NORTH-HOLLAND

Functions That Preserve Families of Positive Semidefinite Matrices

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ABSTRACT

We study various notions of multivariate functions which map families of positive semidefinite matrices or of conditionally positive semidefinite matrices into matrices of the same type.

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1. INTRODUCTION

In this work, we study the following problem. For $k = 1, \ldots, m$, let $A_k = (a_{ij}^k)_{i,j=1}^n$ be distinct $n \times n$ real symmetric (Hermitian) positive semidefinite matrices. Characterize the multivariate real-valued functions $f : \mathbb{R}^m \to \mathbb{R}$ (complex-valued functions $f : \mathbb{C}^m \to \mathbb{C}$) such that the matrix

$$(f(A_1, \ldots, A_m))_{ij} := f\big(a_{ij}^1, \ldots, a_{ij}^m\big), \qquad 1 \le i, j \le n,$$

is also always positive semidefinite (Hermitian positive semidefinite). In the real case for $m = 1$, a solution to this problem was first given by Schoenberg [14], by totally different methods. Similar questions in the real case for $m = 1$ were studied in Christensen and Ressel [1], FitzGerald and Horn [4], and Rudin [11]. The complex case for $m = 1$ was considered by Hertz [5]. Variations of this problem will also be studied here. The motivation for the study of the problem came from certain problems in multivariate interpolation; see Micchelli [8] and Powell [9].

2. THE REAL CASE

We denote by $P_{\mathbb{R}}^n$ the class of all real $n \times n$ positive semidefinite matrices, and by $F_{\mathbb{R}}^m$ the class of all real-valued functions $f : \mathbb{R}^m \to \mathbb{R}$ such that for all $n, f(A_1, \ldots, A_m) \in P_{\mathbb{R}}^n$ whenever $A_1, \ldots, A_m \in P_{\mathbb{R}}^n$. A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be *real entire* if it is real on $\mathbb{R}^m$ and is the restriction to $\mathbb{R}^m$ of an entire function on $\mathbb{C}^m$. We also use $\mathbb{Z}_+^m$ and $\mathbb{R}_+^m$ for all vectors in $\mathbb{Z}^m$, $\mathbb{R}^m$, respectively, with nonnegative coordinates. Finally, for $\mathbf{x} \in \mathbb{R}^m$ and $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$, we use the standard notation $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

The main result of this section is the following.

THEOREM 2.1. *Let $f : \mathbb{R}^m \to \mathbb{R}$. Then $f \in F_{\mathbb{R}}^m$ if and only if f is real entire of the form*

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}, \qquad \mathbf{x} \in \mathbb{R}^m, \tag{2.1}$$

where $c_{\boldsymbol{\alpha}} \ge 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$.

We remark that for $f \in F_{\mathbb{R}}^1$ the positive semidefinite kernel $K(x, y) = f((x, y))$, where x, y are elements in some Hilbert space H, arises as the reproducing kernel of a generalized weighted Fock space which is useful for certain estimation and prediction problems of nonlinear system and signal analysis (de Figueiredo [2]). This paper gives an explicit description of the associated Hilbert space for which K is a reproducing kernel. The case where $H = L^2[0,1]$ and $f(x) = e^x, x \in \mathbb{R}$, is especially important.

It is convenient to divide the proof of the above theorem (especially the necessity) into a series of facts. Firstly, we prove the sufficiency of (2.1), which is elementary. It is based on the following simple lemma.

LEMMA 2.2. *Assume that* $f, g \in F_{\mathbb{R}}^m$, $h \in F_{\mathbb{R}}^1$, *and* $a, b \geq 0$. *Then*

(i) $af + bg \in F_{\mathbb{R}}^m$;
(ii) $fg \in F_{\mathbb{R}}^m$;
(iii) $h \circ f \in F_{\mathbb{R}}^m$, *where* $(h \circ f)(\mathbf{x}) := h(f(\mathbf{x})), \mathbf{x} \in \mathbb{R}^m$;
(iv) *all affine functions nonnegative on* $\mathbb{R}_+^m$ *are in* $F_{\mathbb{R}}^m$;
(v) $F_{\mathbb{R}}^m$ *is closed under pointwise convergence.*

Proof. (i) is a consequence of the cone property of $P_{\mathbb{R}}^n$, while (v) is a consequence of its closure. (ii) follows from the Schur-product theorem. Namely, if $A, B \in P_{\mathbb{R}}^n$, $A = (a_{ij})$, $B = (b_{ij})$, then the Schur product of A and B given by $(a_{ij}b_{ij})$ is in $P_{\mathbb{R}}^n$ (see e.g. Horn and Johnson [6, p. 458]). Both (iii) and (iv) hold essentially by definition. ■

As an immediate consequence, we have:

PROPOSITION 2.3. *If* $f : \mathbb{R}^m \to \mathbb{R}$ *is real entire of the form* (2.1), *with* $c_{\boldsymbol{\alpha}} \geq 0$ *for all* $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$, *then* $f \in F_{\mathbb{R}}^m$.

To prove the necessity portion of Theorem 2.1, we start with

PROPOSITION 2.4. *Assume that* $f \in F_{\mathbb{R}}^m \backslash \{0\}$. *Then:*

(i) $f(\mathbf{x}) \geq 0$ *for* $\mathbf{x} \geq \mathbf{0}$, *and* $f(\mathbf{x}) > 0$ *for* $\mathbf{x} > \mathbf{0}$.
(ii) f *is nondecreasing (elementwise) on* $\mathbb{R}_+^m$.
(iii) *Every* $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m)$, $\eta_k \in \{-1, 1\}$, $k = 1, \ldots, m$, *determines a linear map:* $\eta(\mathbf{x}) := \boldsymbol{\eta}\mathbf{x} := (\eta_1 x_1, \ldots, \eta_m x_m)$, $\mathbf{x} \in \mathbb{R}^m$. *For all such* $\boldsymbol{\eta}$ *we have* $f \pm f \circ \eta \in F_{\mathbb{R}}^m$.
(iv) *For all* $\mathbf{c} \geq \mathbf{0}$ *set* $(E_{\mathbf{c}}f)(\mathbf{x}) := f(\mathbf{x} + \mathbf{c}), \mathbf{x} \in \mathbb{R}^m$. *Then* $E_{\mathbf{c}}f \in F_{\mathbb{R}}^m$.
(v) $f \in C(\mathbb{R}^m)$.

Proof. (i): If $A \in P_{\mathbb{R}}^n$, then $a_{ii} \geq 0, i = 1, \ldots, n$. From this fact we immediately obtain that $f(\mathbf{x}) \geq 0$ for $\mathbf{x} \geq \mathbf{0}$ (i.e., $x_k \geq 0,\ k = 1, \ldots, m$).

For the second claim of (i) we assume to the contrary that $\mathbf{x} > \mathbf{0}$ and $f(\mathbf{x}) = 0$. Since $f \neq 0$, there exists a $\mathbf{y} \in \mathbb{R}^m$ for which $f(\mathbf{y}) \neq 0$. Set

$$A_k = \begin{pmatrix} x_k & y_k \\ y_k & y_k^2/x_k \end{pmatrix}, \qquad k = 1, \ldots, m.$$

Since $A_k \in P_{\mathbb{R}}^2$, $k = 1, \ldots, m$, we have that

$$\begin{pmatrix} f(\mathbf{x}) & f(\mathbf{y}) \\ f(\mathbf{y}) & f(\mathbf{y}^2/\mathbf{x}) \end{pmatrix} \in P_{\mathbb{R}}^2.$$

Therefore, it follows that $0 = f(\mathbf{x})f(\mathbf{y}^2/\mathbf{x}) \geq f^2(\mathbf{y}) > 0$, which is a contradiction.

