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APPLICATIONS OF REPRESENTATION THEOREMS TO PROBLEMS OF CHEBYSHEV APPROXIMATION WITH CONSTRAINTS

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1. INTRODUCTION

Recent years has witnessed a considerable literature concerned with various aspects of best Chebyshev approximation to functions under a variety of auxiliary conditions, such as restricted range approximation (Taylor [15] and references therein), simultaneous approximation (Dunham [2]), approximation with interpolation (Loeb, Morsund, Schumaker and Taylor [7]), approximation of discontinuous functions (Dunham [2], Rosman and Rosenbaum [11]), and approximation with bounded coefficients (Roulier and Taylor [12]). There also have appeared works attempting to unify the theory for Chebyshev systems and unisolvent families of functions (e.g., Chalmers, [1], Lewis [6]).

In this paper, we apply the representation theory developed in Karlin [3], (see also Karlin and Studden [4]), in order to extend, unify, and simplify many of the results characterizing the best Chebyshev approximation for the above class of approximation involving constraints. We attain the best approximation, characterized by alternations, as a limit of

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two unique "polynomials" with one less alternation and opposite orientation. To illustrate, consider the following two theorems.

Theorem 1. (Representation Theorem), (Karlin [3], Karlin and Studden [4].)

Let $\{u_i(t)\}_0^n$ *be a Chebyshev* (T) *system, and* $f(t)$ *and* $g(t)$ *two continuous functions on* $[a,b]$ *such that there exists a polynomial* $v(t) = \sum_{i=0}^{n} a_i u_i(t)$ *for which* $f(t) > v(t) > g(t)$ *for all* $t \in [a,b]$. *Then,*

a) *there exists a unique polynomial* $\underline{u}(t)$ *with the following properties*

(i) $f(t) \geq \underline{u}(t) \geq g(t)$, $t \in [a,b]$

(ii) *there exist* $n+1$ *points* $a \leq x_1 < \ldots < x_{n+1} \leq b$ *such that*

$$\underline{u}(x_i) = \begin{cases} f(x_i) , & i \ \text{even} \\ g(x_i) , & i \ \text{odd} \end{cases} \tag{1.1}$$

b) *Let condition* (ii) *be replaced by* (ii') *obtained by interchanging the functions* f *and* g *in* (1.1). *Then there exists a unique polynomial* $\bar{u}(t)$ *satisfying* (i) *and* (ii').

Let $f(c;t)$ and $g(c;t)$ be a family of continuous functions of $t \in [a,b]$ and $c \in (-\infty,\infty)$. Suppose $f(c;t)$ is non-decreasing in c and $g(c;t)$ is non-increasing in c. Assume the existence of a c_1 and a polynomial $v(t)$ satisfying $f(c_1;t) > v(t) > g(c_1;t)$, $t \in [a,b]$ and a c_2 for which $f(c_2;t) \leq g(c_2;t)$ for some $t \in [a,b]$. Then,

Theorem 2. *Under the above assumptions, let* $c_0 = \inf\{c :$ *there exists a polynomial* $u(t)$ *such that* $f(c;t) > u(t) > g(c;t)$ *for all* $t \in [a,b]\}$.

If c_0 *is such that* $f(c_0;t) > g(c_0;t)$ *for all* $t \in [a,b]$, *then there exists a unique polynomial* $u^*(t)$ *such that* $f(c_0;t) \geq u^*(t) \geq g(c_0;t)$, *and* $u^*(t)$ *alternates at least* $n+1$ *times between* $f(c_0;t)$ *and* $g(c_0;t)$.

The polynomial $u^*(t)$ is obtained as a limit of the $\underline{u}(c;t)$ and $\bar{u}(c;t)$ associated with $f(c;t)$ and $g(c;t)$.

A simple application of the above general theorem provides the existence, uniqueness and characterization of best approximations (with or without weight functions) to continuous functions. Examples which more fully utilize Theorem 2 are illustrated. We shall assume $||f|| = \max_{t \in [a,b]} |f(t)|$, unless otherwise stated.

i) Consider the problem of simultaneous approximation (Dunham [2]). The problem is to find and characterize the polynomial $u(t)$ which minimizes

$$\max(||f-u|| , ||g-u||) .$$

Let $f(t)$ and $g(t)$ be two continuous functions (without loss of generality we may assume that $f(t) \leq g(t)$, $t \in [a,b]$).

Let $f(c;t) = f(t)+c$ and $g(c;t) = g(t)-c$, $c > 0$. From Theorem 2, it follows that unless there exists a polynomial $u(t)$ such that $||f-u|| = ||g-u|| = \frac{1}{2}||f-g||$ (saddle point in the terminology of Dunham [2]), there then exists a unique polynomial $u^*(t)$ satisfying the above minimization and it is uniquely characterized by the property that it alternates at least $n+1$ times between $f(t)+c_0$ and $g(t)-c_0$, for some $c_0 > 0$, where $f(t)+c_0 > g(t)-c_0$, $t \in [a,b]$.

ii) An additional application of Theorem 2 is the earlier case of restricted range approximation considered by Taylor [14]. We wish to approximate a continuous function $f(t)$ by polynomials satisfying $a(t) \leq u(t) \leq b(t)$ where $a(t)$, $b(t)$ are two continuous functions for which $a(t) < b(t)$. We assume the existence of a polynomial $v(t)$ such that $a(t) < v(t) < b(t)$ for all $t \in [a,b]$. Letting $f(c;t) = \min\{f(t)+c, b(t)\}$, $g(c;t) = \max\{f(t)-c, a(t)\}$, and applying Theorem 2, we secure the desired result.

Attributable to this key relationship between representation theorems and problems of best approximation, we refine and extend the technique of Theorem 1 and via the methods of Theorem 2 obtain a host of applications. This approach is basically new. Karlin and Studden [4, p.253] were aware of this connection. However, their approach used properties of the unique best approximation to continuous functions to explicitly obtain results for the $\underline{u}(c;t)$ and $\bar{u}(c;t)$ featured in Theorem 1 (i.e., they approached the problem in the reverse direction).

Theorem 1 may be generalized in various directions. In Section 2 it is shown that Theorem 1 may be extended to include a compact subset of the real line rather than a connected interval, and that the continuous functions $f(t)$ and $g(t)$ may be replaced by lower semi-continuous and upper semi-continuous functions, respectively. It would be useful to extend Theorem 1 to the class of unisolvent families of functions. While this may well be true, we were able to prove this fact only for a subset of the above class.

In Section 3, we indicate a representation theorem which allows contact between $f(t)$ and $g(t)$, and then apply the result to the general case of restricted range approximation. Section 4 presents a new representation theorem constructed to deal with the problems of best approximation with interpolatory data. It is then applied to obtain the result of Loeb, Morsund, Schumaker and Taylor [7].

In Pinkus [10] representation theorems with quite general boundary conditions were developed. These are presented in Section 5. In Section 6 we apply the contents of Section 5 to obtain results on best approximation subject to restricted boundary conditions. The results of Section 6 are mainly new.

The definitions of a Chebyshev (T), Extended Chebyshev (ET), and Extended Complete Chebyshev (ECT) system $\{u_i(t)\}_{i=0}^{n}$ may be found in [4].

2. PROOF OF THEOREM 2 AND EXTENSIONS OF THEOREM 1

Proof of Theorem 2. The uniqueness of $u^*(t)$ satisfying the above properties is a result of the standard zero counting argument, while existence of a $u(t)$ for which $f(c_0;t) \geq u(t) \geq g(c_0;t)$ for all $t \in [a,b]$, is easily proven. The following is a proof of the characterization of $u^*(t)$.

