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A VARIATION ON THE CHEBYSHEV THEORY OF BEST APPROXIMATION

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Summary. A new method of approximation is proposed which maintains much of the essentials of the classical theory of best uniform approximation, while also using an L^q-type measure of approximation.

1. Introduction. The classical Chebyshev theory of best uniform approximation to continuous functions by polynomials of degree $\leqq n$ has a distinct advantage over the corresponding ones for the L^q-norms, $1 \leqq q < \infty$, in that the unique best approximant is characterized by the remarkable geometric property of the error function, exhibiting $n+2$ points of equioscillation. In this paper we propose a new method of approximation which maintains the geometric flavour of Chebyshev's characterization of the best uniform approximant, while also using an L^q-type $(1 \leqq q < \infty)$ measure of approximation. We must, however, pay a price and the main cost is that our 'distance' function is not derived from a norm. However we, perhaps surprisingly, do maintain the uniqueness of a best approximant (when it exists) and we are able to give a fairly simple characterization thereof.

2. Existence. For the sake of simplicity we shall use an L^1-type measure of approximation and shall deal only with approximation from π_n, the set of all real algebraic polynomials of degree $\leqq n$. The same results hold, if we use an L^q-type measure $(1 < q < \infty)$, and consider approximations from a Chebyshev subspace of, say, $C[0, 1]$.

Let $f \in C[0, 1]$. We define two quantities.

$$(1) \quad |||f||| = \max \{ | \int_a^b f(x)dx | : 0 \leqq a \leqq b \leqq 1, f(x) > 0 \text{ on } (a, b), \text{ or } f(x) < 0 \text{ on } (a, b) \}.$$

$$(2) \quad |||f|||_* = \max \{ | \int_a^b f(x)dx | : 0 \leqq a \leqq b \leqq 1, f(x) \geqq 0 \text{ on } (a, b), \text{ or } f(x) \leqq 0 \text{ on } (a,b) \}.$$

In words, $|||f|||$ is the area of the largest positive or negative hump of $f(x)$, while $|||f|||_*$ is the area of the largest non-negative or non-positive hump of $f(x)$. (The maximum in (1) and (2) is attained.)

The following result is easily proved.

Lemma 1. *For every* $f \in C[0, 1]$

a) $|||f|||=0$ *if and only if* $f(x)=0$ *for all* $x \in [0, 1]$, *and similarly for* $|||f|||_*$;

b) *if* $c \in \mathbf{R}$, $|||cf|||=|c|\,|||f|||$, $|||cf|||_*=|c|\,|||f|||_*$;

c) neither $|||\cdot|||$, nor $|||\cdot|||_*$ satisfies the triangle inequality.

Thus, neither $|||\cdot|||$, nor $|||\cdot|||_*$ is a norm. The following result will allow us to deduce existence or non-existence of a best approximant.

Proposition 2. *Assume* $f, f_m \in C[0, 1]$, $m=1, 2, \ldots$, *and that* $f_m \to f$, *uniformly on* $[0, 1]$. *Then* $|||f||| \leqq \underline{\lim}_{m\to\infty} |||f_m||| \leqq \overline{\lim}_{m\to\infty} |||f_m|||_* \leqq |||f|||_*$.

Examples exist, for which strict inequality holds in each of the above inequalities.

Proposition 3. *Let* $f \in C[0, 1]$ *and let* $n \geqq 0$.

a) *There exists a* $p^* \in \pi_n$, *for which* $\inf\{|||f-p|||: p \in \pi_n\}=|||f-p^*|||$;

b) $\inf_{p \in \pi_n} |||f-p|||_*$ *may be unattained.*

3. Characterization and Uniqueness. If I, J are subintervals of $[0, 1]$, then by $I<J$ we mean that $x<y$ for all $x \in I$ and $y \in J$. We now state our first main result.