(ii): Let $\mathbf{x} \geq \mathbf{y} \geq \mathbf{0}$ (i.e., $x_k \geq y_k \geq 0$, $k = 1, \ldots, m$). Since

$$\begin{pmatrix} x_k & y_k \\ y_k & x_k \end{pmatrix} \in P_{\mathbb{R}}^2$$

for each k, we have

$$\begin{pmatrix} f(\mathbf{x}) & f(\mathbf{y}) \\ f(\mathbf{y}) & f(\mathbf{x}) \end{pmatrix} \in P_{\mathbb{R}}^2$$

and therefore $f(\mathbf{x}) \geq f(\mathbf{y})$.

(iii): For $\eta_k \in \{-1, 1\}$, $k = 1, \ldots, m$, we have that

$$\begin{pmatrix} 1 & \eta_k \\ \eta_k & 1 \end{pmatrix} \in P_{\mathbb{R}}^2.$$

Let $A_1, \ldots, A_m \in P_{\mathbb{R}}^n$. Then

$$\begin{pmatrix} A_k & \eta_k A_k \\ \eta_k A_k & A_k \end{pmatrix} \in P_{\mathbb{R}}^{2n}.$$

for each $k = 1, \ldots, m$, and consequently we conclude that

$$C = \begin{pmatrix} f(A_1, \ldots, A_m) & f(\eta_1 A_1, \ldots, \eta_m A_m) \\ f(\eta_1 A_1, \ldots, \eta_m A_m) & f(A_1, \ldots, A_m) \end{pmatrix} \in P_{\mathbb{R}}^{2n}.$$

Set

$$Q_{\pm} = \begin{pmatrix} I & 0 \\ \pm I & I \end{pmatrix} \in \mathbb{R}^{2n \times 2n},$$

where I is the $n \times n$ identity matrix, and observe that

$$D_{\pm} = Q_{\pm}^T C Q_{\pm} \in P_{\mathbb{R}}^{2n}.$$

The principal submatrix of $D_{\pm}$ determined by its first n rows and n columns is a matrix in $P_{\mathbb{R}}^n$. This submatrix is given by

$$2[f(A_1, \ldots, A_m) \pm f(\eta_1 A_1, \ldots, \eta_m A_m)].$$

Thus, we have shown that whenever $A_1, \ldots, A_m \in P_{\mathbb{R}}^n$, then

$$f(A_1, \ldots, A_m) \pm f(\eta_1 A_1, \ldots, \eta_m A_m) \in P_{\mathbb{R}}^n,$$

which simply means that $f \pm f \circ \eta \in F_{\mathbb{R}}^m$.

(iv): Let J denote the $n \times n$ matrix all of whose entries are one. Then $J \in P_{\mathbb{R}}^n$, and for any $A_1, \ldots, A_m \in P_{\mathbb{R}}^n$ and $\mathbf{c} \geq \mathbf{0}, A_k + c_k J \in P_{\mathbb{R}}^n$ for each $k = 1, \ldots, m$. Therefore

$$f(A_1 + c_1 J, \ldots, A_m + c_m J) \in P_{\mathbb{R}}^n.$$

Thus $E_{\mathbf{c}} f \in F_{\mathbb{R}}^m$ for any $\mathbf{c} \geq \mathbf{0}$.

(v): Let $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$. Set

$$A_k = \begin{pmatrix} x_k & \sqrt{x_k y_k} \\ \sqrt{x_k y_k} & x_k \end{pmatrix}, \qquad k = 1, \ldots, m.$$

Each $A_k \in P_{\mathbb{R}}^2$, and therefore, setting $\mathbf{xy} = (x_1 y_1, \ldots, x_m y_m)$, we obtain

$$\begin{pmatrix} f(\mathbf{x}) & f\big((\mathbf{xy})^{1/2}\big) \\ f\big((\mathbf{xy})^{1/2}\big) & f(\mathbf{x}) \end{pmatrix} \in P_{\mathbb{R}}^2,$$

which implies that

$$f(\mathbf{x}) f(\mathbf{y}) \geq f^2\big((\mathbf{xy})^{1/2}\big) \tag{2.2}$$

for all $\mathbf{x}, \mathbf{y} > \mathbf{0}$. Since $f(\mathbf{w}) > 0$ for $\mathbf{w} > \mathbf{0}$ [by (i)], we can define

$$g(\mathbf{x}) = \ln f(e^{\mathbf{x}}),$$

where $e^{\mathbf{x}} = (e^{x_1}, \ldots, e^{x_m})$ for all $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$. From (2.2), we obtain

$$\frac{g(\mathbf{x}) + g(\mathbf{y})}{2} \geq g\left(\frac{\mathbf{x} + \mathbf{y}}{2}\right) \tag{2.3}$$

for all $\mathbf{x}, \mathbf{y} \in R^m$. That is, g is *midconvex* on $\mathbb{R}^m$. From (ii), f is nondecreasing on $\mathbb{R}_+^m$. Thus g is nondecreasing (elementwise) and midconvex on $\mathbb{R}^m$. Any such function (see e.g. Roberts and Varberg [10]) is necessarily convex and hence continuous on the interior of its domain of definition. Thus $g \in C(\mathbb{R}^m)$, which implies that $f \in C(\text{int } \mathbb{R}_+^m)$. In summary, we have observed that if $f \in F_{\mathbb{R}}^m$, then $f \in C(\text{int } \mathbb{R}_+^m)$. Applying (iii) and (iv), it now follows that $f \in C(\mathbb{R}^m)$. ■

We will first prove the result under the further assumption that f is in fact in $C^\infty(\mathbb{R}^m)$.

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_+^m$, we use the notation

$$(D^\alpha f)(\mathbf{x}) = \frac{\partial^{\alpha_1 + \cdots + \alpha_m}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} f(x_1, \ldots, x_m).$$

PROPOSITION 2.5. *Assume $f \in F_{\mathbb{R}}^m \cap C^\infty(\mathbb{R}^m)$. Then*

$$(D^\alpha f)(\mathbf{x}) \geq 0$$

for all $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$ and $\mathbf{x} \geq \mathbf{0}$.

Proof. Let $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_m = r$. For $\mathbf{x} = (x_1, \ldots, x_m) \geq \mathbf{0}$ and for any set of real numbers $\{y_{ik} : i = 1, \ldots, n,\ k = 1, \ldots, m\}$, the matrices $A_k(\varepsilon) = (a_{ij}^k(\varepsilon)),\ i, j = 1, \ldots, n,\ k = 1, \ldots, m$, defined by

$$a_{ij}^k(\varepsilon) = x_k + \varepsilon y_{ik} y_{jk}$$

are in $P_{\mathbb{R}}^n$ for all $\varepsilon \geq 0$. Thus $f(A_1(\varepsilon), \ldots, A_m(\varepsilon)) \in P_{\mathbb{R}}^n$, and for every $\mathbf{w} \in \mathbb{R}^n$,

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^n w_i f\big(a_{ij}^1(\varepsilon), \ldots, a_{ij}^m(\varepsilon)\big) w_j \\ &= \sum_{i,j=1}^n w_i f(x_1 + \varepsilon y_{i1} y_{j1}, \ldots, x_m + \varepsilon y_{im} y_{jm}) w_j. \end{aligned}$$

Since $f \in C^\infty(\mathbb{R}^m)$, we have from Taylor's theorem

$$\begin{aligned} &f(x_1 + \varepsilon y_{i1} y_{j1}, \ldots, x_m + \varepsilon y_{im} y_{jm}) \\ &\quad = f(x_1, \ldots, x_m) + \cdots + \frac{\varepsilon^r}{r!} \sum_{|\beta| = r} \binom{r}{\boldsymbol{\beta}} (y_{i1} y_{j1})^{\beta_1} \cdots (y_{im} y_{jm})^{\beta_m} \\ &\quad \times D^\beta f(x_1, \ldots, x_m) + O(\varepsilon^{r+1}). \end{aligned}$$

