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015

ON SMOOTHEST INTERPOLANTS*

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Abstract. This paper is concerned with the problem of characterizing those functions of minimum L^p-norm on their nth derivative, $1 \leqq p \leqq \infty$, that sequentially take on the given values $(e_i)_1^N$. For $p = \infty$ the unique minimizing function is characterized. For $p < \infty$ fairly explicit necessary conditions are given.

Key words. minimum norm interpolation, Sobolev spaces, splines

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1. Introduction. Let $n \geqq 2$ be fixed. For $p \in (1, \infty]$, $W_p^{(n)}$ will denote the usual Sobolev space of real-valued functions on $[0, 1]$ with $n-1$ absolutely continuous derivatives and nth derivative existing almost everywhere as a function in $L^p[0, 1]$. Equivalently,

$$W_p^{(n)} = \left\{ f: f(x) = \sum_{i=0}^{n-1} a_i x^i + \frac{1}{(n-1)!} \int_0^1 (x-y)_+^{n-1} h(y)\, dy,\ h \in L^p,\ a_i \in R,\ i = 0, 1, \cdots, n-1 \right\}.$$

(Here $a_i = f^{(i)}(0)/i!$, $i = 0, 1, \cdots, n-1$, and $h \equiv f^{(n)}$.) For $p = 1$, rather than considering the analogous $W_1^{(n)}$, we introduce $V^{(n)}$. To define $V^{(n)}$, let M denote the space of real Baire measures on $[0, 1]$. For $\mu \in M$, $\|\mu\|$ will denote the total variation of the measure μ. Then

$$V^{(n)} = \left\{ f: f(x) = \sum_{i=0}^{n-1} a_i x^i + \frac{1}{(n-1)!} \int_0^1 (x-y)_+^{n-1}\, d\mu(y),\ \|\mu\| < \infty \right\}.$$

We will shortly explain our reasons for considering $V^{(n)}$ rather than $W_1^{(n)}$.

Let $e_1, \cdots, e_N$ be given real fixed data, $e_i \neq e_{i+1}$, $i = 1, \cdots, N-1$. Set

$$\Xi_N = \{\mathbf{t}: \mathbf{t} = (t_1, \cdots, t_N),\ 0 \leqq t_1 < \cdots < t_N \leqq 1\}.$$

For each $\mathbf{t} \in \Xi_N$, set

$$W_p^{(n)}(\mathbf{t}; \mathbf{e}) = \{f: f \in W_p^{(n)}, f(t_i) = e_i, i = 1, \cdots, N\}$$

for $p \in (1, \infty]$, and

$$V^{(n)}(\mathbf{t}; \mathbf{e}) = \{f: f \in V^{(n)}, f(t_i) = e_i, i = 1, \cdots, N\}.$$

The following problems are considered in [2] and [6]:

$$\inf \{\|f^{(n)}\|_p: f \in W_p^{(n)}(\mathbf{t}; \mathbf{e})\} \tag{1}$$

for $p \in (1, \infty]$, and

$$\inf \{\|\mu\|: f \in V^{(n)}(\mathbf{t}; \mathbf{e})\}. \tag{2}$$

(It is understood that f and μ in (2) are related as in the definition of $V^{(n)}$.) Later we will describe the solutions to problems (1) and (2). An understanding of their exact form is crucial to a solution of the problems we consider. We do note, however, that there exist $f \in W_p^{(n)}(\mathbf{t}; \mathbf{e})$ and $f \in V^{(n)}(\mathbf{t}; \mathbf{e})$ for which the above infima are in fact attained.

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If we replace the extremum problem (2) by the analogous problem where $W_1^{(n)}$ takes the place of $V^{(n)}$, then this is not necessarily the case, i.e., the infimum need not be attained by some $f \in W_1^{(n)}$. However, the value of the infimum in (2), or in (2) with $W_1^{(n)}$ replacing $V^{(n)}$, is the same. This is one reason for considering $V^{(n)}$ rather than $W_1^{(n)}$. A more detailed discussion of this matter can be found in de Boor [2], and in Fisher and Jerome [5], [6].

In this work, we are interested in solutions to the problems

$$\inf_{\mathbf{t}\in\Xi_N} \inf\{\|f^{(n)}\|_p : f \in W_p^{(n)}(\mathbf{t}; \mathbf{e})\} \tag{3}$$

for $p \in (1, \infty]$, and

$$\inf_{\mathbf{t}\in\Xi_N} \inf\{\|\mu\| : f \in V^{(n)}(\mathbf{t}; \mathbf{e})\}, \tag{4}$$

and in functions for which the infima are attained. Thus, for each fixed $n \geqq 2$, $p \in [1, \infty]$, and data $(e_i)_1^N$, we wish to characterize those functions f that take on the values $(e_i)_1^N$ sequentially, and minimize the L^p-norm of their nth derivative. (Here we are abusing notation in the case where $p = 1$.)

Before continuing, we note three simple facts.

(I) It suffices to assume that $(e_i - e_{i-1})(e_{i+1} - e_i) < 0$, $i = 2, \cdots, N-1$. This follows from continuity considerations. If, for example, $e_{i-1} < e_i < e_{i+1}$ for some $i \in \{2, \cdots, n-1\}$, then we may delete the condition $f(t_i) = e_i$ since f will always attain the value e_i at some point in (t_{i-1}, t_{i+1}).

(II) We may assume that $N > n$. If $N \leqq n$, then for any choice of $\mathbf{t} \in \Xi_N$, there exists a polynomial q of degree $\leqq n-1$ for which $q(t_i) = e_i$, $i = 1, \cdots, N$. Moreover $q^{(n)} \equiv 0$ and our problem is trivially solved.

(III) We always have $t_1 = 0$ and $t_N = 1$. Assume, for example, that $t_N < 1$ for some f which solves (3) or (4). Set $g(x) = f(xt_N)$ for $x \in [0, 1]$. Then g is "admissible" in (3) or (4), and since $g^{(n)}(x) = t_N^n f^{(n)}(xt_N)$, it easily follows that $\|g^{(n)}\|_p < \|f^{(n)}\|_p$ for $p \in (1, \infty]$, with the analogous strict inequality in (4).

