right\},$$

$$Q_{2n-1}^1 = \left\{ \sum_{i=1}^{n} a_i K(x, \xi_i) : a_i > 0, 0 = \xi_1 < \xi_2 < \cdots < \xi_n < 1 \right\}.$$

while we set $Q_0^- = \{0\}$ (i.e., only the zero function) and Q_0^+ is not defined (does not exist). The plus and minus above the Q_n indicates whether a knot exists at zero or not. Note that unlike $\mathscr{P}_n^+$ and $\mathscr{P}_n^-$, the sets Q_n^+ and Q_n^- are *not* closed.

THEOREM 5.2. (I) *Let* $g \in C[0, 1]$ *and assume that*

$$\inf\{\| g - f\|_\infty : f \in Q_m^-\} > \min\{\| g - f\|_\infty, f \in \mathscr{M}_m\}.$$

Then there exists a unique function $f^* \in Q_m^-$ *for which*

$$\inf\{\| g - f\|_\infty : f \in Q_m^-\} = \| g - f^*\|_\infty .$$

f^* *is uniquely characterized by the fact that* $(g - f^*)(x)$ *exhibits exactly* $m + 1$ *points of equioscillation, starting positively.*

(II) *Let* $g \in C[0, 1]$ *and assume that*

$$\inf\{\| g - f\|_\infty : f \in Q_m^+\} > \min\{\| g - f\|_\infty, f \in \mathscr{M}_m\}.$$

Then there exists a unique function $f^* \in Q_m^+$ *for which*

$$\inf\{\| g - f\|_\infty : f \in Q_m^+\} = \| g - f^*\|_\infty .$$

f^* *is uniquely characterized by the fact that* $(g - f^*)(x)$ *exhibits exactly* $m + 1$ *points of equioscillation, starting negatively.*

The above theorem should be viewed as a generalization of Section 5 of Braess [2]. In our theorem, the boundary behavior is detailed.

The existence of f^* of the desired form shall be proven by induction on m. As was the case in the preceding theorems, the proof is via a series of lemmas. We first prove the uniqueness of $f^* \in Q_m^-$ for which $(g - f^*)(x)$ suitably equioscillates. If Q_m^- is replaced by Q_m^+, then the proof is totally analogous.

LEMMA 5.8. *If* $f^* \in Q_m$ *and* $(g - f^*)(x)$ *equioscillates on* $m + 1$ *points, then for all* $f \in \bar{Q}_m^-$, $f \not\equiv f^*$, $\| g - f\|_\infty > \| g - f^*\|_\infty$.

Proof. Assume the existence of an $f \in \bar{Q}_m$ for which $\| g - f\|_\infty \leqslant \| g - f^*\|_\infty$. Thus $m \leqslant S^+((g - f^*) - (g - f)) = S^+(f - f^*)$.

We first consider the case where $m = 2n$. Set $f^*(x) = \sum_{i=1}^n a_i^* K(x, \xi_i^*)$ and $f(x) = \sum_{i=1}^n b_i K(x, \eta_i)$. Therefore

$$f(x) - f^*(x) = \sum_{i=1}^n b_i K(x, \eta_i) - \sum_{i=1}^n a_i^* K(x, \xi_i^*).$$

The functions $\{K(x, \eta_i)\}_{i=1}^n \cup \{K(x, \xi_i)\}_{i=1}^n$ span a Descartes system of order at most $2n = m$. Proposition 2.2(i) then implies that $S^+(f - f^*) \leqslant 2n - 1 = m - 1$. This contradiction proves the lemma for $m = 2n$.

Set $m = 2n - 1$. Thus $f^*(x) = \sum_{i=1}^n a_i^* K(x, \xi_i^*)$ and $f(x) = \sum_{i=1}^n b_i K(x, \eta_i)$, where $\xi_n^* = \eta_n = 1$. Therefore $f(x) - f^*(x) = \sum_{i=1}^{n-1} b_i K(x, \eta_i) - \sum_{i=1}^{n-1} a_i^* K(x, \xi_i^*) + (b_n - a_n^*) K(x, 1)$. Again the functions $\{K(x, \eta_i)\}_{i=1}^{n-1} \cup \{K(x, \xi_i^*)\}_{i=1}^{n-1} \cup \{K(x, 1)\}$ span a Descartes system of order at most $2(n-1) + 1 = m$. We apply Proposition 2.2(i) to obtain a contradiction.