For n sufficiently large, we can choose the $\{y_{ik} : i = 1, \ldots, n,\ k = 1, \ldots, m\}$ and $\{w_i : i = 1, \ldots, n\}$ so that

$$\sum_{i=1}^{n} w_i y_{i1}^{\gamma_1} \cdots y_{im}^{\gamma_m} = 0$$

for all $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{Z}_+^m$ satisfying $|\gamma| \leq r$, except that for $\gamma = \alpha$ we require

$$\sum_{i=1}^{n} w_i y_{i1}^{\alpha_1} \cdots y_{im}^{\alpha_m} \neq 0.$$

(See Remark 2.1 below for a construction of such $\{y_{ik}\}$ and $\{w_i\}$.) Thus,

$$\begin{aligned} 0 \ &\leq \ \sum_{i,j=1}^{n} w_i f(x_1 + \varepsilon y_{i1} y_{j1}, \ldots, x_m + \varepsilon y_{im} y_{jm}) w_j \\ &= \frac{\varepsilon^r}{r!} \binom{r}{\alpha} D^{\alpha} f(x_1, \ldots, x_m) \left(\sum_{i=1}^{n} w_i y_{i1}^{\alpha_1} \cdots y_{im}^{\alpha_m} \right)^2 + O(\varepsilon^{r+1}). \end{aligned}$$

Letting $\varepsilon \downarrow 0$, it then follows that $D^{\alpha} f(\mathbf{x}) \geq 0$. ■

REMARK 2.1. There are numerous methods of constructing the $\{y_{ik} : i = 1, \ldots, n,\ k = 1, \ldots, m\}$ and $\{w_i : i = 1, \ldots n\}$ as above. Here is one such way for n sufficiently large. Let $a_1, \ldots, a_n$ be any n distinct positive numbers. Choose any m positive numbers $c_1, \ldots, c_m$ such that the values $(\boldsymbol{\beta}, \mathbf{c}) = \sum_{k=1}^{m} \beta_k c_k$ are distinct for all $\boldsymbol{\beta} \in \mathbb{Z}_+^m$ satisfying $|\boldsymbol{\beta}| \leq r$. Now, set

$$y_{ik} = a_i^{c_k}, \qquad i = 1, \ldots, n, \quad k = 1, \ldots, m.$$

Recall that for every positive integer s the functions x^{σ_j}, $j = 1, \ldots, s$, constitute a T-system on $(0, \infty)$ for any distinct $\sigma_1, \ldots, \sigma_s$. That is, the functions x^{σ_j}, $j = 1, \ldots, s$, are linearly independent on every s distinct points in $(0, \infty)$. Thus it follows that the vectors

$$\left\{ \left(y_{11}^{\beta_1} \cdots y_{1m}^{\beta_m}, \ldots, y_{n1}^{\beta_1} \cdots y_{nm}^{\beta_m} \right) : |\boldsymbol{\beta}| \leq r \right\} = \left\{ \left(a_1^{(\boldsymbol{\beta}, \mathbf{c})}, \ldots, a_n^{(\boldsymbol{\beta}, \mathbf{c})} \right) : |\boldsymbol{\beta}| \leq r \right\}$$

are linearly independent in $\mathbb{R}^n$. For

$$n \geq \binom{r+m}{m}$$

this linear independence allows us to construct the requisite $\{w_i : i = 1, \ldots, n\}$.

A function $f \in C^\infty[a,b)$ is said to be *absolutely monotone* if $f^{(k)}(x) \geq 0$ for all $x \in [a,b)$ and each $k \in \mathbb{Z}_+$. This class of functions was introduced by Bernstein (see Widder [15, Chapter IV] for an excellent exposition). If f is absolutely monotone on $[a,b)$, then it is known that f is real analytic thereon. That is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

for all $x \in [a,b)$, and thus f is the restriction to $[a,b)$ of a function analytic in $\{z : |z-a| < b-a\}$.

This result has been extended to domains in $\mathbb{R}^2$ (and the case $\mathbb{R}^m$ follows analogously) by Schoenberg [13]. This leads us to

PROPOSITION 2.6. *If $f \in F_{\mathbb{R}}^m \cap C^\infty(\mathbb{R}^m)$, then f is real entire and*

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}, \tag{2.4}$$

where $c_{\boldsymbol{\alpha}} = (D^{\boldsymbol{\alpha}} f)(\mathbf{0})/\boldsymbol{\alpha}! \geq 0$ for each $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$.

Proof. From Schoenberg [13, Theorem 5.2] and Proposition 2.5, we see that f has the form (2.4) for $\mathbf{x} \geq \mathbf{0}$. To extend this result to all $\mathbf{x} \in \mathbb{R}^m$, we apply (iii) and (iv) of Proposition 2.4. ■

We gather the remaining steps in the proof of Theorem 2.1 under one heading.

Proof of Theorem 2.1. We assume that $f \in F_{\mathbb{R}}^m$, and wish to prove that it is of the form (2.4). From (v) of Proposition 2.4 we have that $f \in C(\mathbb{R}^m)$. Let $\phi \in C_0^\infty(I^m)$, where $I = (-1, 0)$. That is, ϕ is a $C^\infty(\mathbb{R}^m)$ function whose support lies in I^m. Assume that ϕ is a density function so that $\phi(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^m$, and

$$\int_{\mathbb{R}^m} \phi(\mathbf{x})\, d\mathbf{x} = 1.$$

For $\varepsilon > 0$, set

$$f_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon^m} \int_{\mathbb{R}^m} f(\mathbf{x}+\mathbf{y})\phi(-\mathbf{y}/\varepsilon) d\mathbf{y}. \tag{2.5}$$

It is easy to see that $f_\varepsilon \in C^\infty(\mathbb{R}^m)$ and $\lim_{\varepsilon \to 0^+} f_\varepsilon = f$, where the convergence is uniform on compact subsets of $\mathbb{R}^m$.

By definition, $\phi(-\mathbf{y}/\varepsilon) > 0$ only if $0 < y_k < \varepsilon$, $k = 1, \ldots, m$. From (iv) of Proposition 2.4, $E_{\mathbf{y}} f \in F_{\mathbb{R}}^m$ for each such $\mathbf{y}$. Thus, if $A_k = (a_{ij}^k) \in P_{\mathbb{R}}^n$, $k = 1, \ldots, m$, then for each $\mathbf{w} \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} w_i f_\varepsilon\big(a_{ij}^1, \ldots, a_{ij}^m\big) w_j$$
$$= \frac{1}{\varepsilon^m} \int_{\mathbb{R}^m} \left(\sum_{i,j=1}^{m} w_i f\big(a_{ij}^1 + y_1, \ldots, a_{ij}^m + y_m\big) w_j \right) \phi(-\mathbf{y}/\varepsilon)\, d\mathbf{y} \geq 0,$$

implying that $f_\varepsilon \in F_{\mathbb{R}}^m$. Since $f_\varepsilon \in F_{\mathbb{R}}^m \cap C^\infty(\mathbb{R}^m)$, we have from Proposition 2.6 that f_ε real entire and

$$f_\varepsilon(\mathbf{x}) = \sum_{\alpha \in \mathbb{Z}_+^m} c_{\alpha}^{\varepsilon} \mathbf{x}^{\alpha},$$

where $c_{\alpha}^{\varepsilon} \geq 0$ for all $\alpha \in \mathbb{Z}_+^m$. Call the right-hand side of the above equation $h_\varepsilon(\mathbf{x})$. Then h_ε is an entire function on $\mathbb{C}^m$.