Thus in what follows we will always assume that

(a) $(e_i - e_{i-1})(e_{i+1} - e_i) < 0$, $i = 2, \cdots, N-1$;

(b) $N > n$;

(c) $t_1 = 0$, $t_N = 1$.

There always exist functions $f \in W_p^{(n)}$ which solve (3) (or $f \in V^{(n)}$ which solve (4)). The proof of this fact is not difficult and we omit it. It follows from the existence already alluded to in (1) and (2), and from the fact that there exists a $\mathbf{t}^* \in \Xi_N$ (and not in $\bar{\Xi}_N \backslash \Xi_N$) for which the left-most infima in (3) or (4) are attained.

We will prove that solutions to (3) and (4) must be of a particular form, given by solutions to (1) and (2), respectively, and must also "oscillate" strictly between the values $(e_i)_1^N$. To explain what we mean by this latter term, we introduce the following definition.

DEFINITION. Let $(e_i - e_{i-1})(e_{i+1} - e_i) < 0$, $i = 2, \cdots, N-1$, and $0 = t_1 < \cdots < t_N = 1$. Let $f \in C[0, 1]$ satisfy $f(t_i) = e_i$, $i = 1, \cdots, N$. We say that f *oscillates* between the $(e_i)_1^N$ on $(t_i)_1^N$ if f is monotone on $[t_i, t_{i+1}]$ for each $i = 1, \cdots, N-1$. We say that f *oscillates strictly* between the $(e_i)_1^N$ on $(t_i)_1^N$ if f is strictly monotone on $[t_i, t_{i+1}]$ for each $i = 1, \cdots, N-1$.

Because of the nature of the problem, we divide our analysis into three parts, namely, $1 < p < \infty$, $p = \infty$, and $p = 1$. Both $p = 1(V^{(n)})$ and $p = \infty$ may be considered as limiting cases, but they are much more special and will be considered separately.

In § 2 we quite easily prove that solutions to (3) for $p \in (1, \infty)$ must be of a particular form and oscillate strictly between the $(e_i)_1^N$, on some $(t_i)_1^N$. However, we are unable to prove either the uniqueness of the solution or the fact that functions of this particular form are necessarily solutions to (3). In other words, we prove necessary but not sufficient conditions for a solution to (3). We do conjecture, however, that these conditions are sufficient and that (3) has a unique solution.

In § 3 we consider the case $p = \infty$. We somewhat surprisingly are able to explicitly characterize the solution and we show that it is unique. The uniqueness is especially surprising since for fixed $\mathbf{t} \in \Xi_N$ and $p = \infty$, the solution to (1) is not necessarily unique. (While for $p \in (1, \infty)$ the solution to (1) is unique.)

In § 4 we consider the case $p = 1$. We prove that the solution to (2) is unique and is of a particularly simple form (splines of degree $n-1$ with $N-n$ knots). We again prove a necessary condition for the solution to (4). Here again both the full characterization and uniqueness is lacking, except in the case $n = 2$ where it is easily seen that every solution to (2) necessarily oscillates strictly between the $(e_i)_1^N$. For $n \geqq 3$, we conjecture that the characterization leads to a unique solution.

Let us review the history of and motivation behind this problem. The above problem in a multidimensional setting (x still runs over [0, 1], but we are dealing with a d-dimensional vector of single-valued functions and d-dimensional data vectors $\mathbf{e}^i$, $i = 1, \cdots, N$) was discussed by Marin [9] and Töpfer [12]. Physical motivation for this problem comes from problems of geometric curve fitting and design of a trajectory for a robot manipulator (see Marin [9] and Töpfer [12]). Marin explicitly proved existence and uniqueness for the one-dimensional problem in the case $p = 2$ and $n = 2$. Here we are dealing with natural cubic splines (solutions of (1)) and we can explicitly calculate the solution. More recently Scherer and Smith [11] dealt with the problem of existence in the multidimensional setting for the case $p = 2$. It is our hope that the one-dimensional problem considered herein will not only be of interest in and of itself but will also provide insight into the multidimensional problem.

Finally it should be noted that generalizations of the results of this paper exist since many of these results are consequences of the underlying total positivity structure of the problem. However, this is not true of all of the results and generally only weaker versions hold. Thus, for example, we might consider an nth order disconjugate differential equation L on [0, 1], and the problem

$$\inf_{\mathbf{t} \in \Xi_N} \inf \{\|Lf\|_p : f \in W_p^{(n)}(\mathbf{t}; \mathbf{e})\}$$

for $p \in (1, \infty]$, with an analogue of (4) for $p = 1$. The main result of § 2, Theorem 2.2, will hold in an analogous form except that it is not necessary that $t_1 = 0$, $t_N = 1$, or that every optimal f^* oscillate strictly between the $(e_i)_1^N$ on some $(t_i^*)_1^N$, but only that f^* on $[t_i^*, t_{i+1}^*]$ take on only values between e_i and e_{i+1}, $i = 1, \cdots, N-1$. For $p = 1$, and especially $p = \infty$, the results are substantially weaker than those obtained herein.

Another generalization that can be dealt with using the techniques of this paper is the following. Consider the problem

$$\inf_{\mathbf{t} \in \Xi_N} \inf \left\{ \|h\|_p : \int_0^1 K(t_i, y) h(y)\, dy = e_i,\ i = 1, \cdots, N \right\}$$

(and the analogous problem for $p = 1$) where K is a strictly totally positive kernel. Here again weaker results of the above form are obtained, but only in the case where $e_i e_{i+1} < 0$, $i = 1, \cdots, N-1$.

2. $p\in(1,\infty)$. To understand the solution to (3) we must first consider the problem (1).

