КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'77
София . 1980, с. 403—416

CONSTRUCTIVE FUNCTION THEORY'77
Sofia . 1980, p. 403—416

ON n-WIDTHS IN L^∞. II. SOME RELATED EXTREMAL PROBLEMS

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Summary. In this paper we solve extremal problems of the type

$$\max_{h \in K(\varrho)} \int_0^1 f(x)h(x)dx$$

$$K(\varrho)=\{h : h \in L^\infty[0,1],\ \|h\|_\infty \leqq 1,\ \|Kh\|_\infty \leqq \varrho\}$$

$$(Kh)(x)=\int_0^1 K(x, y)h(y)dy.$$

A motivation for the study of this question comes from Landau inequalities for derivatives of functions on finite intervals. Our methods use results developed by the authors in connection with n-width problems in L^∞.

1. Introduction. In this paper we further develop certain methods and results obtained in our paper [11]. Our purpose here is to use some of the main results of [11] to solve certain extremal problems related to n-widths. These problems arise in the recent work of S. Karlin [5] on inequalities for consecutive derivatives of functions; see also Y. Domar [1]. L. Hormander [2], H. Kallioniemi [3], A. Kolmogorov [6], F. Landau [7], C. Micchelli [10], A. Pinkus [15], A. Sharma, J. Tzimbaliario [16], I. Schoenberg, A. Cavaretta [17], and V. Tichomirov [18] for background material and results on these problems.

Our results both complement and extend the material contained in [5]. In particular, the methods we use in this paper, being based on matrix inequalities, offer an alternative point of view for inequalities for function classes developed by other authors.

Section 2 contains some preliminary background material. Section 3 treats certain matrix extremal problems, while Section 4 is devoted to inequalities for function classes which are derived from the results of Section 3.

2. Preliminaries. The notation of this paper follows that of [11]. For convenience, and in order that this paper may be to a degree self-contained we redefine and restate certain results of [11], necessary in the subsequent analysis.

$A=\|a_{ij}\|_{i=1,\ j=1}^{N\quad M}$ shall denote an $N\times M$ real matrix. Given any integers, $1\leq i_1<\cdots<i_k\leq N$ and $1\leq j_1<\cdots<j_k\leq M$, we set

$$A\begin{pmatrix} i_1,\ldots,i_k \\ j_1,\ldots,j_k \end{pmatrix}=\begin{vmatrix} a_{i_1j_1} & \ldots & a_{i_1j_k} \\ \vdots & & \vdots \\ a_{i_kj_1} & \ldots & a_{i_kj_k} \end{vmatrix}$$

the standard notation for the minors of A.

Definition 2.1. *The matrix A is said to be strictly totally positive of order n (STP_n) if for all $1\leq i_1<\cdots<i_m\leq N$, $1\leq j_1<\cdots<j_m\leq M$, $m\leq n$.*

$$A\begin{pmatrix} i_1,\ldots,i_m \\ j_1,\ldots,j_m \end{pmatrix}>0.$$

If instead all the minors of A of order $\leq n$ are only nonnegative, then A is called totally positive of order n (TP_n).

Definition 2.2. *Let $\mathbf{x}=(x_1,\ldots,x_m)$ be a real vector of m components*

(i) $S^-(\mathbf{x})$ *counts the number of actual sign changes in the sequence $x_1,\ldots,x_m$ with zero terms discarded.*

(ii) $S^+(\mathbf{x})$ *counts the maximum number of sign changes in the sequence $x_1,\ldots,x_m$ where zero terms are assigned values 1 or -1, arbitrarily.*

The following theorem is of central importance in the study of STP_n matrices.

Theorem 2.1. *If A is an $N\times M$ matrix which is STP_n, and if $\mathbf{x}$ is a nontrivial M-dimensional vector satisfying $S^-(\mathbf{x})\leq n-1$, then*

(i) $S^+(A\mathbf{x})\leq S^-(\mathbf{x})$

(ii) *if $S^+(A\mathbf{x})=S^-(\mathbf{x})$, then the first (and last) component of $A\mathbf{x}$ (if zero, then the sign given in determining $S^+(A\mathbf{x})$), agrees in sign with the first (and last) nonzero component of $\mathbf{x}$.*

Remark 2.1. If A is only TP_n, then upon replacing $S^+(A\mathbf{x})$ by $S^-(A\mathbf{x})$ Theorem 2.1 remains valid, see [4, p. 223].

Definition 2.3. *Given $0=j_0<j_1<\cdots<j_s<j_{s+1}=M+1$ and a vector $\mathbf{x}\varepsilon R^M$ we shall say that $\mathbf{x}$ alternates between $j_1,\ldots,j_s$, provided there exists a sign σ, $\sigma^2=1$, such that $x_k=(-1)^{i+1}\sigma$, $j_{i-1}<k<j_i$, $i=1,\ldots,s+1$. (Note that if $\mathbf{x}$ alternates between $j_1,\ldots,j_s$, no constraints are placed on these components.) We shall also say that $\mathbf{x}$ alternates s times if there exists $1\leq j_1<\cdots<j_s\leq M$ such that $\mathbf{x}$ alternates between $j_1,\ldots,j_s$. The terminology $\mathbf{x}$ alternates with positive orientation means that $\sigma=1$ in the above, that is, $x_k=(-1)^{i+1}$, $j_{i-1}<k<j_i$, $i=1,\ldots,s+1$.*

Definition 2.4. *A vector $\mathbf{y}\varepsilon R^N$ equioscillates on $i_1,\ldots,i_n$, $1\leq i_1<\cdots<i_n\leq N$, if there exists a sign σ, $\sigma^2=1$, satisfying $y_{i_k}=\sigma(-1)^{k+1}\|\mathbf{y}\|_\infty$, $k=1,\ldots,n$,* where $\|\mathbf{y}\|_\infty=\max\{|y_m|:1\leq m\leq N\}$. *We shall also say y equioscillates n times if there exist integers $1\leq i_1<\cdots<i_n\leq N$, such that $\mathbf{y}$ equioscillates on $i_1,\ldots,i_n$. $\mathbf{y}$ equioscillates with positive orientation if $\sigma=1$.*

In [11], the following results were proved.