Since f_ε converges uniformly on compact subsets of $\mathbb{R}^m$ and for every $\mathbf{z} \in \mathbb{C}^m$ we have $|h_\varepsilon(z_1, \ldots, z_m)| \leq f_\varepsilon(|z_1|, \ldots, |z_m|)$, the set $\{h_\varepsilon : \varepsilon > 0\}$ forms a normal family on any bounded subset of $\mathbb{C}^m$; see e.g. Rudin [12, p. 272]. Consequently, some subsequence converges, on compact subsets of $\mathbb{C}^m$, to an entire function h. But f and h agree on $\mathbb{R}^m$, since f_ε and h_ε agree on $\mathbb{R}^m$. Therefore f is of the desired form (2.1). ■

In the last part of this section, we present variations of Theorem 2.1 for conditionally positive semidefinite matrices. By $Q_{\mathbb{R}}^n$ we denote the set of all $n \times n$ matrices $A = (a_{ij})_{i,j=1}^n$ which are real symmetric and satisfy

$$\sum_{i,j=1}^{n} w_i a_{ij} w_j \geq 0$$

provided that $\sum_{i=1}^n w_i = 0$. Such matrices are said to be *conditionally positive semidefinite*. Let $G_{\mathbb{R}}^m$ denote the class of functions $f : \mathbb{R}^m \to \mathbb{R}$ for which $f(A_1, \ldots, A_m) \in Q_{\mathbb{R}}^n$ if $A_1, \ldots, A_m \in P_{\mathbb{R}}^n$, for any n.

THEOREM 2.7. *A function $f : \mathbb{R}^m \to \mathbb{R}$ is in $G_{\mathbb{R}}^m$ if and only if f is real entire of the form*

$$f(\mathbf{x}) = \sum_{\alpha \in \mathbb{Z}_+^m} c_{\alpha} \mathbf{x}^{\alpha}, \qquad \mathbf{x} \in \mathbb{R}^m, \tag{2.6}$$

where $c_{\boldsymbol{\alpha}} \geq 0$ *for all* $\boldsymbol{\alpha} \in \mathbb{Z}_+^m \backslash \{\mathbf{0}\}$.

Proof. Assume $f \in G_{\mathbb{R}}^m$. For each $A_1, \ldots, A_m \in P_{\mathbb{R}}^n$ set

$$B_k := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_k & \\ 0 & & & \end{bmatrix}, \qquad k = 1, \ldots, m.$$

Thus $B_1, \ldots, B_m \in P_{\mathbb{R}}^{n+1}$, which implies that $f(B_1, \ldots, B_m) \in Q_{\mathbb{R}}^{n+1}$. It follows that

$$f(A_1, \ldots, A_m) - f(0, \ldots, 0)J \in P_{\mathbb{R}}^n,$$

and so we have established that $f - f(\mathbf{0}) \in F_{\mathbb{R}}^m$. Applying Theorem 2.1 to the function $f - f(\mathbf{0})$ we obtain (2.6).

Conversely, any f of the form (2.6) can be written as $f(\mathbf{x}) = f(\mathbf{0}) + g(\mathbf{x})$ where $g \in F_{\mathbb{R}}^m$, so that $f(A_1, \ldots, A_m) = f(\mathbf{0})J + g(A_1, \ldots, A_m)$. Theorem 2.1 implies $g(A_1, \ldots, A_m) \in P_{\mathbb{R}}^n$ and consequently $f(A_1, \ldots, A_m) \in Q_{\mathbb{R}}^n$. This proves the theorem. ■

For our next result we let $H_{\mathbb{R}}^m$ be the class of functions $f : \mathbb{R}^m \to \mathbb{R}$ for which $f(A_1, \ldots, A_m) \in P_{\mathbb{R}}^n$ if $A_1, \ldots, A_m \in Q_{\mathbb{R}}^n$, for any n.

THEOREM 2.8. *A function* $f : \mathbb{R}^m \to \mathbb{R}$ *is in* $H_{\mathbb{R}}^m$ *if and only if* f *is real entire and* $(D^{\boldsymbol{\alpha}} f)(\mathbf{x}) \geq 0$ *for all* $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$ *and every* $\mathbf{x} \in \mathbb{R}^m$.

REMARK 2.2. As we already pointed out, Schoenberg [13] showed that any $f \in C^\infty(\mathbb{R}^m)$ with $(D^{\boldsymbol{\alpha}} f)(\mathbf{x}) \geq 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$ and $\mathbf{x} \in \mathbb{R}^m$ is real entire. Another equivalent integral representation will be described in the proof of the theorem.

Proof. Suppose first that $f \in H_{\mathbb{R}}^m$. Since $P_{\mathbb{R}}^n \subseteq Q_{\mathbb{R}}^n$, it follows that $f \in F_{\mathbb{R}}^m$. Thus from Theorem 2.1, f is real entire and $(D^{\boldsymbol{\alpha}} f)(\mathbf{x}) \geq 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$ and $\mathbf{x} \in \mathbb{R}_+^m$. To extend these inequalities we observe that whenever $A \in Q_{\mathbb{R}}^n$ it follows that $A + cJ \in Q_{\mathbb{R}}^n$ for any $c \in \mathbb{R}$. Thus it follows that $E_{\mathbf{c}} f \in H_{\mathbb{R}}^m$ and so $E_{\mathbf{c}} f \in F_{\mathbb{R}}^m$. Consequently, $(D^{\boldsymbol{\alpha}} f)(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^m$ and $\boldsymbol{\alpha} \in \mathbb{Z}_+^m$.

Conversely, from Schoenberg [13, Theorem 5.1] (by replacing x with $-x$) it follows that

$$f(\mathbf{x}) = \int_{\mathbb{R}_+^m} e^{(\mathbf{x}, \mathbf{t})} d\sigma(\mathbf{t}), \tag{2.7}$$

where the integral is an improper Stieltjes integral which is absolutely convergent in $\mathbb{R}^m$, and $d\sigma$ is a nonnegative measure on $\mathbb{R}_+^m$. Schoenberg actually proved this only for $m = 2$, but the proof extends to any m. For $m = 1$, this is a famous result of Bernstein; cf. Widder [15].

To make use of this representation we recall that $A \in Q_{\mathbb{R}}^n$ if and only if $e^{tA} \in P_{\mathbb{R}}^n$ for all $t \geq 0$; cf. Donoghue [3]. If $A_1, \ldots, A_m \in Q_{\mathbb{R}}^n$ then $\sum_{j=1}^m t_j A_j \in Q_{\mathbb{R}}^n$ for all $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{R}_+^m$. Hence, $e^{\sum_{j=1}^n t_j A_j} \in P_{\mathbb{R}}^n$, which by (2.7) implies that $f(A_1, \ldots, A_m) \in P_{\mathbb{R}}^n$. This proves the theorem. ∎

Finally, we let $I_{\mathbb{R}}^m$ denote the class of functions $f: \mathbb{R}^m \to \mathbb{R}$ for which $f(A_1, \ldots, A_m) \in Q_{\mathbb{R}}^n$ if $A_1, \ldots, A_m \in Q_{\mathbb{R}}^n$, for any n.