Recall that for $f\in W_p^{(n)}$,

$$f(x)=\sum_{i=0}^{n-1} a_i x^i+\frac{1}{(n-1)!}\int_0^1 (x-y)_+^{n-1}h(y)\,dy, \tag{5}$$

where $a_i=f^{(i)}(0)/i!$, $i=0,1,\cdots,n-1$, and $h\equiv f^{(n)}\in L^p$.

Let $\mathbf{t}\in\Xi_N$ be fixed, $t_1=0$, $t_N=1$, $N\geqq n+1$. $f[t_i,\cdots,t_{i+n}]$ will denote the nth divided difference of f at the points $t_i,\cdots,t_{i+n}$, $i=1,\cdots,N-n$. For $f\in W_p^{(n)}(\mathbf{t};\mathbf{e})$, set

$$E_i=f[t_i,t_{i+1},\cdots,t_{i+n}],\qquad i=1,\cdots,N-n.$$

When we assume $(e_i-e_{i-1})(e_{i+1}-e_i)<0$, $i=2,\cdots,N-1$, it easily follows that $E_iE_{i+1}<0$, $i=1,\cdots,N-n-1$ (since $n\geqq 1$). Applying the nth divided difference at the points $t_i,\cdots,t_{i+n}$ to $f\in W_p^{(n)}(\mathbf{t};\mathbf{e})$ as in (5), we obtain

$$E_i=\int_0^1 M_{i,n}(y)h(y)\,dy,\qquad i=1,\cdots,N-n,$$

where $M_{i,n}$ is a positive multiple (easily computed) of the B-spline of degree $n-1$ with knots $t_i,\cdots,t_{i+n}$, $i=1,\cdots,N-n$. Problem (1) is equivalent to

$$\inf\left\{\|h\|_p:\int_0^1 M_{i,n}(y)h(y)\,dy=E_i,\ i=1,\cdots,N-n\right\}. \tag{6}$$

Problem (6) (see de Boor [2], and Fisher and Jerome [6]) (and thus (1)) has a unique solution of the form

$$h_p(y)=\left|\sum_{i=1}^{N-n} b_iM_{i,n}(y)\right|^{q-1}\operatorname{sgn}\left(\sum_{i=1}^{N-n} b_iM_{i,n}(y)\right) \tag{7}$$

where $1/p+1/q=1$, and

$$E_i=\int_0^1 M_{i,n}(y)h_p(y)\,dy,\qquad i=1,\cdots,N-n. \tag{8}$$

Equation (8) uniquely determines the coefficients $(b_i)_1^{N-n}$ in (7). To obtain the unique solution f_p to (1), we write

$$f_p(x)=\sum_{i=0}^{n-1} a_i x^i+\frac{1}{(n-1)!}\int_0^1 (x-y)_+^{n-1}h_p(y)\,dy$$

and uniquely determine the $(a_i)_0^{n-1}$ so that $f_p(t_i)=e_i$, $i=1,\cdots,n$. From (8) it follows that $f_p\in W_p^{(n)}(\mathbf{t};\mathbf{e})$.

The following notation will prove useful. For $f\in C[a,b]$, let $S(f)$ denote the number of sign changes of f on $[a,b]$, i.e.,

$$S(f)=\sup\{k: a\leqq x_1<\cdots<x_{k+1}\leqq b,\ f(x_i)f(x_{i+1})<0,\ i=1,\cdots,k\}.$$

Of course, if f is either nonnegative or nonpositive on $[a,b]$, then we set $S(f)=0$. Similarly, for a vector $\mathbf{x}\in R^m\backslash\{\mathbf{0}\}$, $S^-(\mathbf{x})$ will denote the number of sign changes of the vector $\mathbf{x}$, i.e.,

$$S^-(\mathbf{x})=\max\{k: 1\leqq i_1<\cdots<i_{k+1}\leqq m,\ x_{i_j}x_{i_{j+1}}<0,\ j=1,\cdots,k\},$$

unless $\mathbf{x}$ is nonnegative or nonpositive in which case $S^-(\mathbf{x})=0$.

Let $Q(x)=\sum_{i=1}^{N-n} b_i M_{i,n}(x)$ where the $(b_i)_1^{N-n}$ are as determined by (8).

PROPOSITION 2.1. *Q has exactly $N-n-1$ sign changes on $(0,1)$, and $\sigma Q^{(n-1)}(x)(-1)^i>0$ for $x\in(t_i,t_{i+1})$, $i=1,\cdots,N-1$, where $\sigma\in\{-1,1\}$, fixed.*

Proof. It is well known (see, e.g., de Boor [3]) that

$$S\left(\sum_{i=1}^{N-n} b_i M_{i,n}\right)\leqq S^-((b_1,\cdots,b_{N-n})).$$

Since $S^-((b_1,\cdots,b_{N-n}))\leqq N-n-1$, it follows that Q has at most $N-n-1$ sign changes on $(0,1)$. Assume Q has k sign changes on $(0,1)$. Then

$$h_p(y)=|Q(y)|^{q-1}\operatorname{sgn}(Q(y))$$

has k sign changes on $(0,1)$. Let $0=\xi_0<\xi_1<\cdots<\xi_{k+1}=1$ be such that $\delta h_p(y)(-1)^j\geqq 0$ for all $y\in[\xi_{j-1},\xi_j]$, $j=1,\cdots,k+1$, where $\delta\in\{-1,1\}$, fixed. Set

$$a_{ij}=\int_{\xi_{j-1}}^{\xi_j} M_{i,n}(y)|h_p(y)|\,dy,\qquad i=1,\cdots,N-n,\quad j=1,\cdots,k+1.$$

From properties of B-splines (see, e.g., de Boor [3]) it follows that $A=(a_{ij})_{i=1,j=1}^{N-n,k+1}$ is a totally positive (TP) matrix. Furthermore,

$$\sum_{j=1}^{k+1} a_{ij}(-1)^j\delta=E_i,\qquad i=1,\cdots,N-n.$$

From the above, $N-n\geqq k+1$. As a consequence of the variation diminishing property of TP matrices (see Karlin [7]), we have

$$S^-((E_1,\cdots,E_{N-n}))\leqq\min\{\operatorname{rank}(A)-1,S^-(-\delta,\delta,\cdots,(-1)^{k+1}\delta)\}.$$

Since $E_iE_{i+1}<0$, $i=1,\cdots,N-n-1$, it follows that the left-hand side equals $N-n-1$ and that $N-n-1\leqq k$. Therefore $k=N-n-1$ and Q has exactly $N-n-1$ sign changes on $(0,1)$.