Theorem 2.2. *Let A be an $N\times M$ STP_{k+1} matrix, where* $0\leq k<\min\{N, M\}$. *Then there exists a unique M-vector* $\mathbf{x}^0$ *satisfying*

a) $\mathbf{x}^0$ *alternates* k *times with positive orientation*,

b) $\|\mathbf{x}^0\|_\infty=1$,

c) $A\mathbf{x}^0$ *equioscillates* $k+1$ *times*.

Theorem 2.3. *Assume A is an $N\times M$ matrix which is* TP_{k+1} (rank $A\geq k+1$), $0\leq k<\min\{N, M\}$, *and such that any s columns of A, $s\leq k$, are linearly independent. Then there exists an M-vector $\mathbf{x}^0$ (not necessarily unique) satisfying* a), b), c) *of Theorem* 2.2. *Furthermore, if $\mathbf{x}^0$ and $\mathbf{y}^0$ both satisfy* a), b) *and* c) *of Theorem* 2.2., *then* $\|A\mathbf{x}^0\|_\infty=\|A\mathbf{y}^0\|_\infty$.

Theorem 2.3. is used in [11] to compute the n-width of the set $\mathcal{A}=\{A\mathbf{x}:\|\mathbf{x}\|_\infty\leq 1\}$, considered as a subset of R^N. To elaborate on this fact further recall that the n-width of $\mathcal{A}$ in R^N is defined as

$$d_n(\mathcal{A};\ R^N)=\inf_{X_n}\ \sup_{\mathbf{x}\varepsilon\mathcal{A}}\ \inf_{\mathbf{y}\varepsilon X_n}\|x-y\|_\infty,$$

where X_n is any n-dimensional subspace of R^N. Any subspace which achieves the infimum above is called an optimal subspace.

Although the following theorem will not be used here, it nevertheless serves to explain the relationship of Theorem 2.3, as well as our subsequent results to the n-width of $\mathcal{A}$.

Theorem 2.4. *Let A satisfy the hypotheses of Theorem 2.3. Then* $d_k(\mathcal{A};\ R^N)=\|A\mathbf{x}^0\|_\infty$ *and* $X_k^0=\{\sum_{i=1}^k c_i\mathbf{a}_{j_i^0}:(c_1,\ldots,c_k)\varepsilon R^k\}$ *is an optimal subspace, where* $j_1^0,\ldots,j_k^0$ *are the columns on which* $\mathbf{x}^0$ *alternates and* $\mathbf{a}_{j_i^0}$ *is the corresponding column vector of* A.

Remark 2.2. The orientation of $A\mathbf{x}^0$ necessarily agrees with that of $\mathbf{x}^0$ by Theorem 2.1., part (ii).

The following easily proven results may also be found in [11], as applications of Theorem 2.1.

Proposition 2.1. *Assume A and $\mathbf{x}^0$ are as in Theorem* 2.2. *Then any M-vector $\mathbf{x}$ which alternates k times satisfies* $\|A\mathbf{x}^0\|_\infty\leq\|A\mathbf{x}\|_\infty$.

Proposition 2.2. *Assume A and $\mathbf{x}^0$ are as in Theorem 2.2. Let $\mathbf{x}$ be any M-vector for which* $\|\mathbf{x}\|_\infty\leq 1$, *and* $(A\mathbf{x})_{i_m}(-1)^m\sigma\geq 0$, $m=1,\ldots,k+1$ *where* $\sigma^2=1$, *for some $k+1$ components* $1\leq i_1<\cdots<i_{k+1}\leq N$. *Then*

$$\min\{|(A\mathbf{x})_{i_m}|:\ 1\leq m\leq k+1\}\leq\|A\mathbf{x}^0\|_\infty.$$

Finally, we conclude this section by stating some properties relating the concepts of Definitions 2.2, 2.3 and 2.4.

Lemma 2.1. *Let $\mathbf{x}$ and $\mathbf{y}$ be M-vectors which alternate s and m times, respectively, and assume* $\|x\|_\infty=\|y\|_\infty=1$. *Then*

a) $S^-(\mathbf{x}-\alpha\,\mathbf{y})\leq s$, *if* $|\alpha|\leq 1$

b) $S^-(\mathbf{x}-\mathbf{y})\leq\min(s, m)$, *if* $\|\mathbf{x}\|_\infty=\|\mathbf{y}\|_\infty$,

c) $S^-(\mathbf{x}-\mathbf{y})\leq s-1$ *if* $s=m$ *and $\mathbf{x}$ and $\mathbf{y}$ have the same orientation.*

Lemma 2.2. *Let $\mathbf{x}$ and $\mathbf{y}$ be N-vectors which equioscillate n and s times, respectively. Then*

a) $S^+(\mathbf{x}-\mathbf{y})\geqq n-1$, if $\|\mathbf{x}\|_\infty\geqq\|\mathbf{y}\|_\infty$,

b) $S^+(\mathbf{x}-\mathbf{y})\geqq\max(n, s)-1$, if $\|\mathbf{x}\|_\infty=\|\mathbf{y}\|_\infty$,

c) $S^+(\mathbf{x}-\mathbf{y})\geqq n$, if $\|\mathbf{x}\|_\infty=\|\mathbf{y}\|_\infty$, $s=n<N$ *and* $\mathbf{x}$, $\mathbf{y}$ *equioscillate with the same orientation.*

3. Matrix Extremal Problems. Part I. Construction of Extremal Vectors. For each integer k, $0\leqq k<\min\{M, N\}=r+1$, we let $\mathbf{x}_k^0$ be the unique vector which satisfies the conditions of Theorem 2.2.