THEOREM 2.9. *A function $f: \mathbb{R}^m \to \mathbb{R}$ is in $I_{\mathbb{R}}^m$ if and only if f is real entire and $(D^{\boldsymbol{\alpha}} f)(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^m$ and $\boldsymbol{\alpha} \in \mathbb{Z}_+^m \backslash \{\mathbf{0}\}$.*

Proof. As in the proof of the previous result, it is easily verified that whenever $f \in I_{\mathbb{R}}^m$ then $E_{\mathbf{c}} f \in I_{\mathbb{R}}^m$ for every $\mathbf{c} \in \mathbb{R}^m$. Since $P_{\mathbb{R}}^n \subseteq Q_{\mathbb{R}}^n$, it follows that if $f \in I_{\mathbb{R}}^m$ then $f \in G_R^m$. Thus $E_{\mathbf{c}} f \in G_{\mathbb{R}}^m$ for every $\mathbf{c} \in \mathbb{R}^m$. From Theorem 2.7 we see that f is real entire and $(D^{\boldsymbol{\alpha}} f)(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^m$ and $\boldsymbol{\alpha} \in \mathbb{Z}_+^m \backslash \{\mathbf{0}\}$.

Conversely, assume f is real entire and $D^{\boldsymbol{\alpha}} f(\mathbf{x}) \geq 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}_+^m \backslash \{\mathbf{0}\}$ and $\mathbf{x} \in \mathbb{R}^m$. Since $f(\mathbf{x})$ is nondecreasing in each variable, the function

$$f_{\mathbf{c}}(\mathbf{x}) := f(\mathbf{x}) - f(\mathbf{c})$$

is also nonnegative on $\{\mathbf{x}: \mathbf{x} \geq \mathbf{c}\}$. Thus from Schoenberg [13, Theorem 5.1] $f_{\mathbf{c}}$ admits the representation

$$f_{\mathbf{c}}(\mathbf{x}) = \int_{\mathbb{R}_+^m} e^{(\mathbf{x}, \mathbf{t})} d\sigma(\mathbf{t}),$$

where $d\sigma$ is a nonnegative measure on $\mathbb{R}_+^m$ and the integral converges absolutely in $\{\mathbf{x}: \mathbf{x} \geq \mathbf{c}\}$. As before in Theorem 2.8, we see that

$$f_{\mathbf{c}}(A_1, \ldots, A_m) \in P_{\mathbb{R}}^n \qquad \text{if} \quad A_1, \ldots, A_m \in Q_{\mathbb{R}}^n,$$

where $\mathbf{c}$ is chosen to be any vector so that $a_{ij}^k \geq c_k$, $1 \leq i, j \leq n$, $1 \leq k \leq m$. But clearly,

$$f(A_1, \ldots, A_m) = f(\mathbf{c})J + f_{\mathbf{c}}(A_1, \ldots, A_m) \in Q_{\mathbb{R}}^n,$$

and so $f \in I_{\mathbb{R}}^m$, which proves the theorem. ∎

3. THE COMPLEX CASE

In this section we investigate the analog of some of the results of the previous section for complex matrices. For this purpose, we let $P_{\mathbb{C}}^n$ denote the class of complex $n \times n$ Hermitian positive semidefinite matrices, and $F_{\mathbb{C}}^m$ the class of functions $f: \mathbb{C}^m \to \mathbb{C}$ for which $f(A_1, \ldots, A_m) \in P_{\mathbb{C}}^n$ whenever $A_1, \ldots, A_m \in P_{\mathbb{C}}^n$ for all n.

The main result of this section is the following.

THEOREM 3.1. *Let $f: \mathbb{C}^m \to \mathbb{C}$. Then $f \in F_{\mathbb{C}}^m$ if and only if f has the form*

$$f(\mathbf{z}) = \sum_{\alpha, \beta \in \mathbb{Z}_+^m} c_{\alpha,\beta} \mathbf{z}^{\alpha} \overline{\mathbf{z}}^{\beta}, \qquad \mathbf{z} \in \mathbb{C}^m, \tag{3.1}$$

where $c_{\alpha,\beta} \geq 0$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}_+^m$, and the power series converges absolutely for all $\mathbf{z} \in \mathbb{C}^m$.

We remark, for the case $f \in F_{\mathbb{C}}^1$, that the positive semidefinite kernel $k(\mathbf{z}, \boldsymbol{\zeta}) = f((\mathbf{z}, \boldsymbol{\zeta}))$, $\mathbf{z}, \boldsymbol{\zeta} \in \mathbb{C}^n$, is the reproducing kernel of certain Hilbert spaces of analytic functions on the unit ball in $\mathbb{C}^n$. This identification is useful for computing n-widths of certain classes of analytic functions; see Micchelli [7].

In proving the sufficiency of (3.1), we utilize an analog of Lemma 2.2.

LEMMA 3.2. *Assume that $f, g \in F_{\mathbb{C}}^m$, $h \in F_{\mathbb{C}}^1$, and $a, b \geq 0$. Then:*

(i) $af + bg \in F_{\mathbb{C}}^m$.
(ii) $fg \in F_{\mathbb{C}}^m$.
(iii) $h \circ f \in F_{\mathbb{C}}^m$.
(iv) *All affine functions of the form $a(z_1, \ldots, z_m) = a_0 + \sum_{j=1}^m a_j z_j + \sum_{j=1}^m b_j \overline{z}_j$ where $a_j \geq 0, j = 0, 1, \ldots, m$, and $b_j \geq 0, j = 1, \ldots, m$, are in $F_{\mathbb{C}}^m$.*
(v) *$F_{\mathbb{C}}^m$ is closed under pointwise convergence.*

On the basis of Lemma 3.2, whose proof parallels the proof of Lemma 2.2, we have:

PROPOSITION 3.3. *If $f: \mathbb{C}^m \to \mathbb{C}$ has the form (3.1) with $c_{\alpha,\beta} \geq 0$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}_+^m$ and the power series converging absolutely for all $\mathbf{z} \in \mathbb{C}^m$, then $f \in F_{\mathbb{C}}^m$.*

The more difficult part of Theorem 3.1 is the proof of the fact if $f \in F_{\mathbb{C}}^m$, then f satisfies (3.1). We start with:

LEMMA 3.4. *Let $f \in F_{\mathbb{C}}^m$, and*

$$f(\mathbf{z}) = u(\mathbf{z}) + iv(\mathbf{z}),$$

where $u(\mathbf{z}) = \mathrm{Re}\, f(\mathbf{z})$, and $v(\mathbf{z}) = \mathrm{Im}\, f(\mathbf{z})$. Then for $\mathbf{z} \in \mathbb{C}^m$

(i) $f(\overline{\mathbf{z}}) = \overline{f(\mathbf{z})}$,
(ii) $u(\overline{\mathbf{z}}) = u(\mathbf{z})$,
(iii) $v(\overline{\mathbf{z}}) = -v(\mathbf{z})$,
(iv) $u \in F_{\mathbb{C}}^m$,
(v) $f|_{\mathbb{R}^m} = u|_{\mathbb{R}^m} \in F_{\mathbb{R}}^m$.

Proof. (i): If $A = (a_{jl}) \in P_{\mathbb{C}}^n$, then $a_{jl} = \overline{a}_{lj}$. Thus, if $A_k = (a_{jl}^k) \in P_{\mathbb{C}}^n$, $k = 1, \ldots, m$, we must have

$$f(\overline{a}_{lj}^1, \ldots, \overline{a}_{lj}^m) = f(a_{jl}^1, \ldots, a_{jl}^m) = \overline{f(a_{jl}^1, \ldots, a_{lj}^m)}.$$

Thus $f(\overline{\mathbf{z}}) = \overline{f(\mathbf{z})}$ for all $z \in \mathbb{C}^m$.