Since $k=N-n-1$, we have that $S^-((b_1,\cdots,b_{N-n}))=N-n-1$, and thus $b_i(-1)^i\sigma>0$, $i=1,\cdots,N-n$, for some $\sigma\in\{-1,1\}$, fixed. It is well known that the $(n-1)$st derivative of $M_{i,n}(x)$ strictly alternates in sign as we go from (t_j,t_{j+1}) to (t_{j+1},t_{j+2}), $j=1,\cdots,i+n-2$. In particular,

$$M_{i,n}^{(n-1)}(x)(-1)^{i+j}>0,\quad x\in(t_j,t_{j+1}),\quad j=i,\cdots,i+n-1.$$

Thus for $x\in(t_j,t_{j+1})$,

$$Q^{(n-1)}(x)=\sum_{i=1}^{N-n} b_i M_{i,n}^{(n-1)}(x)=\sigma(-1)^j\sum_{i=1}^{N-n}|b_i M_{i,n}^{(n-1)}(x)|.$$

Since $b_i\neq 0$ for all i, and $M_{i,n}^{(n-1)}(x)\not\equiv 0$ on (t_j,t_{j+1}) for some i, it follows that $\sigma(-1)^jQ^{(n-1)}(x)>0$ on (t_j,t_{j+1}), $j=1,\cdots,N-1$. This proves the proposition. $\square$

With the above proposition we easily prove the following theorem.

THEOREM 2.2. *Let $p\in(1,\infty)$, and let $f^*\in W_p^{(n)}$ be a solution of* (3). *There exists a $\mathbf{t}^*=(t_1^*,\cdots,t_N^*)$, $0=t_1^*<\cdots<t_N^*=1$ such that $f^*\in W_p^{(n)}(\mathbf{t}^*;\mathbf{e})$. Furthermore,*

(a) $$f^{*(n)}(y)=\left|\sum_{i=1}^{N-n} b_i^* M_{i,n}(y)\right|^{q-1}\operatorname{sgn}\left(\sum_{i=1}^{N-n} b_i^* M_{i,n}(y)\right)$$

where $1/p+1/q=1$, $M_{i,n}$ is a positive multiple of the B-*spline of degree $n-1$ with knots*

$t_i^*, \cdots, t_{i+n}^*$, $i=1, \cdots, N-n$, and the $(b_i^*)_1^{N-n}$ satisfy

$$\int_0^1 M_{i,n}(y) f^{*(n)}(y)\, dy = f[t_i^*, \cdots, t_{i+n}^*], \qquad i=1, \cdots, N-n.$$

(b) *f^* oscillates strictly between the $(e_i)_1^N$ on $(t_i^*)_1^N$.*

Proof. Let f^* solve (3). Existence implies that there exists a $\mathbf{t}^*$ as above for which $f^* \in W_p^{(n)}(\mathbf{t}^*; \mathbf{e})$. Since f^* must also solve (1) for $\mathbf{t}^*$, it follows that f^* necessarily satisfies (a). It remains to prove that (b) holds.

Since $(e_i - e_{i-1})(e_{i+1} - e_i) < 0$, $i=2, \cdots, N-1$, f^* has at least $N-2$ interior extrema, i.e., $f^{*\prime}$ has at least $N-2$ distinct zeros. From Proposition 2.1, $f^{*(n)}$ does not vanish on any subinterval of $[0, 1]$ and has exactly $N-n-1$ sign changes. From Rolle's theorem applied to $f^{*\prime}$, it follows that $f^{*\prime}$ has exactly $N-2$ (simple) zeros in $(0, 1)$. Let $0 < s_2 < \cdots < s_{N-1} < 1 (s_1 = 0, s_N = 1)$ denote the unique extrema of f^*. Thus f^* oscillates strictly between the $(f^*(s_i))_1^N$ on $(s_i)_1^N$. It remains to prove that $s_i = t_i^*$, $i=2, \cdots, N-1$. Note that $t_{j-1}^* < s_j < t_{j+1}^*$ for $j=2, \cdots, N-1$.

Assume $s_i \neq t_i^*$ for some $i \in \{2, \cdots, N-1\}$. Consider problem (1) at the points $(s_i)_1^N$ with the values $e_i^1 = f^*(s_i)$, $i=1, \cdots, N$. There is a unique solution to this new problem which we denote by g^*. It follows from Proposition 2.1 that $g^* \not\equiv f^*$. Furthermore f^* is "admissible" for this problem. Thus $\|g^{(n)}\|_p < \|f^{*(n)}\|_p$. From continuity considerations, there exist points $0 \leqq w_1 < \cdots < w_N \leqq 1$ such that $g^*(w_i) = e_i$, $i = 1, \cdots, N$. Thus g^* is "admissible" in (3). However, this contradicts the minimality property of f^*. Thus $s_i = t_i^*$, $i=2, \cdots, N-1$, and f^* oscillates strictly between the $(e_i)_1^N$ on $(t_i^*)_1^N$. □

On the basis of the above result, it is natural to ask whether the solution to (3) is unique, and in particular, whether there is a unique function satisfying (a) and (b) of Theorem 2.2. Marin [9] showed by construction that there is a unique function satisfying (a) and (b) in the particular case $n = p = 2$.