For each $c > c_0$, there exists a $v(t)$ for which $f(c;t) > v(t) > g(c;t)$, by the definition of c_0. Thus, by Theorem 1, we have the existence of unique polynomials $\bar{u}(c;t)$ and $\underline{u}(c;t)$, each alternating n times between $f(c;t)$ and $g(c;t)$, with opposite orientation.

Let $c \downarrow c_0$ and choose convergent subsequences. Since $\underline{u}(c;t)$ and $\bar{u}(c;t)$ are continuous functions of c and since $f(c_0;t) > g(c_0;t)$ for all $t \in [a,b]$, the resulting functions $\underline{u}(t)$ and $\bar{u}(t)$ must alternate n times between $f(c_0;t)$ and $g(c_0;t)$ with opposite orientation.

If $\underline{u}(t) \not\equiv \bar{u}(t)$, there then exists a polynomial $u(t)$ for which $f(c_0;t) > u(t) > g(c_0;t)$, contradicting the definition of c_0 (see Lemma 3.1). Thus $\underline{u}(t) \equiv \bar{u}(t)$, and it follows that $u^*(t) \equiv \underline{u}(t) \equiv \bar{u}(t)$ must alternate at least $n+1$ times between $f(c_0;t)$ and $g(c_0;t)$.

Q.E.D.

Remark 2.1 . It is possible, since we do not demand that $f(c;t)$ strictly increases in c, for all $t \in [a,b]$, and similarly for $g(c;t)$, that $f(c;t) \geq u^*(t) \geq g(c;t)$ for some $c < c_0$. The theorem implies that for any $c \leq c_0$, the only polynomial that may lie between $f(c;t)$ and $g(c;t)$ is $u^*(t)$ and it satisfies the alternation property if $f(c;t) > g(c;t)$ for all $t \in [a,b]$.

Remark 2.2 . If c_0 is such that there exists a $\tilde{t} \in [a,b]$ for which $f(c_0;\tilde{t}) = g(c_0;\tilde{t})$, then uniqueness is, in general, lacking. Examples are easily constructed.

Proposition 2.1 . Let $f(t)$, $g(t)$ be two continuous functions on $K \subseteq [a,b]$, K compact containing at least $n+2$ points. Let $\{u_i(t)\}_{i=0}^n$ be a T-system on $[a,b]$. Assume that there exists a polynomial $v(t)$ such that $f(t) > v(t) > g(t)$ for all $t \in K$. Then the results of Theorem 1 extend with K in place of $[a,b]$.

Proposition 2.2 . Let $f^*(t)$, $g^*(t)$ be bounded lower semi-continuous and upper semi-continuous functions, respectively. Assume $f^*(t) > g^*(t)$ for all $t \in [a,b]$, and that there exists a polynomial $v(t)$ such that $f^*(t) > v(t) > g^*(t)$ for all $t \in [a,b]$. Then, the results of Theorem 1 hold for $f^*(t)$ and $g^*(t)$.

Both Propositions 2.1 and 2.2 may be obtained by the analysis used in the proof of Theorem 1, or can be deduced from an application of a limiting process on the results of Theorem 1.

Theorem 1 does not extend to the case of unisolvent families of functions due to the fact that zeros may not be counted with full multiplicity (this is necessary in the proof of Theorem 1). As such, we restrict ourselves to the following class.

Definition 2.1 . Let P be a parameter space. A family of functions $F(A,x)$, $x \in [a,b]$, $A \in P$, is called Extended Unisolvent of degree $n+1$ if $F(A,x) \in C^{n+1}[a,b]$ for all $A \in P$, and there exists a unique function which interpolates any given $n+1$ Hermite data.

For the above family of functions it is not difficult to show that Theorem 1 remains valid.

3. APPLICATIONS OF THEOREM 2

Example 3.1 . The first immediate result of Theorem 2 is the classical characterization and uniqueness of the best approximation, and best one-sided approximation, by polynomials $u(t) = \sum_{i=0}^{n} a_i u_i(t)$, where $\{u_i(t)\}_{i=0}^{n}$ is a T-system on the smallest connected component containing K , to a continuous function defined on a compact set K of $\mathbb{R}$. In what follows, the best approximation is defined with respect to the Chebyshev (L_∞) norm, $||f|| = \max_{t \in K} |f(t)|$. However, this is done simply for ease of notation since we may also define the best approximation with respect to the generalized weight function introduced by Morsund [9], defined as follows:

The error of approximation is measured in terms of the function $W(t;y)$ defined on $K \times \mathbb{R}$, where

(i) $\operatorname{sgn} W(t;y) = \operatorname{sgn} y$

(ii) W is continuous

(iii) for each $t \in K$, W is a strictly monotone increasing function of y , with $\lim_{|y| \to \infty} |W(t;y)| = \infty$.

The polynomial $u^*(t)$ is said to be a best approximation of $f(t)$ with respect to W if

$$\sup_{t\in K} |W(t;f(t)-u^*(t))| \leq \sup_{t\in K} |W(t;f(t)-u(t))|$$

for all polynomials $u(t) = \sum_{i=0}^{n} a_i u_i(t)$, n fixed.

In place of setting $f(c;t) = f(t)+c$, and $g(c;t) = f(t)-c$ as in the case of the best approximation to $f(t)$, and applying Theorem 2, we define $f(c;t) = f(t)+c(t)$, and $g(c;t) = f(t)+b(t)$, where $W(t;c(t)) = c$, $t \in K$, and $W(t,b(t)) = -c$, $t \in K$, $c > 0$. Note that on a connected interval, $c(t)$ and $b(t)$ are continuous functions of t .

<u>Example 3.2 .</u> In Section 1 we indicated how Theorem 2 may be applied to the problem of simultaneous approximation of two continuous functions. From Proposition 2.2, this result may be extended to the case where $g(t) \geq f(t)$ and $g(t)$ is a bounded upper semi-continuous function, while $f(t)$ is a bounded lower semi-continuous function. This fact may be used in the consideration of best approximating bounded measurable functions (see Dunham [2], Rosman and Rosenbaum [11]).

<u>Example 3.3 .</u> <u>Restricted Range Approximation</u>

In order to extend the discussion on restricted range approximation in Section 1 to the case where $a(t) \leq b(t)$ for all $t \in [a,b]$, we make use of a generalization of Theorem 1. The result we state without proof, but include the proof of the following lemma. The proof of the theorem is identical with that found in Karlin and Studden [4,p.74].

<u>Lemma 3.1 .</u> <u>Let</u> $\{u_i(t)\}_{i=0}^{n}$ <u>be an ET-system on</u> $[a,b]$, <u>and let</u> $f(t)$, $g(t)$ <u>be continuous functions on</u> $[a,b]$ <u>such that</u>

1) $f(t) \geq g(t)$ for all $t \in [a,b]$.

2) For each $\tilde{t}$ such that $f(\tilde{t}) = g(\tilde{t})$, $f(t)$, $g(t) \in C^n[\tilde{t}-\varepsilon,\tilde{t}+\varepsilon]$, some $\varepsilon > 0$, (where at a and at b , only the one-sided derivative is assumed).

3) $Z_{[a,b]}(f-g) = m \leq n$, where $Z_{[a,b]}(f)$ counts the number of zeros of $f(t)$ in $[a,b]$, including multiplicities.