We shall in the course of the proof of the existence of f^* assume, unless otherwise stated, that

$$\inf\{\| g - f\|_\infty : f \in Q_m^\pm\} > \min\{\| g - f\|_\infty : f \in \mathscr{M}_\infty\} \tag{3}$$

for a particular choice of $+$, $-$, and m. Note that if (3) holds for Q_m^+ or Q_m^-, then it also holds for Q_{m-1}^+ and Q_{m-1}^-.

To start the induction, we first prove the result for Q_0.

LEMMA 5.9. *Theorem* 5.2 *holds for* Q_0.

Proof. We assume (3) holds for Q_0^-. Since $g \in C[0, 1]$, there exists a point $x^* \in [0, 1]$ for which $|g(x^*)| = \| g\|_\infty$. If $g(x^*) = -\| g\|_\infty$, then we contradict (3) by Theorem 5.1. Thus $g(x^*) = \| g\|_\infty$ and there exists no $x \in [0, 1]$ for which $g(x) = -\| g\|_\infty$.

The proof of the result for Q_1^- and Q_1^+ is instructive and yet simple. Since the proofs in these two cases are analogous, we only prove the theorem for Q_1^-.

LEMMA 5.10. *Theorem* 5.2 *is valid for* Q_1^-.

Proof. We assume (3) to hold for Q_1^- (and thus for Q_0^-). Now,

$$Q_1^- = \{aK(x, 1) : a > 0\}.$$

Since $K(x, y)$ is STP, $K(x, 1) > 0$ and

$$\min\{K(x, 1) : 0 \leqslant x \leqslant 1\} > 0.$$

Because (3) holds for Q_0^-, there exists a $y \in [0, 1]$ for which $g(y) = \| g\|_\infty$, while there does not exist an x for which $g(x) = -\| g\|_\infty$. Consider $g(x) - aK(x, 1)$ as a varies from zero to infinity. Since $\lim_{a \uparrow \infty}(g(x) - aK(x, 1)) = -\infty$ uniformly for $x \in [0, 1]$, there must exist an $a^* \in (0, \infty)$, and points $x_1, x_2 \in [0, 1]$ satisfying

$$g(x_i) - a^*K(x_i, 1) = (-1)^{i+1} \| g(\cdot) - a^*K(\cdot, 1)\|_\infty,$$

$i = 1, 2$. If $x_1 > x_2$, or if there exist more than two points of equioscillation, then from Theorem 5.1, we contradict (3). Thus $x_1 < x_2$ and we are finished.

Because of the different nature of Q_{2n}^- and Q_{2n}^+, we found it necessary, in order to advance the induction, to develop two different methods of proof. The first of these methods may be used to advance the induction for Q_{2n}^+, Q_{2n-1}^+ and Q_{2n-1}^-. The second method is suitable for proving the result for Q_{2n}^-, Q_{2n-1}^+ and Q_{2n-1}^-. As such we shall prove the theorem for Q_{2n}^- and for Q_{2n}^+. The cases Q_{2n-1}^+ and Q_{2n-1}^- may be proven by either method.

LEMMA 5.11. *Assume that Theorem 5.2 is valid for Q_k^+ and Q_k^-, $k \leqslant 2n-1$, and that (3) holds for Q_{2n}^+. Then Theorem 5.2 holds for Q_{2n}^+.*

Proof. Set $g_a(x) = g(x) - aK(x, 1)$. We shall consider the best approximation to $g_a(x)$ (for $a \geqslant 0$) from Q_{2n-1}^+. Set

$$a_0 = \inf\{a : a \geqslant 0, \inf\{\| g_a - f\|_\infty : f \in Q_{2n-1}^+\} = \min\{\| g_a - f\|_\infty : f \in \mathscr{M}_\infty\}\}.$$

We first claim that $a_0 < \infty$. Assume that this is not the case. Let $f_a(x) \in Q_{2n-1}^+$ denote the unique best approximant to $g_a(x)$ from Q_{2n-1}^+. Thus $f_a(x) > 0$ for all $x \in [0, 1]$ and all $a > 0$, and $f_a(x)/a$ is the unique best approximant to $g(x)/a - K(x, 1)$ from Q_{2n-1}^+. Since $\min_{0 \leqslant x \leqslant 1} K(x, 1) > 0$, there exists an $\tilde{a} > 0$, sufficiently large, such that $g(x)/\tilde{a} - K(x, 1) \leqslant 0$ for all $x \in [0, 1]$. For every nonpositive function, the unique best approximant from $\mathscr{M}_\infty$ is the zero function. $f_{\tilde{a}}(x)/\tilde{a}$ is not the zero function. This contradiction implies $a_0 < \infty$.