Remark. For the case $n=1$, it is easily seen that every solution to (1) a spline of degree one with simple knots at $t_2, \cdots, t_{N-1}$. Thus it oscillates strictly between the $(e_i)_1^N$ on $(t_i)_1^N$ for any choice of $\mathbf{t} \in \Xi_N$. However, a bit of calculation shows that the solution to (3) is in fact unique. The optimal choice of $\mathbf{t}^*$ is given by $t_1^* = 0$, $t_N^* = 1$, and

$$t_i^* = \sum_{j=1}^{i-1} |e_{j+1} - e_j| \bigg/ \sum_{j=1}^{N-1} |e_{j+1} - e_j|, \qquad i=2, \cdots, N-1.$$

Note that this unique choice is independent of $p \in (1, \infty)$.

3. $p = \infty$. For fixed $0 = t_1 < \cdots < t_N = 1$, $N > n \geqq 2$, and $(e_i - e_{i-1})(e_{i+1} - e_i) < 0$, $i = 2, \cdots, N-1$, the problem (1) for $p = \infty$, i.e.,

$$\inf\{\|f^{(n)}\|_\infty : f \in W_\infty^{(n)}(\mathbf{t}; \mathbf{e})\}, \tag{9}$$

may have many solutions. There is always at least one solution of particular interest. It is a perfect spline of degree n with exactly $N-n-1$ knots, i.e., a function P of the form

$$P(x) = \sum_{i=0}^{n-1} a_i x^i + \frac{c}{n!}\left[x^n + 2\sum_{i=1}^{N-n-1} (-1)^i (x - \xi_i)_+^n\right]$$

where $\xi_0 = 0 < \xi_1 < \cdots < \xi_{N-n-1} < \xi_{n-n} = 1$ (see, e.g., Karlin [8]). Note that $|P^{(n)}(x)| = |c|$ for all $x \in [0, 1] \backslash \{\xi_1, \cdots, \xi_{N-n-1}\}$.

The main idea used in the proof of Theorem 2.2 does not carry over to the case $p = \infty$ since the g^* constructed therein is generally identically equal to the f^*. However, much research has been done on perfect splines and we will use some of those results to prove not only an analogue of Theorem 2.2, but also the uniqueness of our solution.

We first state two deep results due to Bojanov. Recall that we always assume that $N > n \geqq 2$ and $(e_i - e_{i-1})(e_{i+1} - e_i) < 0$, $i = 2, \cdots, N-1$.

THEOREM 3.1 (Bojanov [1]). *There exists a unique perfect spline P^* of degree n with $N-n-1$ knots, and a unique set of points $0 = t_1^* < \cdots < t_N^* = 1$ for which*

(i) $P^*(t_i^*) = e_i$, $i = 1, \cdots, N$;

(ii) $P^{*\prime}(t_i^*) = 0$, $i = 2, \cdots, N-1$.

THEOREM 3.2 (Bojanov [1]). *Let $P^*(\cdot\,; \mathbf{e})$ denote the unique perfect spline as given in Theorem 3.1, where we indicate the dependence on $\mathbf{e} = (e_1, \cdots, e_N)$. In a neighborhood of every $\mathbf{e}$ satisfying $(e_i - e_{i-1})(-1)^i > 0$, $i = 2, \cdots, N$, we have that $\|P^{*(n)}(\cdot\,; \mathbf{e})\|_\infty$ is a strictly increasing function of each $(-1)^i e_i$, $i = 1, \cdots, N$.*

On the basis of Theorems 3.1 and 3.2, we immediately obtain the following result.

THEOREM 3.3. *There exists a unique perfect spline P^* of degree n with $N-n-1$ knots which satisfies (3) for $p = \infty$. P^* is uniquely characterized by the fact that it oscillates strictly between the $(e_i)_1^N$ on some $(t_i^*)_1^N$.*

Proof. The only fact that is not an immediate consequence of Theorems 3.1 and 3.2 is the fact that P^* is strictly monotone on $[t_i^*, t_{i+1}^*]$ for each $i = 1, \cdots, N-1$. However, a simple Rolle's theorem argument shows that $P^{*\prime}$ has exactly $N-2$ zeros. Thus P^* is strictly monotone on $[t_i^*, t_{i+1}^*]$, $i = 1, \cdots, N-1$. □

In the above theorem, uniqueness is proved only for the class of perfect splines. There is more that is true, namely, Theorem 3.4.

THEOREM 3.4. *The perfect spline of Theorem 3.3 is the unique solution to (3) for $p = \infty$.*

Proof. Let P^* be as in Theorem 3.3 with $P^*(t_i^*) = e_i$, $i = 1, \cdots, N$, $0 = t_1^* < \cdots < t_N^* = 1$. Assume $f \in W_\infty^{(n)}(\mathbf{t}; \mathbf{e})$ for some $\mathbf{t} \in \Xi_N$. There exists a perfect spline P of degree n with $N-n-1$ knots for which $P(t_i) = f(t_i) = e_i$, $i = 1, \cdots, N$, and $\|P^{(n)}\|_\infty \leqq \|f^{(n)}\|_\infty$. If $\mathbf{t} \neq \mathbf{t}^*$, then from Theorem 3.3, $\|P^{*(n)}\|_\infty < \|P^{(n)}\|_\infty$. Thus if $f \in W_\infty^{(n)}$, f is "admissible" in (3), and $\|f^{(n)}\|_\infty = \|P^{*(n)}\|_\infty$, then it necessarily follows that $f \in W_\infty^{(n)}(\mathbf{t}^*; \mathbf{e})$.

We next prove that $f'(t_i^*) = 0$, $i = 2, \cdots, N-1$ for any f as above. Assume $f'(t_j^*) \neq 0$ for some $j \in \{2, \cdots, N-1\}$. Replace t_j^* by $s_j \in (t_{j-1}^*, t_{j+1}^*)$ so that if $g \in W_\infty^{(n)}$, $g(t_i^*) = e_i$, $i = 1, \cdots, N$, $i \neq j$, and $g(s_j) = f(s_j)$, then g attains the value e_j at least twice in (t_{j-1}^*, t_{j+1}^*), and g is "admissible" in (3). Let P be a perfect spline of degree n with $N-n-1$ knots such that $P(t_i^*) = e_i$, $i = 1, \cdots, N$, $i \neq j$, and $P(s_j) = f(s_j)$. Then $P \not\equiv P^*$. Thus $\|P^{(n)}\|_\infty \leqq \|f^{(n)}\|_\infty$ and from Theorem 3.3, $\|P^{*(n)}\|_\infty < \|P^{(n)}\|_\infty$. This contradicts the minimality property of f. Thus $f'(t_i^*) = 0$, $i = 2, \cdots, N-1$.