Let $\varrho_m=\|A\mathbf{x}_m^0\|_\infty$, $m=0,1,\ldots, k$. Then we have

Lemma 3.1. *Let A be an $N\times M$ STP_{k+1} matrix, where $0<k\leqq\min\{N, M\}=r+1$. Then $\varrho_k<\varrho_{k-1}$.*

Proof. Assume that $\varrho_k\geqq\varrho_{k-1}$, that is $\|A\mathbf{x}_k^0\|_\infty\geqq\|A\mathbf{x}_{k-1}^0\|_\infty$. Since $A\mathbf{x}_k^0$ eguioscillates $k+1$ times, $S^+(A(\mathbf{x}_k^0-\mathbf{x}_{k-1}^0))\geqq k$. However, $\mathbf{x}_{k-1}^0$ alternates $k-1$ times, and thus $S^-(\mathbf{x}_k^0-\mathbf{x}_{k-1}^0)\leqq k-1$, by Lemma 2.1, b). From Theorem 2.1, and since $\mathbf{x}_k^0-\mathbf{x}_{k-1}^0\neq 0$, $S^+(A(\mathbf{x}_k^0-\mathbf{x}_{k-1}^0))\leqq S^-(\mathbf{x}_k^0-\mathbf{x}_{k-1}^0)$. The contradiction is immediate and the lemma is proved.

Our principal goal in this part of the paper is to construct two unique one-parameter families of M-vectors $\mathbf{x}_k^1(\varrho)$ and $\mathbf{x}_k^2(\varrho)$, for $\varrho_k<\varrho<\varrho_{k-1}$, $1\leqq k\leqq r$, such that $\mathbf{x}_k^s(\varrho)$, $s=1,2$, alternates k times with positive orientation, while $A\mathbf{x}_k^s(\varrho)$ equioscillates k times with orientation $(-1)^{s+1}$, $s=1,2$. (Sometimes we will write $\mathbf{x}^s(\varrho)$ for $\mathbf{x}_k^s(\varrho)$.) Specifically, we will prove

Theorem 3.1. *Let ϱ be prescribed, $\varrho_k<\varrho<\varrho_{k-1}$. Assume A is an $N\times M$ STP_{k+1} matrix and $0<k\leqq r$. Then there exist unique $\mathbf{x}_k^1(\varrho)$ and $\mathbf{x}_k^2(\varrho)$ satisfying*

(i) $\mathbf{x}_k^s(\varrho)$ *alternates k times with positive orientation*, $s=1,2$;

(ii) $\|\mathbf{x}_k^s(\varrho)\|_\infty=1$ *and* $\|A\mathbf{x}_k^s(\varrho)\|_\infty=\varrho$, $s=1,2$;

(iii) $A\mathbf{x}_k^s(\varrho)$ *equioscillates k times with orientation* $(-1)^{s+1}$, $s=1,2$, i. e., *there exist* $1\leqq i_1^s(\varrho)<\cdots<i_k^s(\varrho)\leqq N$, $s=1,2$ *for which* $(A\mathbf{x}_k^s(\varrho))_{i_m^s(\varrho)}=(-1)^{m+s}\varrho$, $m=1,\ldots, k$.

Proof. Let

$$B(\varrho)=\begin{bmatrix} & & 0\\ & A & \vdots\\ & & 0\\ 0 & \ldots\ 0 & \varrho\end{bmatrix}.$$

Thus $B(\varrho)$ is the $(N+1)\times(M+1)$ matrix $B(\varrho)=\|b_{ij}(\varrho)\|_{i=1,\ j=1}^{N+1,\ M+1}$, where

$$b_{ij}(\varrho)=\begin{cases} a_{ij}, & i=1,\ldots, N,\ j=1,\ldots, M\\ \varrho, & i=N+1\quad j=M+1\\ 0, & \text{otherwise.}\end{cases}$$

For $\varrho > 0$, $B(\varrho)$ is TP_{k+1} and any r columns, $r \leqq k$ are linearly independent Thus, from Theorem 2.3., there exists for each $\varrho > 0$, an $M+1$-vector $\mathbf{x}^0(\varrho) = (((\mathbf{x}^0(\varrho))_1, \ldots, (\mathbf{x}^0(\varrho))_{M'}(\mathbf{x}^0(\varrho))_{M+1})$ satisfying

a) $\mathbf{x}^0(\varrho)$ alternates k times with positive orientation,

b) $\| \mathbf{x}^0(\varrho) \|_\infty = 1$;

c) $B(\varrho)\mathbf{x}^0(\varrho)$ equioscillates $k+1$ times.

($\mathbf{x}^0(\varrho)$ is, in fact, unique. However this result is unnecessary in the analysis of the theorem.)