Statements (ii) and (iii) are immediate consequences of (i). As for (iv), we use (iii) and (iv) of Lemma 3.2 to conclude that $\overline{f(\cdot)} \in F_{\mathbb{C}}^m$. Consequently (i) of Lemma 3.2 implies that

$$\frac{f(\cdot) + \overline{f(\cdot)}}{2} \in F_{\mathbb{C}}^m.$$

Thus $u \in F_{\mathbb{C}}^m$. Finally, to prove (v) we note that from (iii), whenever $\mathbf{x} \in \mathbb{R}^m$ it follows that $v(\mathbf{x}) = 0$. Thus $f|_{\mathbb{R}^m} = u|_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}$. Since $P_{\mathbb{R}}^n \subseteq P_{\mathbb{C}}^n$, we easily conclude that $u|_{\mathbb{R}^m} \in F_{\mathbb{R}}^m$. ∎

We will initially consider $u(\mathbf{z}) = \mathrm{Re}\, f(\mathbf{z})$. Note that $u \in F_{\mathbb{C}}^m$ while $u|_{\mathbb{R}^m} \in F_{\mathbb{R}}^m$, and for each $\mathbf{z} \in \mathbb{C}^m$, $u(\mathbf{z}) \in \mathbb{R}$. For $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{C}^m$, we use the notation $|\mathbf{d}| = (|d_1|, \ldots, |d_m|)$, $\|\mathbf{d}\|_\infty = \max\{|d_j| : 1 \le j \le m\}$, and also $\|\mathbf{d}\|_2 = (\sum_{j=1}^m |d_j|^2)^{1/2}$.

PROPOSITION 3.5. *Let $\mathbf{c} \ge |\mathbf{d}|$, $\mathbf{c}, \boldsymbol{\theta} \in \mathbb{R}^m$, and $\mathbf{d} \in \mathbb{C}^m$. Then the function g defined by*

$$g(\mathbf{z}) = 2u(\mathbf{z} + \mathbf{c}) \pm [u(e^{i\boldsymbol{\theta}}\mathbf{z} + \mathbf{d}) + u(e^{i\boldsymbol{\theta}}\mathbf{z} + \overline{\mathbf{d}})], \tag{3.2}$$

where $e^{i\boldsymbol{\theta}}\mathbf{z} = (e^{i\theta_1} z_1, \ldots, e^{i\theta_m} z_m)$ for $\mathbf{z} = (z_1, \ldots, z_m)$, is in $F_{\mathbb{C}}^m$.

This result and its proof are complex analogs of (iii) and (iv) of Proposition 2.4.

Proof. Let $A_k \in P_{\mathbb{C}}^n$, $k = 1, \ldots, m$. For $\theta_k \in \mathbb{R}$,

$$\begin{pmatrix} 1 & e^{i\theta_k} \\ e^{-i\theta_k} & 1 \end{pmatrix} \in P_{\mathbb{C}}^2,$$

and for $c_k \geq |d_k|$,

$$\begin{pmatrix} c_k & d_k \\ \overline{d}_k & c_k \end{pmatrix} \in P_{\mathbb{C}}^2.$$

Thus

$$\begin{pmatrix} A_k + c_k J & e^{i\theta_k} + d_k J \\ e^{-i\theta_k} A_k + \overline{d}_k J & A_k + c_k J \end{pmatrix} \in P_{\mathbb{C}}^{2n},$$

where, as earlier, J is the $n \times n$ matrix all of whose entries are one. Thus the matrix

$$\begin{pmatrix} u(A_1 + c_1 J, \ldots, A_m + c_m J) & u(e^{i\theta_1} A_1 + d_1 J, \ldots, e^{i\theta_m} A_m + d_m J) \\ u(e^{-i\theta_1} A_1 + \overline{d}_1 J, \ldots, e^{-i\theta_m} A_m + \overline{d}_m J) & u(A_1 + c_1 J, \ldots, A_m + c_m J) \end{pmatrix}$$

which we will denote as C is in $P_{\mathbb{C}}^{2n}$.
Set

$$Q_{\pm} = \begin{pmatrix} I & 0 \\ \pm I & I \end{pmatrix} \in \mathbb{R}^{2n \times 2n},$$

where I is the $n \times n$ identity matrix. Then

$$D_{\pm} = Q_{\pm}^T C Q_{\pm} \in P_{\mathbb{C}}^{2n}.$$

The principal submatrix of $D_{\pm}$ determined by its first n rows and n columns is in $P_{\mathbb{C}}^n$. It is given by

$$2u(A_1 + c_1 J, \ldots, A_m + c_m J) \pm [u(e^{i\theta_1} A_1 + d_1 J, \ldots, e^{i\theta_m} A_m + d_m J) \\ + u(e^{-i\theta_1} A_1 + \overline{d}_1 J, \ldots, e^{-i\theta_m} A_m + \overline{d}_m J)].$$

This proves that the function g defined by (3.2) is in $F_{\mathbb{C}}^m$. ■

Let us draw some conclusions from Proposition 3.5. Using (ii) of Lemma 3.4, it follows that for $\mathbf{x} \in \mathbb{R}^m$

$$u(e^{-i\boldsymbol{\theta}}\mathbf{x} + \overline{\mathbf{d}}) = u(\overline{e^{i\boldsymbol{\theta}}\mathbf{x} + \mathbf{d}}) = u(e^{i\boldsymbol{\theta}}\mathbf{x} + \mathbf{d}).$$

Thus for $\mathbf{x} \in \mathbb{R}^m$ the function g defined by (3.2) simplifies to $g(\mathbf{x}) = 2[u(\mathbf{x} + \mathbf{c}) \pm u(e^{i\boldsymbol{\theta}}\mathbf{x} + \mathbf{d})]$. Furthermore, by Lemma 3.4 (v) we have $g|_{\mathbb{R}^m} \in F_{\mathbb{R}}^m$ if $\mathbf{c} \geq |\mathbf{d}|$ and $\boldsymbol{\theta} \in \mathbb{R}^m$. We write this statement as

$$u(\cdot + c) \pm u(e^{i\boldsymbol{\theta}} \cdot + \mathbf{d}) \in F_{\mathbb{R}}^m. \tag{3.3}$$

Next we introduce the function $U(\mathbf{x}, \mathbf{y}) = u(\mathbf{x} + i\mathbf{y})$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Also, by $U_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{z})$, $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, we mean explicitly for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{Z}_+^m$ the derivative

$$U_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{z}) = \frac{\partial^{\beta_1 + \cdots + \beta_m}}{\partial y_1^{\beta_1} \cdots \partial y_m^{\beta_m}} \frac{\partial^{\alpha_1 + \cdots + \alpha_m}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} U(\mathbf{x}, \mathbf{y}).$$

The exact order of differentiation among the $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ is, as will be shown, not important. However, in what follows we always assume that the partial derivatives are first with respect to $\mathbf{x}$ and then with respect to $\mathbf{y}$. We also use the notation $u_\alpha(\mathbf{z}) := U_{\boldsymbol{\alpha},0}(\mathbf{x}, \mathbf{y})$, where $\mathbf{z} = \mathbf{x} + i\mathbf{y}$.

Our goal is to prove

PROPOSITION 3.6. *Let $f \in F_{\mathbb{C}}^m$. Then the function U defined above has the form*

$$U(\mathbf{x}, \mathbf{y}) = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{Z}_+^m} e_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\alpha}} \mathbf{y}^{\boldsymbol{\beta}}, \tag{3.4}$$

and the power series converges absolutely for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2m}$. In other words, $U : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is real analytic.