Assume $f \not\equiv P^*$. Then $f \not\equiv P^*$ on (t_j^*, t_{j+1}^*) for some $j \in \{1, \cdots, N-1\}$. Since $(P^* - f)(t_i^*) = 0$, $i = j, j+1$, $(P^* - f)'(x)$ must change sign on (t_j^*, t_{j+1}^*). Thus for $\sigma > 0$, sufficiently small, $(P^* - (1-\sigma)f)'(x)$ has a sign change in (t_j^*, t_{j+1}^*). Furthermore $(P^* - (1-\sigma)f)'(t_i^*) = 0$, $i = 2, \cdots, N-1$. Thus $(P^* - (1-\sigma)f)'(x)$ has at least $N-1$ distinct zeros in $[0, 1]$. Since $|P^{*(n)}(x)| \geqq |f^{(n)}(x)| > (1-\sigma)|f^{(n)}(x)|$ almost everywhere on $[0, 1]$, it follows from Rolle's theorem that $(P^* - (1-\sigma)f)^{(n)}(x)$ has at least $N-n$ sign changes on $[0, 1]$. But $P^{*(n)}(x)$, and thus $(P^* - (1-\sigma)f)^{(n)}(x)$, has exactly $N-n-1$ sign changes thereon. This contradiction proves the theorem. □

Remark. For $N = n+1$, P^* is the unique polynomial of degree n that satisfies $P^*(t_i^*) = e_i$, $i = 1, \cdots, n+1$, and $P^{*\prime}(t_i^*) = 0$, $i = 2, \cdots, n$. Such polynomials have been considered previously (see, e.g., Davis [4] and Mycielski and Paszkowski [10]). It seems that it was not previously noted that such polynomials satisfy an extremal property with respect to their nth derivative.

Remark. In the case $n = 1$ both Theorems 3.3 and 3.4 are valid. However, the proofs are somewhat different. The unique solution is identical for all $p \in (1, \infty]$ (see the remark at the end of § 2), and is a perfect spline with knots $(t_i^*)_2^{N-1}$.

4. $p=1$. We first consider in some detail solutions to (2) for fixed $0=t_1<t_2<\cdots<t_N=1$ with $N>n\geqq 2$ and $(e_i-e_{i-1})(e_{i+1}-e_i)<0$, $i=2,\cdots,N-1$. We recall that $f\in V^{(n)}(\mathbf{t};\mathbf{e})$ if

$$f(x)=\sum_{i=0}^{n-1}a_ix^i+\frac{1}{(n-1)!}\int_0^1(x-y)_+^{n-1}\,d\mu(y), \tag{10}$$

where $\|\mu\|<\infty$, and $f(t_i)=e_i$, $i=1,\cdots,N$. We are concerned with the problem

$$\min\{\|\mu\|: f\in V^{(n)}(\mathbf{t};\mathbf{e})\} \tag{11}$$

where μ is associated with f as in (10).

As in § 2, set

$$E_i=f[t_i,t_{i+1},\cdots,t_{i+n}],\qquad i=1,\cdots,N-n.$$

Thus (11) is equivalent to

$$\min\left\{\|\mu\|:\int_0^1 M_{i,n}(y)\,d\mu(y)=E_i,\, i=1,\cdots,N-n\right\}. \tag{12}$$

Since $n\geqq 1$, we have $E_iE_{i+1}<0$, $i=1,\cdots,N-n-1$.

It is well known that $(M_{i,n})_1^{N-n}$ is a weak Chebyshev (WT-) system on $[0,1]$. Thus there exists a nontrivial

$$h(y)=\sum_{i=1}^{N-n}c_iM_{i,n}(y)$$

and points $0<\xi_1<\cdots<\xi_{N-n}<1$ such that

$$h(\xi_i)=(-1)^i\|h\|_\infty,\qquad i=1,\cdots,N-n.$$

Without loss of generality, we normalize h so that $\|h\|_\infty=1$. Before showing how we use h to construct a solution to (12), let us consider h and the points of equi-oscillation $(\xi_i)_1^{N-n}$ in more detail.

PROPOSITION 4.1. *Let h be as above. Then we have the following*:

(i) *h is unique.*

(ii) *$c_i(-1)^i>0$, $i=1,\cdots,N-n$.*

(iii) *The $(\xi_i)_1^{N-n}$ are uniquely defined.*

(iv) *If $n\geqq 3$, then $t_{i+1}<\xi_i<t_{i+n-1}$, $i=1,\cdots,N-n$.*

Proof. For $n=2$, h is continuous and piecewise linear with knots $t_2,\cdots,t_{N-1}$, and satisfies $h(0)=h(1)=0$. h is easily seen to exist and satisfy (i), (ii), and (iii) with $\xi_i=t_{i+1}$, $i=1,\cdots,N-2$.

Assume $n\geqq 3$. By construction h has at least $N-n-1$ sign changes. From Proposition 2.1 and the proof thereof, it follows that h has exactly $N-n-1$ sign changes, $h^{(n-1)}$ strictly changes sign at each t_i, $i=2,\cdots,N-1$, and $c_i(-1)^i>0$, $i=1,\cdots,N-n$.