For $a \in [0, a_0)$ ($a_0 > 0$ by continuity considerations) set

$$f_a(x) = \sum_{i=1}^{n} a_i(a)\, K(x, \xi_i(a)),$$

where $a_i(a) > 0$, $i = 1,..., n$, and $0 = \xi_1(a) < \cdots < \xi_n(a) < 1$. The function $(g_a - f_a)(x)$ equioscillates on exactly $2n$ points, starting negatively. Since $\|f_a\|_\infty \leqslant 2\, \|g_a\|_\infty$, $f_a(x)$ is uniformly bounded for $a \leqslant a_0$ and thus the coefficients $a_i(a)$ are uniformly bounded for $a < a_0$. Therefore as $a \uparrow a_0$, there exists at least a subsequence along which

$$f_a(x) \to f_{a_0}(x) = \sum_{i=1}^{n} a_i(a_0)\, K(x, \xi_i(a_0)),$$

where $a_i(a_0) \geqslant 0$, $i = 1,..., n$, and $0 = \xi_1(a_0) \leqslant \cdots \leqslant \xi_n(a_0) \leqslant 1$. Furthermore, $(g_{a_0} - f_{a_0})(x)$ equioscillates on at least $2n$ points, starting negatively.

We claim that $f_{a_0}(x) \in Q_{2n-1}^+$, i.e., $a_i(a_0) > 0$, $i = 1,..., n$, and $0 =$

$\xi_1(a_0) < \cdots < \xi_n(a_0) < 1$. Since $(g_{a_0} - f_{a_0})(x) = g(x) - (f_{a_0}(x) + a_0 K(x, 1))$, and $f_{a_0}(x) + a_0 K(x, 1) \in \bar{Q}_{2n}^+$, any other conclusion will contradict (3).

Since $f_{a_0}(x) \in Q_{2n-1}^+$ and

$$\inf\{\| g_{a_0} - f \|_\infty : f \in Q_{2n-1}^+\} = \min\{\| g_{a_0} - f \|_\infty : f \in \mathscr{M}_\infty\},$$

$(g_{a_0} - f_{a_0})(x)$ must equioscillate at least $2n$ times, starting positively. This fact together with the previous fact which said that $(g_{a_0} - f_{a_0})(x)$ equioscillates on at least $2n$ points, starting negatively, implies that $(g_{a_0} - f_{a_0})(x)$ actually equioscillates on at least $2n + 1$ points. Now $(g_{a_0} - f_{a_0})(x) = g(x) - (f_{a_0}(x) - a_0 K(x, 1))$, and $f^*(x) = f_{a_0}(x) + a_0 K(x, 1) \in Q_{2n}^+$. If $(g - f^*)(x)$ equioscillates on more than $2n + 1$ points or on $2n + 1$ points, starting positively, then we again contradict (3) by Theorem 5.1. This proves Theorem 5.2 for Q_{2n}^+.

LEMMA 5.12. *Assume that Theorem* 5.2 *is valid for* Q_k^+ *and* Q_k^-, $k \leqslant 2n - 1$, *and that* (3) *holds for* Q_{2n}^-. *Then Theorem* 5.2 *holds for* Q_{2n}^-.

Proof. Until now, we have always considered our approximations from functions of the form $f(x) = \int_0^1 K(x, y)\, d\mu(y)$. Since $K(x, y)$ is STP on $[0, 1] \times [0, 1]$, it is also STP on $[0, 1] \times [0, d]$ for all $d \in (0, 1]$. Set

$$\mathscr{M}_\infty(d) = \left\{ f(x) : f(x) = \int_0^d K(x, y)\, d\mu(y), \mu \in \mathscr{B}, \mu \geqslant 0 \right\},$$

the measure defined on $[0, d]$.

It is certainly true that everything we have proven so far in this section is valid if we replace $\mathscr{M}_\infty$ by $\mathscr{M}_\infty(d)$ except, of course, that our knots now lie in $[0, d]$. Let $Q_m^+(d)$ and $Q_m^-(d)$ be defined analogously to Q_m^+ and Q_m^- except that we are now considering the interval $[0, d]$ rather than the interval $[0, 1]$. Since we have assumed that Theorem 5.2 is valid for Q_k^+ and Q_k^- for $k \leqslant 2n - 1$, it also remains valid for $Q_k^+(d)$ and $Q_k^-(d)$ for $k \leqslant 2n - 1$.

Set

$$\begin{aligned} d_0 = \sup\{d : d \leqslant 1, \inf\{\| g - f \|_\infty : f \in Q_{2n-1}^-(d)\} \\ = \min\{\| g - f \|_\infty : f \in \mathscr{M}_\infty(d)\}\}. \end{aligned}$$

As a reminder,

$$Q_{2n-1}^-(d) = \left\{ \sum_{i=1}^n a_i K(x, \xi_i) : a_i > 0, 0 < \xi_1 < \cdots < \xi_n = d \right\}.$$

Since (3) holds, then by continuity considerations it follows that $d_0 < 1$, while, as is easily seen, $d_0 \geqslant 0$.