Assume that there exist two distinct polynomials $v_1(t)$ and $v_2(t)$ such that $f(t) \geq v_i(t) \geq g(t)$, $t \in [a,b]$, $i = 1,2$. Then, there exists a polynomial $v(t)$ such that $f(t) \geq v(t) \geq g(t)$ for all $t \in [a,b]$, and $Z_{[a,b]}(f-v) = Z_{[a,b]}(v-g) = m$.

Proof. Let $\{t_i\}_{i=1}^k$ be the distinct points at which $f(t) = g(t)$, and let $\{\mu_i\}_{i=1}^k$ be the respective multiplicities of the zeros of $f(t_i)-g(t_i)$, $i = 1,\ldots,k$. Note that μ_i is even if $t_i \in (a,b)$, $\sum_{i=1}^{k} \mu_i = m$.

The set of polynomials lying between $f(t)$ and $g(t)$ is a closed convex set, and if the multiplicity of the zero of $(f-v_1)(t_i)$ is δ_i and the multiplicity of the zero of $(f-v_2)(t_i)$ is λ_i , where $f(t) \geq v_i(t) \geq g(t)$, $i = 1,2$, $v_1(t) \not\equiv v_2(t)$, then $(f - \frac{v_1+v_2}{2})(t_i)$ has a zero of multiplicity $\min\{\delta_i,\lambda_i\}$ at t_i . In this way we may easily define polynomials $\tilde{v}_1(t)$ and $\tilde{v}_2(t)$ which have the least amount of contact with $f(t)$ and $g(t)$, respectively. Let

$$u^*(t) = \frac{\tilde{v}_1(t)+\tilde{v}_2(t)}{2} .$$

We show that if $u^*(t)$ does not satisfy the statement of the lemma, then a contradiction ensues.

Assume $(f-u^*)(t)$ has a zero of multiplicity δ_i at t_i and $(u^*-g)(t)$ has a zero of multiplicity λ_i at t_i, $i = 1,\ldots,k$. Then $\min\{\delta_i,\lambda_i\} = \mu_i$, $i = 1,\ldots,k$, and define $\max\{\delta_i,\lambda_i\} = \nu_i$, $i = 1,\ldots,k$.

Assume $u^*(t)$ alternates ℓ times between $f(t)$ and $g(t)$ where we disregard the points $\{t_i\}_{i=1}^k$. Let $\{s_i\}_{i=1}^{\ell+1}$ be $\ell+1$ points of alternation.

Now, if $\sum_{i=1}^k v_i+\ell \geq n+1$, then a contradiction ensues since by assumption there exists a polynomial $u(t)$, $f(t) \geq u(t) \geq g(t)$, and $u(t) \not\equiv u^*(t)$. But, by definition of $u^*(t)$, $u(t) - u^*(t)$ have zeros of multiplicity ν_i at t_i, $i = 1,\ldots,k$ and must have at least ℓ other zeros in $[a,b]$, since ν_i is even, $t_i \in (a,b)$. Thus $u^*(t) \equiv u(t)$.

Since $\sum_{i=1}^k \nu_i+\ell \leq n$, it is easily shown that $\ell = 0$ by perturbing $u^*(t)$ by a polynomial with ℓ zeros in $(a,b)\setminus\{t_i\}_{i=1}^k$ which interlace the $\ell+1$ points of alternation $\{s_i\}_{i=1}^{\ell+1}$. By the minimum property of $u^*(t)$, $\ell = 0$.

The same idea is used to show that $\nu_i = \mu_i$, $i = 1,\ldots,k$, and the lemma is proven.

Q.E.D.

The theorem now follows.

<u>Theorem 3.1 .</u> <u>Let the assumption of Lemma</u> 3.1 <u>hold. Then,</u>

a) <u>there exists a unique polynomial</u> $\bar{u}(t)$ <u>satisfying</u>

(i) $f(t) \geq \bar{u}(t) \geq g(t)$, $t \in [a,b]$

(ii) <u>There exist</u> $n-m+1$ <u>points</u>, $a \leq s_1 < \ldots < s_{n-m+1} \leq b$ <u>such that</u>

$$\bar{u}(s_i) = \begin{cases} f(s_i) & i \ \underline{\text{odd}} \\ g(s_i) & i \ \underline{\text{even}} \end{cases}$$

where we remove the points t_i with multiplicity μ_i, $i = 1,\ldots,k$. That is to say that if $s_i = t_j$, some i, j and if, for example, i is odd, the zero $(f-\bar{u})(t_j)$ has multiplicity $> \mu_j$, etc. ....

b) Let condition (ii) be replaced by (ii') by interchanging the functions $f(t)$ and $g(t)$. Then there exists a unique polynomial $\underline{u}(t)$ satisfying (i) and (ii').

The extension of Propositions 2.1 and 2.2 to the above case is more technical in nature and is omitted.

To return to the problem of restricted range approximation, we assume that $a(t)$, $b(t)$ are upper semi-continuous and lower semi-continuous functions, respectively, (not necessarily bounded), that $a(t) \leq b(t)$ and $a(t)$, $b(t)$ satisfy conditions 2) and 3) of Lemma 3.1. We also assume that $\{u_i(t)\}_{i=0}^{n}$ is an ET-system on $[a,b]$.

Let $f(t) \in C[a,b]$, and $a(t) \leq f(t) \leq b(t)$ for all $t \in [a,b]$. Applying the analysis of Theorems 2 and 3.1 together with Example 3.2 and the discussion of restricted range approximation in Section 1, and assuming that $f(t)$ is not a polynomial $u(t)$, we obtain the uniqueness and characterization of the polynomial of best approximation to $f(t)$ from polynomials $u(t)$ for which $a(t) \leq u(t) \leq b(t)$, where we assume the existence of at least two distinct polynomials lying between $a(t)$ and $b(t)$.

This polynomial of best approximation is characterized by the fact that it alternates at least $n-m+1$ times between $f(c_0;t) = \min\{f(t) + c_0, b(t)\}$ and $g(c_0;t) = \max\{f(t)-c_0, a(t)\}$, in the sense of Theorem 3.1 (and Proposition 2.2), where $Z_{[a,b]}(b(t)-a(t)) = m$.

We may also assume $f(t)$ is not a continuous function (see Example 3.2). Note the ε-Interpolatory approximation (see Taylor [15]) is a specific example of the above. (See also Laurent [5], where we assume that the approximating subspace is a T-system.)

4. REPRESENTATION THEOREMS WITH INTERPOLATION, AND INTERPOLATORY APPROXIMATION

The results of this section are known except for the representation theorem itself. In this section the above methods are applied to the work of Loeb, Morsund, Schumaker and Taylor [7].