Let us denote the components of alternation of $\mathbf{x}^0(\varrho)$ by $\mathbf{j}^0(\varrho) = (j_1^0(\varrho), \ldots, j_k^0(\varrho))$, and the components of equioscillation of $B(\varrho) \cdot \mathbf{x}^0(\varrho)$ by $\mathbf{i}^0(\varrho) = (i_1^0(\varrho), \ldots, i_{k+1}^0(\varrho))$.

The proof of the theorem is divided into a series of lemmas.

Lemma 3.2. *If $\varrho_k < \varrho < \varrho_{k-1}$ then $\| B(\varrho)\mathbf{x}^0(\varrho) \|_\infty = \varrho$.*

Proof. Define $\mathbf{z} = ((\mathbf{x}_k^0)_1, \ldots, (\mathbf{x}_k^0)_M, (-1)^k)$. Since $\mathbf{x}_k^0$, from Theorem 2.2, alternates k times with positive orientation so does $\mathbf{z}$. From Proposition 2.1, $\| B(\varrho)\mathbf{x}^0(\varrho) \|_\infty \leqq \| B(\varrho)\mathbf{z} \|_\infty$. Now $|(B(\varrho)\mathbf{z})_i| = |(A\mathbf{x}_k^0)_i| \leqq \varrho_k$, $i = 1, \ldots, N$ and $|(B(\varrho)\mathbf{z})_{N+1}| = \varrho$. But $\varrho > \varrho_k$ and thus we conclude half of the lemma,

$$\| B(\varrho)\mathbf{x}^0(\varrho) \|_\infty \leqq \varrho. \tag{3.1}$$

Now, let $\widehat{\mathbf{z}} = ((\mathbf{x}_{k-1}^0)_1, \ldots, (\mathbf{x}_{k-1}^0)_M, (-1)^k)$. From Theorem 2.3, (c), there exists $1 \leqq i_1 < \cdots < i_k \leqq N$ satisfying

$$(B(\varrho)\widehat{\mathbf{z}})_{i_m} = (A\mathbf{x}_{k-1}^0)_{i_m} = (-1)^{m+1}\varrho_{k-1}, \quad m = 1, \ldots, k.$$

By our construction $(B(\varrho)\widehat{\mathbf{z}})_{N+1} = (-1)^k\varrho$. Thus using Proposition 2.2. and $\varrho < \varrho_{k-1}$, we obtain

$$\varrho \leqq \| B(\varrho)\mathbf{x}^0(\varrho) \|_\infty \tag{3.2}$$

which, combined with (3.1), proves the lemma.

For completeness, we note the following easily proven lemmas.

Lemma 3.3. *For $\varrho \leqq \varrho_k$, $\| B(\varrho)\mathbf{x}^0(\varrho) \|_\infty = \varrho_k$ and $\mathbf{x}^0(\varrho) = ((\mathbf{x}_k^0)_1, \ldots, (\mathbf{x}_k^0)_M, (-1)^k)$.*

Lemma 3.4. *For $\varrho \geqq \varrho_{k-1}$, $\| B(\varrho)\mathbf{x}^0(\varrho) \|_\infty = \varrho_{k-1}$ and $\mathbf{x}^0(\varrho) = ((\mathbf{x}_{k-1}^0)_1, \ldots, (\mathbf{x}_{k-1}^0)_M, (-1)^k\varrho_{k-1}/\varrho)$.*

Returning to the proof of Theorem 3.1, we have

Lemma 3.5. *Let $\varrho_k < \varrho < \varrho_{k-1}$ and suppose $1 \leqq i_1^0(\varrho) < \cdots < i_{k+1}^0(\varrho) \leqq N+1$, $1 \leqq j_1^0(\varrho) < \cdots < j_k^0(\varrho) \leqq M+1$ are the components of equioscillation of $B(\varrho)\mathbf{x}^0(\varrho)$ and alternation of $\mathbf{x}^0(\varrho)$, respectively. Then $i_{k+1}^0(\varrho) = N+1$ and $j_k^0(\varrho) \leqq M$.*

Proof. Let $\mathbf{w}^0(\varrho)$ be the restriction of $\mathbf{x}^0(\varrho)$ to its first M components. Thus $(B(\varrho)\mathbf{x}^0(\varrho))_i = (A\mathbf{w}^0(\varrho))_i$, $i = 1, \ldots, N$. If $j_k^0(\varrho) = M+1$ then $\mathbf{w}^0(\varrho)$ alternates $k-1$ times. Thus from Proposition 2.1, $\varrho_{k-1} = \| A\mathbf{x}_{k-1}^0 \|_\infty \leqq \| A\mathbf{w}^0(\varrho) \|_\infty$, contradicting the fact that $\| A\mathbf{w}^0(\varrho) \|_\infty \leqq \varrho$. Thus $j_k^0(\varrho) \leqq M$ and $\mathbf{w}^0(\varrho)$ alternates k times.

Now, if $i_{k+1}^0(\varrho) \leqq N$, then $A\mathbf{w}^0(\varrho)$ equioscillates $k+1$ times. Proposition 2.2 implies that

$$\| A\mathbf{w}^0(\varrho) \|_\infty \leqq \| A\mathbf{x}_k^0 \|_\infty = \varrho_k,$$

contradicting $\|Aw^0(\varrho)\|_\infty=\varrho$ and $\varrho>\varrho_k$. Thus $i^0_{k+1}(\varrho)=N+1$ and the lemma is proven.

Since $\|w^0(\varrho)\|_\infty=1$ and $w^0(\varrho)$ has positive orientation $(Aw^0(\varrho))_{i^0_m(\varrho)}=(B(\varrho)x^0(\varrho))_{i^0_m(\varrho)}=(-1)^{m+1}\varrho$, $m=1,\ldots,k$, it follows that $w^0(\varrho)\equiv x^1(\varrho)$ satisfies the requirements of Theorem 3.1.