For $d > d_0$ let $f_d \in Q_{2n-1}^-(d)$ denote the unique function for which

$$\inf\{\|g - f\|_\infty : f \in Q_{2n-1}^-(d)\} = \|g - f_d\|_\infty .$$

Thus $f_d(x) = \sum_{i=1}^n a_i(d)\, K(x, \xi_i(d))$, where $a_i(d) > 0$, $i = 1,\dots, n$, $0 < \xi_1(d) < \cdots < \xi_n(d) = d$, and $(g - f_d)(x)$ exhibits exactly $2n$ points of equioscillation on $[0, 1]$, starting positively.

We first prove that $d_0 > 0$. If $d_0 = 0$, then it follows that $g(x) - aK(x, 0)$ exhibits, for some $a \geqslant 0$, at least $2n$ points of equioscillation. This contradicts (3). Thus $d_0 > 0$. As in Lemma 5.11, it is easy to see that letting $d \downarrow d_0$ we obtain a function $f_{d_0}(x) = \sum_{i=1}^n a_i(d_0)\, K(x, \xi_i(d_0))$, where $a_i(d_0) \geqslant 0$, $i = 1,\dots, n$, $0 \leqslant \xi_1(d_0) < \cdots < \xi_n(d_0) = d_0$. $(g - f_{d_0})(x)$ exhibits at least $2n$ points of equioscillation, starting positively, and $\inf\{\|g - f\|_\infty : f \in Q_{2n-1}^-(d_0)\} = \|g - f_{d_0}\|_\infty$. If $f_{d_0} \notin Q_{2n-1}^-(d_0)$, i.e., either knots coalesce together or to endpoints or $a_i(d_0) = 0$, then we contradict (3). Thus $a_i(d_0) > 0$, $i = 1,\dots, n$ and $0 < \xi_1(d_0) < \cdots < \xi_n(d_0) = d_0 < 1$. Since $\inf\{\|g - f\|_\infty : f \in Q_{2n-1}^-(d_0)\} = \min\{\|g - f\|_\infty : f \in \mathscr{M}_\infty(d_0)\}$, $(g - f_{d_0})(x)$ exhibits at least $2n$ points of equioscillation, starting negatively. As in previous proofs, this implies that $(g - f_{d_0})(x)$ exhibits at least $2n + 1$ points of equioscillation. Note, however, that $f_{d_0} \in Q_{2n}^-$ since $d_0 < 1$, i.e., $\xi_n(d_0)$ is no longer an endpoint. Any conclusion other than the desired one contradicts (3) via Theorem 5.1.

This proves Theorem 5.2.

The knots of the unique approximants obtained in Theorem 5.2 exhibit various interlacing properties. Assume that $\inf\{\|g - f\|_\infty : f \in Q_m^\pm\} > \min\{\|g - f\|_\infty : f \in \mathscr{M}_\infty\}$. Let f_m^+ and f_m^- denote the unique best approximants from Q_m^+ and Q_m^-, respectively, to g.

PROPOSITION 5.3. *Under the above assumptions, the knots of f_m^+ and f_m^- strictly interlace.*

Proof. Assume that $m = 2n - 1$. Then $f_{2n-1}^+(x) = \sum_{i=1}^n a_i K(x, \xi_i)$, where $a_i > 0$, $0 = \xi_1 < \cdots < \xi_n < 1$, and $f_{2n-1}^-(x) = \sum_{i=1}^n b_i K(x, \eta_i)$, where $b_i > 0$, $0 < \eta_1 < \cdots < \eta_n = 1$. Independent of which of the two quantities $\|g - f_{2n-1}^+\|_\infty$ and $\|g - f_{2n-1}^-\|_\infty$ is larger,

$$\begin{aligned} 2n - 1 &\leqslant S^+((g - f_{2n-1}^-) - (g - f_{2n-1}^+)) \\ &= S^+(f_{2n-1}^+ - f_{2n-1}^-) \\ &= S^+\left(\sum_{i=1}^n a_i K(x, \xi_i) - \sum_{i=1}^n b_i K(x, \eta_i)\right). \end{aligned}$$

Since $\{K(x, \xi_i)\}_{i=1}^n \cup \{K(x, \eta_i)\}_{i=1}^n$ spans a Descartes system of order at most $2n$, it follows that $S^+(\sum_{i=1}^n a_i K(x, \xi_k) - \sum_{i=1}^n b_i K(x, \eta_i)) \leqslant 2n - 1$.