Theorem 4.1. Let $\{u_i(t)\}_{i=0}^{n}$ be an ET-system on $[a,b]$, and let $f(t)$, $g(t)$ be two continuous functions on $[a,b]$. We are given $\{x_i\}_{i=1}^{k}$, $a \leq x_1 < \dots < x_k \leq b$, positive integers $\{m_i\}_{i=1}^{k}$, $\sum_{i=1}^{k} m_i = m \leq n$, and real numbers $\{a_{ij}\}_{i=1}^{k}{}_{j=0}^{m_i-1}$. Assume that there exists a polynomial $v(t)$ such that $f(t) > v(t) > g(t)$ for all $t \in [a,b]$, and $v^{(j)}(x_i) = a_{ij}$, $j = 0,1,\dots,m_i-1$; $i = 1,\dots,k$. Then,

a) there exists a unique polynomial $\bar{u}(t)$ such that

(i) $f(t) \geq \bar{u}(t) \geq g(t)$, $t \in [a,b]$

(ii) $\bar{u}^{(j)}(x_i) = a_{ij}$, $j = 0,1,\dots,m_i-1$; $i = 1,\dots,k$,

(iii) there exist $n-m+1$ points $\{y_i\}_{i=1}^{n-m+1}$, $a \leq y_1 < \dots < y_{n-m+1} \leq b$ such that $\bar{u}(t)$ satisfies the following generalized oscillation property. If

$$(-1)^{i-1} \prod_{\ell=1}^{k} (x_\ell - y_i)^{m_\ell} > 0 \text{ , then } \bar{u}(y_i) = f(y_i)$$

$$(-1)^{i-1} \prod_{\ell=1}^{k} (x_\ell - y_i)^{m_\ell} < 0 \text{ , then } \bar{u}(y_i) = g(y_i) \text{ ,}$$

$i = 1,\ldots,n-m+1$.

b) There exists a unique polynomial $\underline{u}(t)$ satisfying (i), (ii) and (iii') where (iii') is obtained from (iii) by interchanging the roles of the functions $f(t)$ and $g(t)$.

Proof. Let $\tilde{f}(t) = f(t)-v(t)$ and $\tilde{g}(t) = v(t)-g(t)$. Then the statement of the theorem is reduced to the case where $a_{ij} = 0$, $j = 0,1,\ldots,m_i-1$; $i = 1,\ldots,k$. The method of proving existence of $\bar{u}(t)$ and $\underline{u}(t)$ is a simple variation on the proofs of existence for the other representation theorems (cf. [3], [4], [10]).

Uniqueness is proven as follows. Consider part a) of the theorem and assume $u_1(t)$, $u_2(t)$ are two distinct solutions. Let $\{y_i\}_{i=1}^{n-m+1}$, and $\{z_i\}_{i=1}^{n-m+1}$ be their associated points in (iii), respectively.

By a consideration of the points $\{y_i\}_{i=1}^{n-m+1}$ and by property (iii), it is readily seen that $u_1(t)-u_2(t)$ has a zero, aside from those given in (ii), in each of $[y_i,y_{i+1}]$, $i = 1,\ldots,n-m$, where if $u_1(t)-u_2(t)$ has a double zero at y_i , $i = 1,\ldots,n-m+1$, we allot one zero to the interval $[y_{i-1},y_i]$ and one zero to $[y_i,y_{i+1}]$. If $u_1(t)-u_2(t)$ has a single zero at y_i (note that $f(t)$, $g(t)$ are continuous, but not necessarily differentiable), then $u_1(t)-u_2(t)$ has a zero in $[y_{i-1},y_i)$ or $(y_i,y_{i+1}]$, and we allot the zero at

y_i to the adjacent interval lacking a zero. The analogous result holds for the points $\{z_i\}_{i=1}^{n-m+1}$.

If $u_1(t)-u_2(t)$ has $n-m+1$ zeros in $[a,b]$, aside from the m zeros given in (ii), then $u_1(t) \equiv u_2(t)$. We show the existence of an extra zero in addition to the $n-m$ given above.

Assume, without loss of generality, that $z_1 \leq y_1$. If $z_1 < y_1$, then considering $u_1(t)$ and $u_2(t)$ at z_1 and y_1, we obtain an additional zero of $u_1(t)-u_2(t)$ in $[z_1,y_1]$, with the same convention as above with respect to zeros at z_1,y_1, and thus $u_1(t) = u_2(t)$. If $z_1 = y_1$ and $u_1(z_1) = u_2(z_1) = f(z_1)$, while $u_1(t) > u_2(t)$, $t \in (z_1,z_1+\varepsilon)$, some $\varepsilon > 0$, then $u_1(t)-u_2(t)$ has a zero at $z_1 = y_1$ and an additional zero in $(y_1,y_2]$. Uniqueness follows.

Q.E.D.

Remark 4.1. Every representation theorem of the above form solves a series of extremal problems. The interested reader may consult [4] or [10] for analogous cases.

Remark 4.2. The results of Theorem 4.1 extend, as in Section 2, to closed subsets K of $[a,b]$, and to functions $f(t)$, $g(t)$ as in Proposition 2.2. Note that while we may deal with a compact subset K of $[a,b]$, we still demand that $\{u_i(t)\}_{i=0}^{n}$ is an ET-system on $[a,b]$, and as such, the points $\{x_i\}_{i=1}^{k}$ need not be in K.

As in the previous sections, the above representation theorem easily leads to a characterization, in terms of alternations, of a problem of best approximation.

Theorem 4.2 . ([7]). Let $f(t)$ be a continuous function on $[a,b]$, and let $\{u_i(t)\}_{i=0}^{n}$ be an ET-system in $[a,b]$. Let $\{x_i\}_{i=1}^{k}$, $a \leq x_1 < \ldots < x_k \leq b$, positive integers $\{m_i\}_{i=1}^{k}$, $\sum_{i=1}^{k} m_i = m \leq n$, and real numbers $\{a_{ij}\}$, $j = 0,\ldots,m_i-1$; $i = 1,\ldots,k$, be given such that $f(x_i) = a_{i0}$, $i = 1,\ldots,k$. Then the polynomial of best Chebyshev approximation to $f(t)$ from the class of polynomials $u(t)$ satisfying $u^{(j)}(x_i) = a_{ij}$, $j = 0,\ldots,m_i-1$; $i = 1,\ldots,k$, is unique and is characterized by the property that it alternates $n-m+1$ times between $f(t)+c_0$ and $f(t)-c_0$ for some $c_0 > 0$. (Assume $f(t)$ is not a polynomial of the above form.) That is to say that there exist $n-m+2$ points $\{y_i\}_{i=1}^{n-m+2}$, $a \leq y_1 < \ldots < y_{n-m+2} \leq b$, satisfying part (iii) or (iii') of Theorem 4.1.

Remark 4.3 . If $m_i = 1$, $i = 1,\ldots,k$, we may assume that $\{u_i(t)\}_{i=0}^{n}$ is a T-system.

Remark 4.4 . The condition $f(x_i) = a_{i0}$, $i = 1,\ldots,k$, may be weakened and the results of the theorem still obtain if

$$\min_{u} ||f-u|| = c_0 > \max_{i=1,\ldots,k} |f(x_i) - a_{i0}| ,$$

where the above minimum is taken over the class of all polynomials.

Remark 4.5 . Theorem 4.2 may be generalized to considering the generalized weight function introduced by Morsund [8], (see Example 3.1), to considering a closed subset K of $[a,b]$ (see Remark 4.2), and to the class of functions $f(t)$ considered in Proposition 2.2, (see Remark 4.2 and Example 3.2).

Remark 4.6 . The problem of approximating a function $f(t)$ by polynomials satisfying $v^{(j)}(x_i) = a_{ij}$, $j=0,1,\ldots,m_i-2$; $i = 1,\ldots,k$, and $\underline{b}_i \leq v^{(m_i-1)}(x_i) \leq \bar{b}_i$, $i = 1,\ldots,k$, where $v(x_i) = f(x_i)$ may also be dealt with by the use of Theorem 4.1 and the analysis of Theorem 6.2.