The construction of $x^2(\varrho)$ is achieved in a totally analogous manner by considering the matrix

$$F(\varrho)=\begin{bmatrix} \varrho & 0 \ldots 0 \\ 0 & \\ \vdots & A \\ 0 & \end{bmatrix}.$$

We omit the straightforward details.

To complete the proof of Theorem 3.1 it remains to prove the uniqueness of $x^1(\varrho)$ and $x^2(\varrho)$.

Lemma 3.6. *The vectors* $x^1(\varrho)$ *and* $x^2(\varrho)$ *constructed above are unique.*

Proof. Assume the existence of two distinct M-vectors w and z satisfying (i), (ii), (iii) of Theorem 3.1 with $s=1$. From Lemma 2.1, c), $S^-(w-z)\leqq k-1$, while Lemma 2.2 c), implies $S^+(A(w-z))\geqq k$. This contradicts Theorem 2.1 and completes the proof of the lemma as well as Theorem 3.1.

Let $i^s(\varrho)=(i^s_1(\varrho),\ldots,i^s_k(\varrho))$ and $j^s(\varrho)=(j^s_1(\varrho),\ldots,j^s_k(\varrho))$ denote the components of equioscillation of $Ax^s(\varrho)$ and alternation of $x^s(\varrho)$, $s=1,2$, respectively.

Remark 3.1. It should be noted that the uniqueness of $x^s(\varrho)$, $s=1,2$ does not necessarily imply the uniqueness of $i^s(\varrho)$ or $j^s(\varrho)$.

We now describe certain interlacing properties which hold for the components $i^s(\varrho)$ and $j^s(\varrho)$, $s=1,2$ when $\varrho_k\leqq\varrho<\varrho_{k-1}$. Recall that $x^1(\varrho_k)=x^2(\varrho_k)=x^0_k$, $k=0,1,\ldots,r$, and thus we shall dispense with the superscript notation which distinguishes these vectors for $\varrho=\varrho_k$.

Proposition 3.1. *Let A be as in Theorem* 3.1. *Then*

(i) $j_m(\varrho_k)\leqq j_m(\varrho_{k-1})\leqq j_{m+1}(\varrho_k)$, $m=1,\ldots,k-1$.

(ii) $i_m(\varrho_k)\leqq i_m(\varrho_{k-1})\leqq i_{m+1}(\varrho_k)$, $m=1,\ldots,k$.

Proof. Since x^0_{k-1} alternates $k-1$ times $S^-(x^0_{k-1}\pm x^0_k)\leqq k-1$. Ax^0_{k-1} equioscillates on some k components and $\varrho_{k-1}>\varrho_k$. Thus $S^+(A(x^0_{k-1}\pm x^0_k))\geqq k-1$. From Theorem 2.1 it follows that

$$S^-(x^0_{k-1}\pm x^0_k)=S^+(A(x^0_{k-1}\pm x^0_k))=k-1.$$

It is readily seen that in order that $S^-(x^0_{k-1}\pm x^0_k)=k-1$, (i) must be valid.

Now consider the vectors $z_k=x^0_k\pm rx^0_{k-1}$, $r=\varrho_k/\varrho_{k-1}$, and Az_k. Since $r<1$, $S^-(z_k)\leqq k$. However, $S^+(Az_k)\geqq k$, by Lemma 2.2. A). Therefore $S^-(z_k)=S^+(Az_k)=k$. To insure that $S^+(A(x^0_k\pm rx^0_{k-1}))=k$, (ii) must hold. The proposition is proven.

Proposition 3.2. *For* $\varrho_{k-1}>\varrho'\geqq\varrho>\varrho_k$, *and*

(1) $j^1_m(\varrho)\leqq j^1_m(\varrho')$, $m=1,\ldots,k$,

(2) $i^1_m(\varrho)\leqq i^1_m(\varrho')$, $m=1,\ldots,k$.

Proof. Again Lemma 2.1 implies $S^-(\mathbf{x}^1(\varrho')-\mathbf{x}^1(\varrho))\leqq k-1$, while since $\varrho'>\varrho$ and $A\mathbf{x}^1(\varrho)$ equioscillates k times, $S^+(A(\mathbf{x}^1(\varrho')-\mathbf{x}^1(\varrho)))\leqq k-1$. Thus Theorem 2.1, (i), implies $S^+(A(\mathbf{x}^1(\varrho')-\mathbf{x}^1(\varrho)))=S^-(\mathbf{x}^1(\varrho')-\mathbf{x}^1(\varrho))=k-1$. Moreover, the vector $A(\mathbf{x}^1(\varrho')-\mathbf{x}^1(\varrho))$ is necessarily positively orientated, and utilizing Theorem 2.1, (ii) applied to the vector $\mathbf{x}^1(\varrho')-\mathbf{x}^1(\varrho)$, the statement (1) then follows.

To prove (2), we consider the vectors $(\varrho/\varrho')\mathbf{x}^1(\varrho')$ and $\mathbf{x}^1(\varrho)$. Since $\varrho'>\varrho$, $S^-(\mathbf{x}^1(\varrho)-(\varrho/\varrho')\mathbf{x}^1(\varrho'))\leqq k$. However, by Lemma 2.2, c) $S^+(A(\mathbf{x}^1(\varrho)-(\varrho/\varrho')\mathbf{x}^1(\varrho')))\geqq k$. Thus equality holds above, and again applying Theorem 2.1, (ii), to $A(\mathbf{x}^1(\varrho)-(\varrho/\varrho')\mathbf{x}^1(\varrho'))$, the result of (2) holds since the vector $\mathbf{x}^1(\varrho)-(\varrho/\varrho')\mathbf{x}^1(\varrho')$ is necessarily positively oriented. Proposition 3.2 is proven.