For each $j\in\{1,\cdots,N\}$, $(M_{i,n})_{i=1,i\neq j}^{N-n}$ is a WT-system on $[0,1]$. Since h equi-oscillates at $N-n$ points, it follows that $-c_j^{-1}\sum_{i=1,i\neq j}^{N}c_iM_{i,n}$ is a best approximant to $M_{j,n}$ on $[0,1]$ in the uniform norm from span $\{M_{i,n}\}_{i=1,i\neq j}^{N-n}$. Furthermore, the error in the best approximation is exactly $|c_j|^{-1}$. If $\tilde h=\sum_{i=1}^{N-n}d_iM_{i,n}$ satisfies $\|\tilde h\|_\infty=1$, and $\tilde h(\eta_i)=(-1)^i$, $i=1,\cdots,N-n$ for some $0<\eta_1<\cdots<\eta_{N-n}<1$, then it follows as above that $(-1)^jd_j^{-1}=|d_j|^{-1}=|c_j|^{-1}=(-1)^jc_j^{-1}$. Thus $d_j=c_j$ for each j proving the uniqueness of h. (A different proof of the uniqueness of h follows from the analysis in the proof of Theorem 4.2.)

Since

$$\sum_{i=1}^{N-n} c_i M_{i,n}(\xi_j) = (-1)^j, \qquad j=1,\cdots,N-n,$$

and $(M_{i,n}(\xi_j))_{i,j=1}^{N-n}$ is TP, it follows that this matrix is nonsingular and thus $\xi_i \in (t_i, t_{i+n})$, $i=1,\cdots,N-n$. However, we wish to prove more, namely, $\xi_i \in (t_{i+1}, t_{i+n-1})$, $i=1,\cdots,N-n$. To this end we use Rolle's theorem and the fact that $h'(\xi_i)=0$, $i=1,\cdots,N-n$. Since $n \geqq 3$, h' is continuous and vanishes on $(0, \xi_i]$ at the points $\xi_1,\cdots,\xi_i$. Furthermore $h^{(j)}(0)=0, j=0,1,\cdots,n-2$. When we apply Rolle's theorem, it follows that $h^{(n-2)}$ has at least $i+1$ distinct zeros in $[0,\xi_i]$, and $h^{(n-1)}$ has at least i sign changes in $(0,\xi_i)$ (since $h^{(n-1)}$ does not vanish identically on any subinterval). But $h^{(n-1)}$ changes sign exactly at $t_2,\cdots,t_{N-1}$. Thus $t_{i+1}<\xi_i$. Similarly we prove that $\xi_i < t_{i+n-1}$.

It remains to prove (iii) for $n \geqq 3$. Property (iii) follows if we can show that h' has no zeros in $(0,1)$ other than $(\xi_i)_1^{N-n}$. This fact may be proven by a simple Rolle's theorem argument. Alternatively, we can argue as follows:

$$h'(y) = \sum_{i=1}^{N-n+1} d_i M_{i,n-1}(y)$$

where $\operatorname{supp} M_{i,n-1} = (t_i, t_{i+n-1})$, $i=1,\cdots,N-n+1$. If $h'(\xi)=0$ for some $\xi \in (0,1)\backslash\{\xi_1,\cdots,\xi_{N-n}\}$, then by setting $\{\eta_1,\cdots,\eta_{N-n+1}\}=\{\xi_1,\cdots,\xi_{N-n},\xi\}$ where $0<\eta_1<\cdots<\eta_{N-n+1}<1$, it follows from (iv) that $t_i<\eta_i<t_{i+n-1}$, $i=1,\cdots,N-n+1$. But this implies that $h'\equiv 0$, which is a contradiction. □

We can now construct a unique solution to (11).

THEOREM 4.2. *Let h and $(\xi_i)_1^{N-n}$ be as given above. Set*

$$\mu_t = \sum_{j=1}^{N-n} b_j \delta_{\xi_j}$$

where the $(b_j)_1^{N-n}$ are chosen so that

$$\sum_{j=1}^{N-n} b_j M_{i,n}(\xi_j) = E_i, \qquad i=1,\cdots,N-n.$$

Then $\|\mu_t\| = \sum_{j=1}^{N-n} |b_j|$ and μ_t is the unique solution to (11).

Proof. Let μ_t be as given above. Such a μ_t exists and is unique since $(M_{i,n}(\xi_j))_{i,j=1}^{N-n}$ is nonsingular. Since $E_i(-1)^i\sigma>0$, $i=1,\cdots,N-n$ for some $\sigma\in\{-1,1\}$, fixed, it follows from the total positivity of $(M_{i,n}(\xi_j))_{i,j=1}^{N-n}$ that $b_j(-1)^j\sigma>0$, $j=1,\cdots,N-n$. Furthermore since $c_i(-1)^i>0$, $i=1,\cdots,N-n$, it is easily seen that

$$\sum_{j=1}^{N-n} |b_j| = \left|\sum_{i=1}^{N-n} c_i E_i\right| = \sum_{i=1}^{N-n} |c_i E_i|.$$

For any μ, $\|\mu\|<\infty$, satisfying

$$\int_0^1 M_{i,n}(y)\, d\mu(y) = E_i, \qquad i=1,\cdots,N-n,$$

we have

$$\left|\sum_{i=1}^{N-n} c_i E_i\right| = \left|\int_0^1 h(y)\, d\mu(y)\right| \leqq \|h\|_\infty \|\mu\|.$$

Since $\|h\|_\infty = 1$,

$$\|\mu_t\| = \sum_{j=1}^{N-n} |b_j| = \left|\sum_{i=1}^{N-n} c_i E_i\right| \leqq \|\mu\|,$$

which implies that μ_t is a solution to (11). Assume $\|\mu_t\| = \|\mu\|$. Then necessarily

$$\left|\int_0^1 h(y)\, d\mu(y)\right| = \|h\|_\infty \|\mu\|.$$

By Proposition 4.1(iii), h attains its norm only at the values $(\xi_j)_1^{N-n}$. Thus μ has support only on the $(\xi_j)_1^{N-n}$. Since $(M_{i,n}(\xi_j))_{i,j=1}^{N-n}$ is nonsingular, it follows that $\mu \equiv \mu_t$. □

We now turn our attention to the problem as stated in (4), namely,

$$\min_{t \in \Xi_N} \min \{\|\mu\| : f \in V^{(n)}(\mathbf{t}; \mathbf{e})\}. \tag{13}$$