We begin the proof with

LEMMA 3.7. *Let $f \in F_{\mathbb{C}}^m, u = \operatorname{Re} f$, and $U(\mathbf{x}, \mathbf{y}) = u(\mathbf{x} + i\mathbf{y})$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Then:*

(i) *$U(\cdot, \cdot)$ is in $C(\mathbb{R}^{2m})$.*
(ii) *For any $\mathbf{x} \in \mathbb{R}^m, U(\mathbf{x}, \cdot)$ is real analytic.*
(iii) *For any $\mathbf{y} \in \mathbb{R}^m, U(\cdot, \mathbf{y})$ is real analytic.*
(iv) *For any $\boldsymbol{\alpha} \in \mathbb{Z}_+^m \backslash \{\mathbf{0}\}, u_{\boldsymbol{\alpha}} \in F_{\mathbb{C}}^m$.*
(v) *For any $x \in \mathbb{R}^m, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}_+^m, U_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{x}, \cdot)$ is real analytic.*

Proof. For our first claim we let $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2m}$ and choose $\mathbf{c} \in \mathbb{R}^m$ such that $\mathbf{c} > |\mathbf{a} + i\mathbf{b}|$. Then from (3.3) the function

$$U(\mathbf{x} + \mathbf{c}, \mathbf{0}) \pm U(\mathbf{a} + (\cos\boldsymbol{\theta})\mathbf{x}, \mathbf{b} + (\sin\boldsymbol{\theta})\mathbf{x})$$

is in $F^m_{\mathbb{R}}$. In particular, from Proposition 2.4(ii) it is nondecreasing (elementwise) on $\mathbb{R}^m_+$. Hence it follows that

$$|U(\mathbf{a} + (\cos\boldsymbol{\theta})\mathbf{x}, \mathbf{b} + (\sin\boldsymbol{\theta})\mathbf{x}) - U(\mathbf{a}, \mathbf{b})| \le U(\mathbf{x} + \mathbf{c}, \mathbf{0}) - U(\mathbf{c}, \mathbf{0})$$

for $\mathbf{x} \ge \mathbf{0}$. Since $U(\mathbf{x}, \mathbf{0})$ is a continuous function of $\mathbf{x}$ [see Proposition 2.4 (v)], the continuity of U at $(\mathbf{a}, \mathbf{b})$ easily follows.

To prove (ii), let $\mathbf{x}$ be any vector in $\mathbb{R}^m$. In (3.3) choose $\mathbf{d} = \mathbf{x}, \theta_j = \pi/2,\ j = 1, \dots, m$, and $\mathbf{c} \ge |\mathbf{x}|$. It then follows that

$$u(\cdot + \mathbf{c}) \pm u(i \cdot + \mathbf{x}) \in F^m_{\mathbb{R}}.$$

By (v) of Lemma 3.4, $u(\cdot + \mathbf{c}) = E_{\mathbf{c}} u|_{\mathbb{R}^m}$ is in $F^m_{\mathbb{R}}$, and thus is real analytic. Thus for $\mathbf{x} \in \mathbb{R}^m, U(\mathbf{x}, \cdot) = u(i \cdot + \mathbf{x})$ is real analytic. Next, we choose $\boldsymbol{\theta} = \mathbf{0}, \mathbf{d} = i\mathbf{y}$, and $\mathbf{c} \ge |\mathbf{y}|$ in (3.3) to get

$$u(\cdot + \mathbf{c}) \pm u(\cdot + i\mathbf{y}) \in F^m_{\mathbb{R}},$$

and so, as before, $U(\cdot, \mathbf{y}) = u(\cdot + i\mathbf{y})$ is real analytic for any fixed $\mathbf{y} \in \mathbb{R}^m$, which proves (iii). For the next claim, we first note from (iii) that $u_{\boldsymbol{\alpha}}(\mathbf{z})$ exists for all $\mathbf{z} \in \mathbb{C}^m$. To prove $u_{\boldsymbol{\alpha}}$ is actually in $F^m_{\mathbb{C}}$ for any $\boldsymbol{\alpha} \in \mathbb{Z}^m_+$ it suffices to demonstrate this for $\boldsymbol{\alpha} = \mathbf{e}_k = (0, \dots, 0, 1, \dots, 0)$, where $\mathbf{e}_k$ is the kth unit vector, $1 \le k \le m$. The general case follows from this case by induction on $|\boldsymbol{\alpha}|$.

Let h be a positive number, and choose $\mathbf{d} = \boldsymbol{\theta} = \mathbf{0}$ and $\mathbf{c} = h\mathbf{e}_k$ in (3.2). Then, since h is positive, Proposition 3.5 implies that $h^{-1}[E_{\mathbf{c}}u(\cdot) - u(\cdot)] \in F^m_{\mathbb{C}}$. Letting $h \to 0^+$ and using the fact that $F^m_{\mathbb{C}}$ is closed [see Lemma 3.2 (v)], we have $u_{\mathbf{e}_k} \in F^m_{\mathbb{C}}$.

There remains (v). However, from (ii) we conclude that since the real-valued function $u_{\boldsymbol{\alpha}}$ is in $F^m_{\mathbb{C}}$, we have that for any $\mathbf{x} \in \mathbb{R}^m, U_{\boldsymbol{\alpha},\mathbf{0}}(\mathbf{x}, \cdot)$ is real analytic and hence so too is $U_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{x}, \cdot)$. ■

We now have the needed information to prove Proposition 3.6.

Proof of Proposition 3.6. From Lemma 3.7(iii) we have

$$U(\mathbf{x}, \mathbf{y}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^m_+} \frac{U_{\boldsymbol{\alpha},\mathbf{0}}(\mathbf{0}, \mathbf{y})}{\boldsymbol{\alpha}!} \mathbf{x}^{\boldsymbol{\alpha}}$$

valid for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Now, from (v) of Lemma 3.7 $U_{\boldsymbol{\alpha},\mathbf{0}}(\mathbf{0}, \cdot)$ is real analytic.

Thus,

$$U(\mathbf{x},\mathbf{y}) = \sum_{\alpha\in\mathbb{Z}_+^m}\left(\sum_{\beta\in\mathbb{Z}_+^m}\frac{U_{\alpha,\beta}(\mathbf{0},\mathbf{0})}{\beta!}\mathbf{y}^\beta\right)\frac{\mathbf{x}^\alpha}{\alpha!}$$

is also valid for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^m$.

The remainder of the proof consists in showing that the double sum converges uniformly and absolutely for $\mathbf{x},\mathbf{y}$ in any bounded subset of $\mathbb{C}^m$. To this end, we apply (3.3) to the real-valued function u_α which lies in $F_{\mathbb{C}}^m$ for the choice $\mathbf{d}=\mathbf{c}=\mathbf{0}$ and $\boldsymbol{\theta}=(\pi/2,\dots,\pi/2)$. We conclude again, using Theorem 2.1 and Lemma 3.7, that the function.

$$U_{\alpha+\beta,\mathbf{0}}(\cdot,\mathbf{0}) \pm U_{\alpha,\beta}(\mathbf{0},\cdot)$$

is in $F_{\mathbb{R}}^m$. Hence it follows that

$$|U_{\alpha,\beta}(\mathbf{0},\mathbf{0})| \le U_{\alpha+\beta,\mathbf{0}}(\mathbf{0},\mathbf{0}).$$

Also, for any $R>0$, by the Cauchy integral formula there is an $M_R\in(0,\infty)$ such that

$$0\le U_{\alpha,\mathbf{0}}(\mathbf{0},\mathbf{0}) \le \frac{M_R\alpha!}{R^{|\alpha|}}.$$

Therefore, if $\|\mathbf{x}\|_\infty, \|\mathbf{y}\|_\infty \le cR$ for some constant $c\in(0,\frac{1}{4})$, then

$$\begin{aligned}\left|\frac{U_{\alpha,\beta}(\mathbf{0},\mathbf{0})}{\alpha!\beta!}\mathbf{x}^\alpha\mathbf{y}^\beta\right| &\le \left|\frac{U_{\alpha+\beta,\mathbf{0}}(\mathbf{0},\mathbf{0})}{\alpha!\beta!}\mathbf{x}^\alpha\mathbf{y}^\beta\right|\\ &\le M_R\frac{(\alpha+\beta)!}{\alpha!\beta!} \le M_R(2c)^{|\alpha|+|\beta|}.\end{aligned}$$