Remark 4.7 . Both Theorems 4.1 and 4.2 may be generalized to include non-Hermite given data if the corresponding interpolation problem remains poised with any $n-m+1$ additional Hermite data. Thus, for instance, we may assume we are given Hermite and even block data.

5. REPRESENTATION THEOREMS WITH BOUNDARY CONSTRAINTS

The results of this section may be found in [10]. To understand the statement of the representation theorem, certain facts are needed. These are outlined below.

Pólya (see Karlin and Studden [4, p.379]) pointed out the following characterization of ECT-systems.

Theorem 5.1 . *Let* $u_i(t) \in C^n[a,b]$ *obey the initial conditions*

$$u_i^{(p)}(a) = 0 \ , \ p=0,1,\ldots,i-1 \ ; \ i = 1,\ldots,n \ . \tag{5.1}$$

Then $\{u_i(t)\}_{i=0}^n$ *is an ECT-system on* $[a,b]$ *if and only if*

$$u_i(t) = w_0(t) \int_a^t w_1(\xi_1) \int_a^{\xi_1} w_2(\xi_2) \ldots \int_a^{\xi_{i-1}} w_i(\xi_i) d\xi_i \ldots d\xi_1 \ ,$$

$i = 0,1,\ldots,n$, *where* $w_0(t)$, $w_1(t)$, ... , $w_n(t)$ *are* $n+1$ *strictly positive functions on* $[a,b]$ *such that* $w_k(t)$ *is of continuity class* $C^{n-k}[a,b]$.

Remark 5.1 . If the conditions at the endpoint $t = a$ are not satisfied by an ECT-system $\{u_i(t)\}_{i=0}^n$, then effecting a non-singular linear transformation, we can determine a new ECT-system $\{u_i(t)\}_{i=0}^n$ which satisfies (5.1). Thus, we assume without loss of generality that the ECT-system $\{u_i(t)\}_{i=0}^n$ satisfies the initial conditions. Note that if $a = 0$, and $w_i(t) = \frac{1}{i+1}$, $i = 0,1,\ldots,n$, then $u_i(t) = t^i$, $i = 0,1,\ldots,n$.

Associated with an ECT-system $\{u_i(t)\}_{i=0}^n$ is a natural system of first order differential operators $D_j f = \frac{d}{dt} \frac{f(t)}{w_j(t)}$, $j = 0,1,\ldots,n$, where the $\{w_i(t)\}_{i=0}^n$ are those exhibited in Theorem 5.1. Define $D^j = D_{j-1},\ldots,D_0$, $j = 1,\ldots,n+1$, $D^0 = I$. Accordingly,

$$D^j\, u_j(t) = w_j(t) \quad , \quad j = 0,1,\ldots,n$$

$$D^k\, u_j(t) = 0 \quad , \quad \text{for } k > j$$

and by virtue of the initial conditions (5.1), $D^k\, u_j(a) = \delta_{kj}\, w_j(a)$, $j = 0,1,\ldots,n$. For the powers $\{t^k\}_{k=0}^n$, $D^j = \frac{1}{j!} \frac{d^j}{dt^j}$.

The next representation theorem is concerned with polynomials satisfying homogeneous boundary conditions of the form:

$$\mathcal{A}_k : \sum_{j=0}^{n} A_{ij}\, D^j\, u(a) = 0 \quad , \quad i = 1,\ldots,k \quad ,$$

$$\mathcal{B}_m : \sum_{j=0}^{n} B_{ij}\, D^j\, u(b) = 0 \quad , \quad i = 1,\ldots,m \quad ,$$

where the matrices $\tilde{A} = \left|\left|A_{ij}\ (-1)^j\right|\right|_{i=1\ j=0}^{k\quad n}$, and

$B = ||B_{ij}||_{i=1}^{m}\,{}_{j=0}^{n}$ obey Postulate I below. The collection of all such polynomials will be denoted by $\mathcal{U}(\mathfrak{A}_k \cap \mathfrak{B}_m)$.

Postulate I.

(i) $0 \leq k,m \leq n$, and $k+m \leq n$.

(ii) There exists $\{i_1,\dots,i_m\}$, $\{j_1,\dots,j_k\}$ such that

$$B\begin{pmatrix}1 ,\dots, m\\ i_1,\dots,i_m\end{pmatrix} \quad \tilde{A}\begin{pmatrix}1 ,\dots, k\\ j_1,\dots,j_k\end{pmatrix} \neq 0 \text{ , and}$$

$i_\nu \leq j'_{\nu+1+n-(m+k)}$, $\nu = 1,\dots,m$, where $\{j'_\nu\}_{\nu=1}^{n-k+1}$ is the complementary ordered set of indices to $\{j_\nu\}_{\nu=1}^{k}$ in $\{\ell\}_{\ell=0}^{n}$.

(iii) For all $\{i_1,\dots,i_m\}$, $\{j_1,\dots,j_k\}$ satisfying (ii),

$$\operatorname{sgn} B\begin{pmatrix}1 ,\dots, m\\ i_1,\dots,i_m\end{pmatrix} = \varepsilon_m(B) \text{ , and } \operatorname{sgn} \tilde{A}\begin{pmatrix}1 ,\dots, k\\ j_1,\dots,j_k\end{pmatrix} = \varepsilon_k(\tilde{A}),$$

i.e., the $m \times m$ and $k \times k$ subdeterminants from (ii) have constant signs, respectively.

The possibility that boundary conditions apply at one endpoint only is not excluded.

Definition 5.1 . If $u(t)$ in the class $\mathcal{U}(\mathfrak{A}_k \cap \mathfrak{B}_m)$ entails $u(b) = Du(b) = \dots = D^{\beta-1}u(b) = 0$, while there exists a $u(t) \in \mathcal{U}(\mathfrak{A}_k \cap \mathfrak{B}_m)$ for which $D^\beta u(b) \neq 0$, then we say that the class of polynomials $\mathcal{U}(\mathfrak{A}_k \cap \mathfrak{B}_m)$ has a zero of degree β at b , denoted by $\beta = N_b(\mathfrak{A}_k, \mathfrak{B}_m)$. Similarly, we define $N_a(\mathfrak{A}_k, \mathfrak{B}_m)$.

We now outline the procedure used for the addition of a zero at an endpoint.

Assume $N_b(\mathfrak{A}_k, \mathfrak{B}_m) = \beta$, and $m+k < n$. Let

$$B' = \| B'_{ij} \|_{i=1}^{m+1} \, {}_{j=0}^{n} \ , \quad \text{where}$$

$$B'_{ij} = \begin{cases} \delta_{\beta j} & i = 1 \quad ; \quad j = 0,1,\ldots,n \\ B_{i-1,j} & i=2,\ldots,m+1 \quad ; \quad j = 0,1,\ldots,n \end{cases}$$

and let $\mathcal{B}'_{m+1} : \sum_{j=0}^{n} B'_{ij} D^j u(b) = 0 \ , \quad i = 1,\ldots,m+1 .$

We call this construction the addition of a zero at b . A similar construction may be executed at a to obtain A' and $\mathcal{O}\!\mathcal{l}'_{k+1}$. It is shown in [10] that subject to the condition that B and $\tilde{A}$ satisfy Postulate I, and $m+k < n$, then B' and $\tilde{A}$ and B and $\tilde{A}'$ also satisfy Postulate I.