Equality must therefore obtain in all the above inequalities. Since $a_i, b_i > 0$, $i = 1,..., n$, equality in the last inequality holds by Proposition 2.2(ii) only if the $\{\xi_i\}_{i=1}^n$ and $\{\eta_i\}_{i=1}^n$ strictly interlace. Because $\xi_1 = 0$, $\eta_n = 1$, we in fact have

$$\xi_1 < \eta_1 < \xi_2 < \cdots < \eta_{n-1} < \xi_n < \eta_n.$$

This same argument proves the proposition for $m = 2n$.

A similar argument may also be used to prove the following result.

PROPOSITION 5.4. *Let the assumptions of Proposition 5.3 hold and suppose that $m \geqslant 2$. Then the knots of both f_m^+ and f_m^- strictly interlace the knots of both f_{m-1}^+ and f_{m-1}^-, except where equality must hold at an endpoint.*

This next fact concerning the interlacing of knots is proven in exactly the same manner as was Proposition 5.3 (see Braess [1, Theorem 6.3] for a similar result).

PROPOSITION 5.5. *Let the assumptions of Proposition 5.3 hold and suppose that $m \geqslant 3$. Then the interior knots of f_m^- strictly interlace the interior knots of f_{m-2}^-, and the interior knots of f_m^+ strictly interlace those of f_{m-2}^+.*

The next theorem we present without proof since its proof follows much the pattern of the proof of Theorem 5.1.

We are given a fixed interval $[a, b]$, where $0 < a < b < \infty$. Set

$$\mathscr{A} = \left\{ f(x) : f(x) = \sum_{n=0}^{\infty} a_n x^n, a_n \geqslant 0, \text{ all } n \right\}.$$

Our problem is, for given $g \in C[a, b]$, to solve

$$\inf\{\| g - f \|_{\infty} : f \in \mathscr{A} \}.$$

THEOREM 5.6. *For every $g \in C[a, b]$ there exists a unique best approximant $f^* \in \mathscr{A}$. If $g \notin \mathscr{A}$ then $f^*(x) = \sum_{k=1}^{N} a_{i_k}^* x^{i_k}$, where N is a nonnegative (finite) integer, $a_{i_k}^* > 0$, $k = 1,..., N$, and $0 \leqslant i_1 < \cdots < i_N < \infty$. $f^*(x)$ is uniquely characterized by the following conditions: Set*

$$\begin{aligned} b_i &= -1 \qquad i \notin \{i_1,..., i_N\} \\ &= 0 \qquad i \in \{i_1,..., i_N\}. \end{aligned}$$

Then $(g - f^*)(x)$ exhibits at least $S^+(b_0, b_1,...) + 1$ points of equioscillation, and if the number of equioscillations is exactly $S^+(b_0, b_1,...) + 1$ then the *sign pattern of the vector* $\mathbf{b} = (b_0, b_1, ...)$ *(that given in determining* $S^+(b_0, b_1, ...)$*) and the sign pattern of the equioscillations agree.*

Remark. Having begun this paper with a discussion of the result of Sattes on the Sobolev space $W_\infty^{(r)}$, we feel it necessary to discuss, at least superficially, the situation wherein $K(x, y)$ is totally positive rather than strictly totally positive and, also the case of $B_\infty^{(r)}$. The difference between the results obtained in the last three sections and the results in the case of the same minimum problem where the kernel $K(x, y)$ is only totally positive is simply that we lose the uniqueness of the best approximant. The fact that there exists a best approximant which satisfies the conditions given in Theorems 3.1, 4.1, 4.2, 4.3, 5.1, and 5.2 is a result of a standard smoothing technique (see Karlin [6]) and the fact that for totally positive kernels Theorem 2.1 remains valid if $S^+(\mu)$ is replaced by $S^-(\mu)$.

The set $B_\gamma^{(r)}$ is not quite of this form. However, every $f \in B_\gamma^{(r)}$ may be written as

$$f(x) = \sum_{i=0}^{r-1} a_i x^i + \frac{1}{(r-1)!} \int_0^1 (x-y)_+^{r-1} f^{(r)}(y)\, dy.$$

The kernel $K(x, y) = (x - y)_+^{r-1}$ is totally positive. The difference here is due to the existence of the functions $1, x, ..., x^{r-1}$. These functions alter our result (aside from the uniqueness already lost) only in that there will now be an additional r points of equioscillation (see, e.g., [9] where a similar situation occurs).

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