From Theorem 4.2 we know that any solution must be of the specific form given therein. We will prove that any solution must also oscillate strictly between the $(e_i)_1^N$ on some $(t_i)_1^N$. For $n = 2$, this result is simple, and yet disappointing. For any $\mathbf{t} \in \Xi_N$, $t_1 = 0 < t_2 < \cdots < t_N = 1$, construct μ_t and the associated $f(x) = a_0 + a_1 x + \sum_{j=1}^{N-2} b_j (x - \xi_j)_+^1$. As noted earlier $\xi_j = t_{j+1}, j = 1, \cdots, N-2$. Since $(e_i - e_{i-1}) \times (e_{i+1} - e_i) < 0$, $i = 2, \cdots, N-1$, it is easily seen that for any choice of $\mathbf{t} \in \Xi_N$, f oscillates strictly between the $(e_i)_1^N$ on $(t_i)_1^N$. Thus our result clearly holds, but obviously this condition is not sufficient in (13). However, it is possible, by calculation, to verify that the solution to (13) is unique.

We now turn our attention to the case $n \geqq 3$. Here it is unclear as to whether the following necessary conditions are also sufficient for a solution to (13), and also as to whether uniqueness holds.

THEOREM 4.3. *Let $n \geqq 3$ and let f^* be any solution to* (13). *There exists a* $\mathbf{t}^* = (t_1^*, \cdots, t_N^*)$, $0 = t_1^* < \cdots < t_N^* = 1$ *such that* $f^* \in V^{(n)}(\mathbf{t}^*; \mathbf{e})$. *Furthermore,*

(a) $f^*(x) = \sum_{i=0}^{n-1} a_i x^i + \sum_{j=1}^{N-n} b_j (x - \xi_j)_+^{n-1}$

where the $(b_j)_1^{N-n}$ *and* $(\xi_j)_1^{N-n}$ *are as in Theorem* 4.2 *with respect to* $(t_i^*)_1^N$.

(b) f^* *oscillates strictly between the* $(e_i)_1^N$ *on* $(t_i^*)_1^N$.

Proof. Let f^* be a solution to (13). There then exists a $\mathbf{t}^*$ as above for which $f^* \in V^{(n)}(\mathbf{t}^*; \mathbf{e})$. Since f^* must also solve (11) for $\mathbf{t}^*$, it follows that f^* necessarily satisfies (a). It remains to prove (b).

We first prove that $f^{*\prime}(t_i^*) = 0$, $i = 2, \cdots, N-1$. Assume to the contrary that $f^{*\prime}(t_k^*) \neq 0$ for some $k \in \{2, \cdots, N-1\}$. Without loss of generality we will assume that $(e_k - e_{k-1}) > 0$. Thus there exists an s_k as near as we wish to t_k^* for which $f(s_k) > e_k$. Recall from Proposition 4.1 that $t_{i+1}^* < \xi_i < t_{i+n-1}^*$, $i = 1, \cdots, N-n$. Taking s_k close to t_k^*, this implies the existence of a unique

$$g(x) = \sum_{i=0}^{n-1} c_i x^i + \sum_{j=1}^{N-n} d_j (x - \xi_j)_+^{n-1}$$

that satisfies $g(t_i^*) = e_i$, $i = 1, \cdots, N$, $i \neq k$, and $g(s_k) = e_k$. Furthermore sgn d_j = sgn $b_j = (-1)^j \sigma$, $j = 1, \cdots, N-n$, where $\sigma \in \{-1, 1\}$, fixed. g is "admissible" in (13) with $\|\mu\| = \sum_{j=1}^{N-n} |d_j|$. We will prove that $\sum_{j=1}^{N-n} |d_j| < \sum_{j=1}^{N-n} |b_j|$, contradicting the minimality of f^*. To this end, note that

$$(f^* - g)(x) = \sum_{i=0}^{n-1} (a_i - c_i) x^i + \sum_{j=1}^{N-n} (b_j - d_j)(x - \xi_j)_+^{n-1}$$

satisfies $(f^*-g)(t_i^*)=0$, $i=1,\cdots,N$, $i\neq k$, and $(f^*-g)(s_k)>0$. The conditions $t_{i+1}^*<\xi_i<t_{i+n-1}^*$, $i=1,\cdots,N-n$, easily imply that f^*-g vanishes only at $(t_i^*)_{i=1,i\neq j}^N$. Thus, in particular, $(f^*-g)(t_k^*)>0$.

Set $(f^*-g)[t_i^*,\cdots,t_{i+n}^*]=F_i$, $i=1,\cdots,N-n$. Then

$$F_i=\sum_{j=1}^{N-n}(b_j-d_j)M_{i,n}(\xi_j),\qquad i=1,\cdots,N-n$$

and $F_i(-1)^i\sigma\geqq 0$, $i=1,\cdots,N-n$. Furthermore the $(F_i)_1^{N-n}$ are not all zero. From the total positivity of $(M_{i,n}(\xi_j))_{i,j=1}^{N-n}$ it follows that $(b_j-d_j)(-1)^j\sigma\geqq 0, j=1,\cdots,N-n$, and not all the $(b_j-d_j)_1^{N-n}$ are zero (in fact all are nonzero). Thus

$$\sum_{j=1}^{N-n}|b_j|=\sum_{j=1}^{N-n}b_j(-1)^j\sigma>\sum_{j=1}^{N-n}d_j(-1)^j\sigma=\sum_{j=1}^{N-n}|d_j|.$$

Therefore $f^{*\prime}(t_i^*)=0$, $i=2,\cdots,N-1$.

It remains to prove that f^* is strictly monotone on $[t_i^*,t_{i+1}^*]$ for each $i=1,\cdots,N-1$. Using the fact that $t_{i+1}^*<\xi_i<t_{i+n-1}^*$, $i=1,\cdots,N-n$, it follows that $f^{*\prime}(x)$ has no zeros in $[0,1]$ other than $t_2^*,\cdots,t_{N-1}^*$. Thus f^* oscillates strictly between the $(e_i)_1^N$ on $(t_i^*)_1^N$. $\square$

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