With this we may now define

<u>Definition 5.2 .</u> <u>We say that</u> $\mathcal{U}(\mathcal{O}\!\mathcal{l}_k \cap \mathcal{B}_m)$ <u>satisfies Property J if</u> $m+k < n$, <u>and</u>

1) $N_a(\mathcal{O}\!\mathcal{l}_k, \mathcal{B}_m) = N_a(\mathcal{O}\!\mathcal{l}_k, \mathcal{B}'_{m+1})$

<u>or</u> 2) $N_b(\mathcal{O}\!\mathcal{l}_k, \mathcal{B}_m) = N_b(\mathcal{O}\!\mathcal{l}'_{k+1}, \mathcal{B}_m)$.

1) and 2) are, in fact, equivalent statements as is shown in [10].

With the above definitions we present the following representation theorem.

<u>Theorem 5.2 .</u> <u>Let</u> f <u>and</u> g <u>be two continuous functions on</u> $[a,b]$ <u>such that there exists a polynomial</u> $v(t)$ <u>lying strictly between</u> $f(t)$ <u>and</u> $g(t)$, i.e., $f(t) > v(t) > g(t)$, $t \in [a,b]$, <u>where</u> $v(t)$ <u>satisfies</u>

$$\text{(5.2)} \quad \begin{aligned} &\sum_{j=0}^{n} A_{ij} D^j u(a) = \alpha_i \ , \quad i = 1,\ldots,k \\ &\sum_{j=0}^{n} B_{ij} D^j u(b) = \beta_i \ , \quad i = 1,\ldots,m \ , \ \underline{\text{and}} \end{aligned}$$

the homogeneous equations of the above form satisfy Postulate I. Then,

a) there exists a polynomial $\bar{u}(t)$ such that

(i) $f(t) \geq \bar{u}(t) \geq g(t) \quad , \quad t \in [a,b]$

(ii) $\bar{u}(t)$ satisfies (5.2).

(iii) There are $n-(m+k)+1$ points $s_1 < \ldots < s_{n-(m+k)+1}$ in $[a,b]$ such that

$$\bar{u}(s_i) = \begin{cases} f(s_i) & i \text{ odd} \\ g(s_i) & i \text{ even}. \end{cases}$$

$\bar{u}(t)$ is unique if any one of the following hold.

(1) $N_a(\mathfrak{A}_k, \mathcal{B}_m) + N_b(\mathfrak{A}_k, \mathcal{B}_m) > 0$

(2) $N_a(\mathfrak{A}_k, \mathcal{B}_m) + N_b(\mathfrak{A}_k, \mathcal{B}_m) = 0$, and $\bar{u}(a) \neq f(a)$,

or $\bar{u}(b) \neq f(b)$, $n-(m+k)+1$ odd
$\bar{u}(b) \neq g(b)$, $n-(m+k)+1$ even .

(3) Property J holds for $\mathcal{U}(\mathfrak{A}_k \cap \mathcal{B}_m)$.

b) Let condition (iii) and (2) be replaced by (iii') and (2') where $f(t)$ and $g(t)$ are interchanged. Then there exists a polynomial $\underline{u}(t)$ satisfying (i), (ii) and (iii'), and it is unique if (1), (2') or (3) hold.

Proof. [10, Theorem 3.3].

Remark 5.2 . The results of Theorem 5.2 extend, as in Section 2, to a closed subset K of $[a,b]$, and to functions $f(t)$, $g(t)$ as in Proposition 2.2.

Remark 5.3 . Theorems 1 and 2, together with Theorem 5.1, have an additional application in the theory of best polynomial approximation. Let $\{u_i(t)\}_{i=0}^n$ be an ECT-system on

$[a,b]$, satisfying the initial conditions (5.1). For $c > c_0$, let $\bar{u}(c;t) = \sum_{i=0}^{n} a_i(c)u_i(t)$ and $\underline{u}(c;t) = \sum_{i=0}^{n} b_i(c)u_i(t)$ be as defined in Theorem 1. Let $u(t) = \sum_{i=0}^{n} d_i u_i(t)$ be any polynomial such that $f(c;t) \geq u(t) \geq g(c;t)$ for all $t \in [a,b]$. Then, $(-1)^i a_i(c) \geq (-1)^i d_i \geq (-1)^i b_i(c)$, $i = 0,1,\ldots,n$. Obviously $(-1)^i a_i(c) \downarrow (-1)^i a_i^*$, and $(-1)^i b_i(c) \uparrow (-1)^i a_i^*$, $i=0,1,\ldots,n$, where $u^*(t) = \sum_{i=0}^{n} a_i^* u_i(t)$ is as in Theorem 2.

6. APPLICATIONS OF REPRESENTATION THEOREMS WITH BOUNDARY CONDITIONS

I. Let the conditions of Section 5 hold. We wish to approximate $f(t)$ by polynomials $u(t) = \sum_{i=0}^{n} a_i u_i(t)$ which satisfy (5.2), where Postulate I obtains.

Let $v(t)$ be any polynomial satisfying (5.2). Then the above problem is equivalent to approximating $f(t)-v(t)$ by polynomials $u(t) \in \mathcal{U}(\mathfrak{A}_k \cap \mathcal{B}_m)$. Let $f(t)-v(t) = \tilde{f}(t)$ and assume $\tilde{f}(t)$ is not a polynomial in $\mathcal{U}(\mathfrak{A}_k \cap \mathcal{B}_m)$.

The analysis follows that given in the earlier sections. As such, we state the results, without proof.

Existence is readily shown, and let

$$\min_{u \in \mathcal{U}(\mathfrak{A}_k \cap \mathcal{B}_m)} ||\tilde{f}-u|| = c_0 > 0 \; .$$

<u>Case 1:</u> $N_a(\mathfrak{A}_k, \mathcal{B}_m) > 0$ and $|\tilde{f}(a)| < c_0$, and $N_b(\mathfrak{A}_k, \mathcal{B}_m) > 0$ and $|\tilde{f}(b)| < c_0$.

Then there exists a unique polynomial of best approximation $u^*(t)$ from $\mathcal{U}(\mathfrak{A}_k \cap \mathcal{B}_m)$ to $\tilde{f}(t)$ and it is characterized by the property that $u^*(t)$ alternates $n-m-k+1$ times between $\tilde{f}(t) + c_0$ and $\tilde{f}(t) - c_0$.

<u>Case 2:</u> $N_a(\mathfrak{A}_k, \mathcal{B}_m) > 0$ and $|\tilde{f}(a)| = c_0$, or $N_b(\mathfrak{A}_k, \mathcal{B}_m) > 0$ and $|\tilde{f}(b)| = c_0$.

Uniqueness does not necessarily follow. Counterexamples are easily constructed.

<u>Case 3:</u> $N_a(\mathfrak{A}_k, \mathcal{B}_m) = N_b(\mathfrak{A}_k, \mathcal{B}_m) = 0$, and $\mathcal{U}(\mathfrak{A}_k \cap \mathcal{B}_m)$ satisfies Property J.

Uniqueness of the best approximation and its characterization in terms of $n-m-k+1$ alternations follow.

<u>Case 4:</u> $N_a(\mathfrak{A}_k, \mathcal{B}_m) = N_b(\mathfrak{A}_k, \mathcal{B}_m) = 0$, and Property J is not satisfied.

In this case uniqueness need not hold. Counterexamples may be constructed. However, if there exists a polynomial which alternates $n-m-k+2$ times between $\tilde{f}(t)+c_0$ and $\tilde{f}(t)-c_0$, or if a polynomial alternates $n-m-k+1$ times between $\tilde{f}(t)+c_0$ and $\tilde{f}(t)-c_0$, but a or b is not a point of alternation, then uniqueness follows.