The vector $\mathbf{x}^1(\varrho)$, is by construction a uniquely determined continuous function of ϱ for $\varrho_k\leqq\varrho\leqq\varrho_{k-1}$ and $\mathbf{x}^1(\varrho)\to\mathbf{x}^0_k$ as $\varrho\downarrow\varrho_k$, while $\mathbf{x}^1(\varrho)\to\mathbf{x}^0_{k-1}$ as $\varrho\uparrow\varrho_{k-1}$.

We summarize these facts together with Propositions 3.1 and 3.2 in

Proposition 3.3. *As* ϱ *increases from* ϱ_k *to* ϱ_{k-1},

(1) $j^1_m(\varrho)$ *increases from* $j_m(\varrho_k)$ *to* $j_m(\varrho_{k-1})$, $m=1,\ldots,k-1$, *while* $j^1_k(\varrho)$ *increases from* $j_k(\varrho_k)$ *and disappears for* $\varrho=\varrho_{k-1}$.

(2) $i^1_m(\varrho)$ *increases from* $i_m(\varrho_k)$ *to* $i_m(\varrho_{k-1})$, $m=1,\ldots,k$.

In a totally analogous fashion (note that $\mathbf{x}^2(\varrho)\to\mathbf{x}^0_k$ as $\varrho\downarrow\varrho_k$ and $\mathbf{x}^2(\varrho)\to\mathbf{x}^0_{k-1}$ as $\varrho\uparrow\varrho_{k-1}$) we have

Proposition 3.4. *As* ϱ *increases from* ϱ_k *to* ϱ_{k-1},

(1) $j^2_m(\varrho)$ *decreases from* $j_m(\varrho_k)$ *to* $j_{m-1}(\varrho_{k-1})$, $m=2,\ldots,k$, *while* $j^2_1(\varrho)$ *decreases from* $j_1(\varrho_k)$ *and disappears for* $\varrho=\varrho_{k-1}$.

(2) $i^2_m(\varrho)$ *decreases from* $i_{m+1}(\varrho_k)$ *to* $i_m(\varrho_{k-1})$, $m=1,\ldots,k$.

Part II: The Extremal Problem; An Application of Theorem 3.1. Throughout this section, we shall assume the conditions of Theorem 3.1 to hold. In addition, let $\mathbf{a}=(a_1,\ldots,a_M)$ be any M-vector such that the $N+1\times M$ matrix

$$C=\begin{bmatrix}\mathbf{a}\\ A\end{bmatrix}=\|c_{ij}\|^{N}_{i=0}\ {}^{M}_{j=1},$$

where

$$c_{ij}=\begin{cases}a_j, & i=0,\ j=1,\ldots,M\\ a_{ij}, & i=1,\ldots,N,\ j=1,\ldots,M\end{cases}$$

is also STP.

We have associated with A the sequence $\varrho_0>\varrho_1>\cdots>\varrho_r>\varrho_{r+1}=0$, where $r+1=\min\{N,M\}$ and now we shall consider the problem

$$\max\{(\mathbf{a},\mathbf{x}):\mathbf{x}\in\mathcal{A}(\varrho)\},\tag{3.3}$$

where $\mathcal{A}(\varrho)=\{\mathbf{x}:\|\mathbf{x}\|_\infty\leqq 1,\ \|A\mathbf{x}\|_\infty\leqq\varrho\}$ and $(\mathbf{a},\mathbf{x})$ represents the usual inner product of $\mathbf{a}$ and $\mathbf{x}$.

First let us observe that $\max\{(\mathbf{a}, \mathbf{x}): \|\mathbf{x}\|_\infty \leqq 1\} = (\mathbf{a}, \mathbf{x}_0^0)$, where $\mathbf{x}_0^0 = (1, 1, \ldots, 1)$, since $a_j > 0$, $j = 1, \ldots, M$, and this vector is unique. Now, if $\varrho \geqq \varrho_0$, then for $\|\mathbf{x}\|_\infty \leqq 1$ we have $\|A\mathbf{x}\|_\infty \leqq \|A\mathbf{x}_0^0\|_\infty = \varrho_0 \leqq \varrho$ and thus $\mathbf{x}_0^0 \varepsilon \mathcal{A}(\varrho)$. Consequently, (3.3) has the unique trivial solution $\mathbf{x}_0^0$ for $\varrho \geqq \varrho_0$.