Theorem 4. *Let* $f \in C[0, 1]$ *and let* $n \geqq 0$. *There is a unique* $p^* \in \pi_n$ *satisfying* $\inf\{|||f-p|||: p \in \pi_n\}=|||f-p^*|||$. *This* p^* *is characterized by the following property:*

There exist $n+2$ *disjoint (nonempty) open intervals* $I_1<\cdots<I_{n+2}$ *and* $\sigma=+1$ *or* -1, *fixed, such that, for* $k=1, \ldots, n+2$,

a) $(-1)^k\sigma(f-p^*) \geqq 0$ *on* I_k;

b) $(-1)^k\sigma\int_{I_k}(f-p^*)(x)dx \geqq |||f-p^*|||$.

Since $\inf\{|||f-p|||_*: p \in \pi_n\}$ may be unattained, it is surprising, but true that it can be uniquely characterized, if it exists.

Theorem 5. *Let* $f \in C[0, 1]$, *and let* $n \geqq 0$. *Then*

$$\inf\{|||f-p|||_*: p \in \pi_n\} \tag{3}$$

is attained if and only if there exist $n+2$ *disjoint (nonempty) open intervals* $I_1<\cdots<I_{n+2}$, *and* $\sigma=+1$ *or* -1, *fixed, such that for* $k=1, \ldots, n+2$,

a) $(-1)^k\sigma(f-p^*) \geqq 0$ *on* I_k:

b) $(-1)^k\sigma\int_{I_k}(f-p^*)(x)dx=|||f-p^*|||_*$,

where p^* *is the unique minimal polynomial given in Theorem* 4. *If the infimum in* (3) *is attained, then it is attained by* p^* *only.*

4. Additional Results. Given $n \geqq 0$, for each $f \in C[0, 1]$, let $p(f: x) \in \pi_n$ denote the unique best approximant to f from π_n with respect to $|||\cdot|||$. Since $|||\cdot|||$ is not a continuous mapping of $C[0, 1]$ into the reals, it is natural to ask whether $p(f; x)$ may be viewed as a continuous mapping into $C[0, 1]$. We have

Theorem 6. *Let* $f, f_m \in C[0, 1]$, $m=1, 2, \ldots$, *and assume that* f_m *converges uniformly to* f. *Then* $p(f_m; x)$ *converges uniformly to* $p(f; x)$.

One of the advantages in the use of uniform approximation is the fairly simple bounds on the value of the error which were given by de la Vallée Poussin. Analogues of this fundamental result hold here as well.

Theorem 7. *Assume* $f \in C[0, 1]$, $n \geqq 0$ *and* $p \in \pi_n$. *Suppose* $\sigma=\pm 1$, $J_1<\cdots<J_{n+2}$ *are disjoint (nonempty) open intervals of* $[0, 1]$ *and* $(-1)^k\sigma(f-p) \geqq 0$ *on* J_k, $k=1, \ldots, n+2$.

a) *If* $p \neq p^*$ *of Theorem* 4, *then* $|||f-p^*||| > \min_{1 \leq k \leq n+2} (-1)^k \sigma \int_{J_k} (f-p)(x)dx$.

b) $\inf_{q \in \pi_n} |||f-q|||_* > \min_{1 \leq k \leq n+2} (-1)^k \sigma \int_{J_k} (f-p)(x)dx$.

In addition we have the following analogue of Bernstein's comparison theorem.

Theorem 8. *Suppose* $f, g \in C[0, 1]$ *and* $n \geq 0$. *Assume that* $g^{(n+1)}$ *exists,* $f^{(n+1)} > 0$, *and* $|g^{(n+1)}(x)| \leq f^{(n+1)}(x)$ *throughout* $(0, 1)$. *Then* $\min_{p \in \pi_n} |||g-p||| \leq \min_{p \in \pi_n} |||f-p|||$.

For full details see: A. Pinkus and O. Shisha: Variations on the Chebyshev and L^q theories of best approximation. *J. Approx. Theory,* **35,** 1982, 148-168.

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