<u>Remark 6.1 .</u> Case 3 is equivalent to saying that $\mathcal{U}(\mathfrak{A}_k \cap \mathcal{B}_m)$ has a basis of dimension $n-m-k+1$ which is a T-system on $[a,b]$. This fact may be proven directly.

<u>Remark 6.2 .</u> The results of Sections 3 and 4 apply here with obvious restrictions.

II. The following is a generalization of a problem considered by Roulier and Taylor [12].

As above, $\{u_i(t)\}_{i=0}^n$ is an ECT-system on $[a,b]$, and let $\mathcal{P}$ be the set of polynomials $u(t)$ for which

$$(6.1)\quad \begin{aligned} \alpha_i &\leq \sum_{j=0}^{n} A_{ij}\, D^j\, u(a) \leq \beta_i \quad , \quad i = 1,\ldots,k \quad , \\ \delta_i &\leq \sum_{j=0}^{n} B_{ij}\, D^j\, u(b) \leq \gamma_i \quad , \quad i = 1,\ldots,m \quad , \end{aligned}$$

where the homogenous boundary conditions

$$\sum_{j=0}^{n} A_{ij}\, D^j\, u(a) = 0 \ , \quad i = i_1,\ldots,i_p$$

$$\sum_{j=0}^{n} B_{ij}\, D^j\, u(b) = 0 \ , \quad i = j_1,\ldots,j_q$$

satisfy Postulate I and Property J for all $p = 0,1,\ldots,k$, $\{i_1,\ldots,i_p\} \subseteq \{1,\ldots,k\}$, and $q = 0,1,\ldots,m$, $\{j_1,\ldots,j_q\} \subseteq \{1,\ldots,m\}$. For the special case $u_i(t) = t^i$, $i = 0,1,\ldots,n$, $a = 0$, and conditions of the form $\alpha_j \leq p^{(i_j)}(0) \leq \beta_j$, $j = 1,\ldots,k$, the problem was considered by Roulier and Taylor [12].

For ease of notation, let

$$(6.2)\quad \begin{aligned} \sum_{j=0}^{n} A_{ij}\, D^j\, u(a) &= a_i \ , \quad i = 1,\ldots,k \\ \sum_{j=0}^{n} B_{ij}\, D^j\, u(b) &= b_i \ , \quad i = 1,\ldots,m \ . \end{aligned}$$

<u>Theorem 6.1 .</u> <u>If</u> $N_a(\mathcal{O}_k,\mathcal{B}_m) = N_b(\mathcal{O}_k,\mathcal{B}_m) = 0$, <u>then for any</u> $f(t) \in C[a,b]$, <u>there exists a unique best approximation to</u> $f(t)$ <u>from</u> $\mathcal{P}$.

<u>Proof.</u> Assume $f(t) \notin \mathcal{P}$. By Case 3 above, for each fixed $\{a_i\}_{i=1}^{k}$, $\{b_i\}_{i=1}^{m}$, there exists a unique polynomial of best approximation to $f(t)$ satisfying (6.2), which is

characterized by the property that it alternates $n-m-k+1$ times between $f(t)+c$ and $f(t)-c$, for some $c > 0$. Denote this polynomial by $u(t;\{a_i\}, \{b_i\})$.

Assume $\min_{u\in \mathcal{P}} ||f-u|| = c_0 > 0$, and $u(t;\{a_i\},\{b_i\})$ and $u(t;\{c_i\},\{d_i\})$ are two best approximations to $f(t)$ from $\mathcal{P}$. (Existence of the best approximation is easily proven.) It then follows that

$$\sum_{i=1}^{k} (a_i-c_i)^2 + \sum_{i=1}^{m} (b_i-d_i)^2 > 0 , \quad \text{and} \quad u(t;\{\lambda a_i+(1-\lambda)c_i\} ,$$

$\{\lambda b_i + (1-\lambda)d_i\}) = u(t;\lambda)$, $0 \leqslant \lambda \leqslant 1$, are also best approximations to $f(t)$ from $\mathcal{P}$.

We argue by induction on $m+k$. For $m+k = 0$, the theorem follows. Assume $m+k > 0$. Since

$$\sum_{i=1}^{k} (a_i-c_i)^2 + \sum_{i=1}^{m} (b_i-d_i)^2 > 0 ,$$ there exists an i_0 such that $a_{i_0} \neq c_{i_0}$ or $b_{i_0} \neq d_{i_0}$. Assume $a_{i_0} \neq c_{i_0}$. Thus, $\alpha_{i_0} < \lambda a_{i_0} + (1-\lambda) c_{i_0} < \beta_{i_0}$, $0 < \lambda < 1$.

Consider the above problem where we drop the condition

$$\alpha_{i_0} \leqslant \sum_{j=0}^{n} A_{i_0 j} D^j u(a) \leqslant \beta_{i_0} .$$

By the induction hypothesis and the conditions of the theorem, there exists a unique polynomial of best approximation to $f(t)$, $u^*(t)$, and $||f-u^*|| = c^* \leqslant c_0$. Due to the uniqueness property of $u^*(t)$, $c^* < c_0$. Consider λ fixed, $0 < \lambda < 1$, and $\tilde{u}(t;\varepsilon) = \varepsilon u^*(t) + (1-\varepsilon) u(t;\lambda)$. For ε sufficiently small, $\varepsilon > 0$, $\tilde{u}(t;\varepsilon) \in \mathcal{P}$. But $||f(t)-\tilde{u}(t;\varepsilon)|| \leqslant \varepsilon \, ||f(t)-u^*(t)||+(1-\varepsilon) \, ||f(t)-u(t;\lambda)|| < c_0$. This contradicts our assumptions and uniqueness follows.

Q.E.D.

Theorem 6.2 . Assume that the conditions of Theorem 6.1 hold and that ℓ is the number of conditions in (6.1) for which the unique polynomial of best approximation $u^*(t)$, from $\mathcal{P}$ to $f(t)$ satisfies

$$(6.3)\quad \begin{aligned} &\alpha_i < \sum_{j=0}^{n} A_{ij} D^j u(a) < \beta_i \\ \text{and}\quad &\delta_i < \sum_{j=0}^{n} B_{ij} D^j u(b) < \gamma_i \ . \end{aligned}$$

Then $u^*(t)$ alternates at least $n-m-k+\ell+1$ times between $f(t) + c_0$ and $f(t) - c_0$, where $||f-u^*|| = c_0 > 0$.

Proof. The method of proof is the same as that used in Theorem 6.1. Let $\{i_1,\ldots,i_r\} \subseteq \{1,\ldots,k\}$, $\{k_1,\ldots,k_s\} \subseteq \{1,\ldots,m\}$ be those indices for which (6.3) does not hold. Consider the above problem with the conditions

$$(6.4)\quad \begin{aligned} &\alpha_{i_\ell} \leq \sum_{j=0}^{n} A_{i_\ell j} D^j u(a) \leq \beta_{i_\ell} \quad , \quad \ell = 1,\ldots,r \\ &\delta_{k_\ell} \leq \sum_{j=0}^{n} B_{k_\ell j} D^j u(b) \leq \alpha_{k_\ell} \quad , \quad \ell = 1,\ldots,s \ . \end{aligned}$$

By Case 3, the polynomial of best approximation must alternate at least $n-r-s+1 = n-m-k+\ell+1$ times. The unique polynomial of best approximation under the conditions (6.4) must be $u^*(t)$. Otherwise a contradiction follows as in Theorem 6.1.