Now, let us extend $\mathbf{x}_k^1(\varrho)$ and $\mathbf{x}_k^2(\varrho)$, for $\varrho_0 \leqq \varrho < \varrho_{-1} = \infty$, by defining $\mathbf{x}_0^1(\varrho) = \mathbf{x}_0^2(\varrho) = \mathbf{x}_0^0$. Then we have

Theorem 3.2. *For* $\mathbf{a}$, A *as above and* $\varrho_k \leqq \varrho < \varrho_{k-1}$, $k = 0, \ldots, r$ $\max\{(\mathbf{a}, \mathbf{x}): \mathbf{x} \varepsilon \mathcal{A}(\varrho)\} = (\mathbf{a}, \mathbf{x}_k^1(\varrho))$ *and the maximum is uniquely achieved.*

Proof. By our previous remarks the theorem is proved when $k = 0$. For $k \geqq 1$, we argue by contradiction and assume the existence of an M-vector $\mathbf{y}$ satisfying $\mathbf{a} \cdot \mathbf{y} \geqq \mathbf{a} \cdot \mathbf{x}_k^1(\varrho)$, where $\mathbf{y} \varepsilon \mathcal{A}(\varrho)$. Now since $\varrho_{k-1} > \varrho \geqq \varrho_k$ we have $S_1^-(\mathbf{x}^1(\varrho) - \mathbf{y}) \leqq k$. From the additional properties possessed by $\mathbf{x}_k^1(\varrho)$ and the fact that $\mathbf{a} \cdot \mathbf{y} \geqq \mathbf{a} \cdot \mathbf{x}_k^1(\varrho)$, we have $S^+(C(\mathbf{x}_k^1(\varrho) - \mathbf{y})) \geqq k$. Thus from Theorem 2.2 (i),

$$S^-(\mathbf{x}^1(\varrho) - \mathbf{y}) = S^+(C(\mathbf{x}^1(\varrho) - \mathbf{y})) = k.$$

(If $\varrho = \varrho_k$, then $S^+(C(\mathbf{x}_k^1(\varrho) - \mathbf{y})) \geqq k+1$ and a contradiction is immediate.) However, since $S^-(\mathbf{x}_k^1(\varrho) - \mathbf{y}) = k$, the vector $\mathbf{x}_k^1(\varrho) - \mathbf{y}$ is positively oriented, while $C(\mathbf{x}_k^1(\varrho) - \mathbf{y})$ is negatively oriented (with respect to S^+), and thus we contradict Theorem 2.1, (ii). The theorem is proven.

A corresponding result holds for $\mathbf{x}_k^2(\varrho)$ and any M-vector $\mathbf{b}$ for which the matrix $\widehat{C} = \begin{bmatrix} A \\ b \end{bmatrix}$ is STP. In this case, the maximum in (3.3) is uniquely achieved by the vector $(-1)^k \mathbf{x}_k^2(\varrho)$.

To complete the analysis of the extremum problem (3.3) we need to study what happens for $0 = \varrho_{r+1} \leqq \varrho < \varrho_r$. We consider this case in the next two lemmas.

Lemma 3.7. *Suppose* $0 = \varrho_{r+1} \leqq \varrho < \varrho_r$ *and* $M \leqq N$, *then* (3.3) *is uniquely solved by* $\mathbf{x}_{r+1}^1(\varrho) = (\varrho/\varrho_r)\mathbf{x}_r^0$.

Proof. Since $r = \min(M, N) - 1 = M - 1$, then for any M-vector with $\|\mathbf{x}\|_\infty = 1$, $\mathbf{x}$ alternates r times. Hence according to Proposition 2.1 $\|A\mathbf{x}\|_\infty \geqq \|A\mathbf{x}_r^0\|_\infty = \varrho_r$. Thus we have proved that $\varrho_r \|\mathbf{x}\|_\infty \leqq \|A\mathbf{x}\|_\infty$ and $\varrho_r = \min\{\|A\mathbf{x}\|_\infty : \|\mathbf{x}\|_\infty = 1\}$. Now the vector $\mathbf{x}_{r+1}^1(\varrho)$ defined above alternates $r+1$ times (trivially so since $\|\mathbf{x}_{r+1}^1(\varrho)\|_\infty = \varrho/\varrho_r < 1$) and $A\mathbf{x}_{r+1}^1(\varrho)$ has $r+1$ equioscillations. Thus the previous argument used in Theorem 3.2 proves this lemma.

Lemma 3.8. *Suppose* $0 = \varrho_{r+1} \leqq \varrho < \varrho_r$ *and* $M > N$, *then there exist two unique vectors* $\mathbf{x}_{r+1}^s(\varrho)$, $s = 1, 2$, *which have the properties described in Theorem* 3.1 *and* $\mathbf{x}_{r+1}^1(\varrho)$ *is the unique solution to* (3.3).

Proof. In this case we claim that the proof of Theorem 3.1 implies the existence of $\mathbf{x}_{r+1}^1(\varrho)$. Since clearly $\mathbf{x}_k^0(\varrho)$ exists even when $k = r+1 < \min\{M+1, N+1\} = r+2$. Now Lemma 3.2 and Lemma 3.5 remain valid even in this case. These facts insure that $\mathbf{x}_{r+1}^1(\varrho)$ has all the essential properties except perhaps $\|\mathbf{x}_{r+1}^1(\varrho)\|_\infty = 1$ (note that when $M \leqq N$ this is not satisfied as indicated by Lemma 3.7 above). However, in this case $k = r+1 = \min\{M, N\} = N$ is strictly less than the dimension of $\mathbf{x}_{r+1}^1(\varrho)$. Thus there is at least one component of $\mathbf{x}^1_{r+1}(\varrho)$ distinct from $j_1(\varrho), \dots, j_{r+1}(\varrho)$. Consequently $\mathbf{x}^1_{r+1}(\varrho)$ has max-norm equal to one.

The construction of $\mathbf{x}^2_{k+1}(\varrho)$ follows similarly by using the matrix $F(\varrho)$ defined earlier. The proof of the extremal property of $\mathbf{x}^1_{r+1}(\varrho)$ (and $\mathbf{x}^2_{r+1}(\varrho)$ as well) proceeds as before.