Since

$$\begin{aligned}\sum_{\alpha,\beta\in\mathbb{Z}_+^m}(2c)^{|\alpha|+|\beta|} &= \sum_{r=0}^\infty (2c)^r\binom{2m+r-1}{r}\\ &\le 2^{2m-1}\sum_{r=0}^\infty(4c)^r<\infty,\end{aligned}$$

we conclude that

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^m}\frac{U_{\alpha,\beta}(\mathbf{0},\mathbf{0})\mathbf{x}^\alpha\mathbf{y}^\beta}{\alpha!\beta!}$$

is absolutely and uniformly convergent for $\mathbf{x}, \mathbf{y}$ in any bounded subset of $\mathbb{C}^m$. ■

Thus far we have proven that if $f \in F^m_{\mathbb{C}}$ and $u = \operatorname{Re} f$, then for $\mathbf{z} = \mathbf{x} + i\mathbf{y},\ \mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^m$
we have

$$u(\mathbf{z}) = U(\mathbf{x}, \mathbf{y}) = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{Z}^m_+} e_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\alpha}} \mathbf{y}^{\boldsymbol{\beta}},$$

where the power series converges absolutely for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2m}$. We now prove this same result for $v = \operatorname{Im} f$.

PROPOSITION 3.8. *Let $f \in F^m_{\mathbb{C}}$ and $v = \operatorname{Im} f$. Set $v(\mathbf{z}) = V(\mathbf{x}, \mathbf{y})$, where $\mathbf{z} = \mathbf{x} + i\mathbf{y}$. Then*

$$V(\mathbf{x}, \mathbf{y}) = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{Z}^m_+} d_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\alpha}} \mathbf{y}^{\boldsymbol{\beta}}, \tag{3.5}$$

and the power series converges absolutely for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2m}$.

Proof. Let $g_k(\mathbf{z}) = z_k$ and $\widehat{g}_k(\mathbf{z}) = \overline{z}_k$ for $k = 1, \ldots, m$. From (iv) of Lemma 3.2, $g_k, \hat{g}_k \in F^m_{\mathbb{C}}$, while from (ii) we obtain $g_k f, \widehat{g}_k f \in F^m_{\mathbb{C}}$. Let $x_k = \operatorname{Re} z_k, y_k = \operatorname{Im} z_k$ for $k = 1, \ldots, m$. Then $\operatorname{Re}[g_k(\mathbf{z}) f(\mathbf{z})] = x_k U(\mathbf{x}, \mathbf{y}) - y_k V(\mathbf{x}, \mathbf{y})$ and $\operatorname{Re}\ [\widehat{g}_k(\mathbf{z}) f(\mathbf{z})] = x_k U(\mathbf{x}, \mathbf{y}) + y_k V(\mathbf{x}, \mathbf{y})$. Therefore from Proposition 3.6 we obtain that

$$y_k V(\mathbf{x}, \mathbf{y}) = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{Z}^m_+} d^k_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\alpha}} \mathbf{y}^{\boldsymbol{\beta}},$$

where the power series converges absolutely for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2m}$ and each $k = 1, \ldots, m$. Setting $y_k = 0$ in the above gives $d^k_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}^m_+$, with $\beta_k = 0$. Hence we conclude that

$$\begin{aligned} V(\mathbf{x}, \mathbf{y}) &= \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{Z}^m_+ \\ \beta_k > 0}} \frac{d^k_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\alpha}} \mathbf{y}^{\boldsymbol{\beta}}}{y_k} \\ &= \sum_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{Z}^m_+} d^k_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\alpha}} \mathbf{y}^{\boldsymbol{\beta}}, \end{aligned} \tag{3.6}$$

which proves Proposition 3.8. ■

We gather together the remaining steps in the proof of Theorem 3.1.

Proof of Theorem 3.1. We assume $f \in F_{\mathbb{C}}^m$. Then

$$f(\mathbf{z}) = U(\mathbf{x}, \mathbf{y}) + iV(\mathbf{x}, \mathbf{y}),$$

and from Propositions 3.6 and 3.8, we have

$$f(\mathbf{z}) = \sum_{\alpha,\beta \in \mathbb{Z}_+^m} b_{\alpha,\beta} \mathbf{x}^{\alpha} \mathbf{y}^{\beta}, \tag{3.7}$$

where the power series is absolutely convergent for all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2m}$. Substitute $x_k = (z_k + \overline{z}_k)/2$ and $y_k = (z_k - \overline{z}_k)/2i, k = 1, \ldots, m$, in (3.7). It is easily checked that from (3.7) we obtain

$$f(\mathbf{z}) = \sum_{\alpha,\beta \in \mathbb{Z}_+^m} c_{\alpha,\beta} \mathbf{z}^{\alpha} \overline{\mathbf{z}}^{\beta}, \tag{3.8}$$

and this power series is absolutely convergent for all $\mathbf{z} \in \mathbb{C}^m$. It remains to prove that $c_{\alpha,\beta} \geq 0$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}_+^m$. This is done as in the proof of Proposition 2.5. For any choice of $\varepsilon > 0$ and numbers $\{y_{jk} : j = 1, \ldots, n,\ k = 1, \ldots, m\} \subset \mathbb{C}$, the matrices

$$A_k = (\varepsilon y_{jk} \overline{y}_{lk}), \qquad j, l = 1, \ldots, n, \quad k = 1, \ldots, m,$$

are in $P_{\mathbb{C}}^n$. Thus

$$\sum_{j,\, l = 1}^{n} w_j f(\varepsilon y_{j1} \overline{y}_{l1}, \ldots, \varepsilon y_{jm} \overline{y}_{lm}) \overline{w}_l \geq 0$$

for all $\mathbf{w} \in \mathbb{C}^n$. Substituting in (3.8), we obtain

$$\sum_{\alpha,\beta \in \mathbb{Z}_+^m} c_{\alpha,\beta} \varepsilon^{|\alpha| + |\beta|} \left| \sum_{j = 1}^{n} w_j y_{j1}^{\alpha_1} \overline{y}_{j1}^{\beta_1} \cdots y_{jm}^{\alpha_m} \overline{y}_{jm}^{\beta_m} \right|^2 \geq 0.$$

This inequality must hold for all $\varepsilon > 0$, $\mathbf{w} \in \mathbb{C}^n$, $\{y_{jk} : j = 1, \ldots, n,\ k = 1, \ldots, m\}$, and $n \in \mathbb{N}$. It is possible, given $\mathbf{p}, \mathbf{q} \in \mathbb{Z}_+^m$, to choose $n, \mathbf{w}$, and $\{y_{jk} : j = 1, \ldots, n,\ k = 1, \ldots, m\}$ such that

$$\sum_{j = 1}^{n} w_j y_{j1}^{\alpha_1} \overline{y}_{j1}^{\beta_1} \cdots y_{jm}^{\alpha_m} \overline{y}_{jm}^{\beta_m} = 0$$

for all $\underline{\alpha}, \underline{\beta} \in \mathbb{Z}_+^m$ satisfying $|\boldsymbol{\alpha}| + |\boldsymbol{\beta}| \leq |\mathbf{p}| + |\mathbf{q}|$ except in the case where $\boldsymbol{\alpha} = \mathbf{p}$ and $\boldsymbol{\beta} = \mathbf{q}$. We then obtain $c_{\mathbf{p},\, \mathbf{q}} \geq 0$ by letting $\varepsilon \downarrow 0$ in (3.9). ■

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