Q.E.D.

Remark 6.3 . Theorem 6.2 provides necessary conditions for the unique polynomial of best approximation from $\mathcal{P}$ to $f(t)$. These conditions are by no means sufficient as is easily seen.

Remark 6.4 . It is natural to ask in what way the conditions (6.1) may be generalized. For instance, in place of the rectangle of values considered, can we require that $\{\sum_{j=0}^{n} A_{ij} D^j u(a)\}_{i=1}^{k} \cup \{\sum_{j=0}^{n} B_{ij} D^j u(b)\}_{i=1}^{m}$ be in some closed convex set in $\mathbb{R}^{k+m}$? This question remains open.

In Theorems 6.1 and 6.2, we demanded that $N_a(\mathcal{O}_k, \mathcal{B}_m) = N_b(\mathcal{O}_k, \mathcal{B}_m) = 0$. To drop these restrictions, we must add certain conditions.

<u>Theorem 6.3 .</u> <u>Let the conditions of Theorem</u> 6.1 <u>hold except that if</u> $N_a(\mathcal{O}_k, \mathcal{B}_m) > 0$, <u>then</u> u(a) <u>is one of the conditions in</u> (6.1) <u>and</u> f(a) <u>lies within the bounds given for</u> u(a) , <u>and if</u> $N_b(\mathcal{O}_k, \mathcal{B}_m) > 0$, <u>then the similar conditions hold at</u> b . <u>Then, there exists a unique best approximation to</u> f(t) <u>from</u> $\mathcal{P}$.

<u>Proof.</u> The proof is identical with the proof of Theorem 6.1 except that it is necessary to use Cases 1, 2 and 3 from part I.

Q.E.D.

<u>Theorem 6.4 .</u> <u>Let the assumptions of Theorem</u> 6.3 <u>hold. Then the results of Theorem</u> 6.2 <u>persist, and if</u> $N_a(\mathcal{O}_k, \mathcal{B}_m) > 0$, <u>and</u> $u^*(a) = f(a)+c_0 = \beta_1$, <u>or</u> $u^*(a) = f(a)-c_0 = \alpha_1$, <u>then there exists an additional point of alternation of</u> $f(t)-u^*(t)$. <u>The same conclusion holds with respect to the endpoint</u> b .

<u>Proof.</u> The proof paraphrases that used for Theorem 6.2. Assume, for sake of convenience, that $N_a(\mathcal{O}_k, \mathcal{B}_m) > 0$, while $N_b(\mathcal{O}_k, \mathcal{B}_m) = 0$. If $\alpha_1 < u^*(a) < \beta_1$, then we may drop this condition as in the proof of Theorem 6.2. If, without loss of generality, $u^*(a) = \beta_1$, and $\beta_1 < f(a)+c_0$, then we maintain this assumption and apply Case 1 of part I, while if $\beta_1 = f(a) + c_0$, we may again drop this condition.

Q.E.D.

III. Consider the following problem, which is not strictly within the terms of reference of this paper:

Let $f(t) \in C^n[a,b]$, and define

$$||f||_M = \max\left\{\{|\sum_{j=0}^{n} A_{ij} D^j f(a)|\}_{i=1}^{k}, \{|\sum_{j=0}^{n} B_{ij} D^j f(b)|\}_{i=1}^{m}, ||f||_\infty\right\}$$

where we assume the conditions of the prior problem to hold. We wish to find a polynomial $u^*(t)$ such that $||f-u||_M \leq \inf_u ||f-u||_M$ over the class of all polynomials $u(t)$.

Assume $f(t)$ is not a polynomial. The existence of a $u^*(t)$ is easily shown. We prove uniqueness and give some necessary conditions which $u^*(t)$ must satisfy.

$$\text{Let} \quad \sum_{j=0}^{n} A_{ij} D^j f(a) = f_i^1 \quad , \quad i = 1,\ldots,k$$

$$\sum_{j=0}^{n} B_{ij} D^j f(b) = f_i^2 \quad , \quad i = 1,\ldots,m \quad ,$$

and consider the problem, from II, of best approximating $f(t)$ by polynomials satisfying

$$f_i^1 - c \leq \sum_{j=0}^{n} A_{ij} D^j u(a) \leq f_i^1 + c \quad , \quad i = 1,\ldots,k$$

$$f_i^2 - c \leq \sum_{j=0}^{n} B_{ij} D^j u(b) \leq f_i^2 + c \quad , \quad i = 1,\ldots,m \quad ,$$

$c \geq 0$. Denote the unique polynomial of best approximation by $u(t;c)$ and let $||f(t)-u(t;c)||_\infty = c_0(c)$. Obviously, $c_0(c)$ is a decreasing function of c , $c_0(0) > 0$, and $c_0(c) = c' = \min_u ||f-u||_\infty$ for all $c \geq c''$, c'' some positive number. From the uniqueness of $u(t;c)$, it follows that $c_0(c)$ is a continuous function of c and hence, there exists a unique point c^* such that $c_0(c^*) = c^*$. $u(t;c^*)$ is the unique best approximation and must satisfy the conditions given in Theorems 6.2 and 6.4.

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REFERENCES

[1] Chalmers, B.L. A unified approach to uniform real approximation by polynomials with linear restrictions. Trans. A.M.S. 166: 309-316. 1972.

[2] Dunham, C.B. Simultaneous Chebyshev approximation of functions on an interval. Proc. A.M.S. 18: 472-477. 1967.

[3] Karlin, S. Representation theorems for positive functions. J. Math. and Mech. 12: 599-618. 1963.

[4] Karlin, S. and W. Studden. Tchebycheff systems: with applications in analysis and statistics. Interscience, New York. 1966.

[5] Laurent, P.J. Approximation uniforme de fonctions continues sur un compact avec constraintes de type inegalite. R.I.P.O. 1: 81-95. 1967.

[6] Lewis, J.T. Approximation with convex constraints. SIAM Review 15: 193-217. 1973.

[7] Loeb, H.L., D.G. Morsund, L.L. Schumaker and G.D. Taylor. Uniform generalized weight function polynomial approximation with interpolation. SIAM J. Numer. Anal. 6: 283-293. 1969.

[8] Morsund, D.G. Chebyshev approximation using a generalized weight function. SIAM J. Numer. Anal. 3: 435-450. 1966.

[9] Morsund, D.G. Computational aspects of Chebyshev approximation using a generalized weight function. SIAM J. Numer. Anal. 5: 126-137. 1968.

[10] Pinkus, A. Representation theorems for Tchebycheffian polynomials with boundary conditions and their applications. Isr. J. Math. 17: 11-34. 1974.

[11] Rosman, B.H. and P.D. Rosenbaum. Chebyshev approximation of regulated functions by unisolvent families. J. Approx. Theory 4: 113-119. 1971.

[12] Roulier, J.A. and G.D. Taylor. Uniform approximation by polynomials having bounded coefficients. Math. Abhandlung 36: 126-135. 1971.

[13] Schumaker, L.L. and G.D. Taylor. On approximation by polynomials having restricted ranges II. SIAM J. Numer. Anal. 6: 31-36. 1969.

[14] Taylor, G.D. On approximation by polynomials having restricted ranges. SIAM J. Numer. Anal. 5: 258-268. 1968.

[15] Taylor, G.D. Approximation by functions having restricted ranges III. J. Math. Anal. Appl. 27: 241-248. 1969.