Remark 3.2. When $\varrho=0$ and $M>N$ then according to Lemma 3.8 the solution of (3.3) is a vector, $\mathbf{x}^1_{r+1}(0)$, with the property that $A\mathbf{x}^1_{r+1}(0)=0$ and $\mathbf{x}^1_{r+1}(0)$ alternates $r+1$ times. Of course, when $\varrho=0$ and $M\leq N$, then $\mathcal{A}(\varrho)$ consists only of the zero vector and this is consistent with the statement of Lemma 3.7 as $\varrho \to 0^+$.

This concludes our discussion of matrix inequalities. We now turn to an extension of these results to integral operators.

4. Integral Operators. In this section, we will consider real-valued continuous kernels $K(x,y)$ defined on the unit square $[0,1]\times[0,1]$ with the property that $K(x,y)$ is strictly totally positive (STP), that is

$$K\begin{pmatrix} x_1, \dots, x_m \\ y_1, \dots, y_m \end{pmatrix} = \begin{vmatrix} K(x_1,y_1) & \dots & K(x_1,y_m) \\ \vdots & & \vdots \\ K(x_m,y_1) & \dots & K(x_m,y_m) \end{vmatrix} > 0,$$

for all $0\leqq x_1<\dots<x_m\leqq 1$, $0\leqq y_1<\dots<y_m\leqq 1$, $m\geqq 1$.

We will make use of the following lemma. A stronger version is proved in [11].

Lemma 4.1. *For any constants* $a_1, \dots, a_n$, $0=y_0<y_1<\dots<y_n<y_{n+1}=1$, *the function*

$$g(x)=\sum_{j=0}^{n}(-1)^j\int_{y_j}^{y_{j+1}}K(x,y)dy+\sum_{j=1}^{n}a_jK(x,y_j)$$

has at most n *distinct zeros in* $[0,1]$.

Proof. Suppose to the contrary that $g(x_i)=0$, $i=1,\dots,n+1$, $0\leqq x_1<\dots<x_{n+1}\leqq 1$. If

$$u(x)=K\begin{pmatrix} x_1, \dots, x_{n+1} \\ x, y_1, \dots, y_n \end{pmatrix}=\sum_{j=1}^{n+1}\beta_jK(x_j,x)$$

then $0=\sum_{j=1}^{n+1}\beta_jg(x_j)=\int_0^1|u(x)|dx$ and a contradiction follows, since $u(x)\not\equiv 0$.

Theorem 4.1. *Let* K *be as above. Then for any* $k\geqq 0$ *there is a unique function*

$$P_k(x)=\sum_{j=0}^{k}(-1)^j\int_{\xi_j}^{\xi_{j+1}}K(x,y)dy$$

$0=\xi_0<\xi_1<\dots<\xi_k<\xi_{k+1}=1$ *which equioscillates* $k+1$ *times on* $[0,1]$, $P_k(\eta_j)=(-1)^{j+1}\|P_k\|_\infty$, $j=1,\dots,k+1$, *where* $0\leqq\eta_1<\dots<\eta_{k+1}\leqq 1$, *and* $\|P_k\|_\infty=\max\{|P_k(x)|: 0\leqq x\leqq 1\}$. *Moreover, if* $\varrho_k=\|P_k\|_\infty$ *then* $\varrho_0>\varrho_1>\dots>\varrho_k>\dots \lim_{k\to\infty}\varrho_k=0$.

Proof. This theorem is essentially proved in [11]. We duplicate the simple details here as they will be used again in our next result.

The proof makes use of the following estimates

$$(4.1)\qquad \left|\frac{1}{N}\sum_{k=0}^{m}(-1)^k\sum_{j=j_k}^{j_{k+1}-1}K\left(x,\frac{j}{N}\right)-\sum_{k=0}^{m}(-1)^k\int_{j_k/N}^{j_{k+1}/N}K(x,y)\,dy\right|$$

where $0=j_0<j_1<\ldots<j_{m+1}=N$

$$\leqq\max\{|K(t,y_1)-K(t,y_2)|:\ |y_1-y_2|\leqq 1/N\}=\omega(K(t,\cdot);\ 1/N)$$

and

$$(4.2)\qquad \left|\sum_{j=0}^{m-1}(-1)^j\int_{j/m}^{(j+1)/m}K(x,y)dy\right|\leqq 2^{-1}\omega(K(t,\cdot);\ m^{-1})+m^{-1}\|K\|_\infty$$

$$\|K\|_\infty=\max\{|K(x,y):\ (x,y)\varepsilon[0,1]\times[0,1]\}.$$

The second inequality follows from the identity

$$2\sum_{j=0}^{m-1}(-1)^j\int_{j/m}^{(j+1)/m}K(x,y)dy=\sum_{j=0}^{m-2}(-1)^j\int_{j/m}^{(j+1)/m}(K(x,y)-K(x,y+1/m))dy$$

$$+\int_0^{1/m}K(x,y)dy+(-1)^{m-1}\int_{1-m^{-1}}^{1}K(x,y)dy.$$

Now, for each large integer N we apply Theorem 2.2 to the matrix $a_{ij}=N^{-1}K(i/N,\ j/N)$, $i,\ j=0,1,\ldots,N$. Thus there is an N-vector $\mathbf{x}_k^0(N)$ which alternates k times, at $0\leqq j_1^N<\cdots<j_k^N\leqq N$, $\|\mathbf{x}_k^0(N)\|_\infty=1$ and $A\mathbf{x}_k^0(N)$ equioscillates at $0\leqq i_1^N<\cdots<i_{k+1}^N\leqq N$. There is a subsequence such that $j_p^{N'}/N\to\xi_s$, $s=1,\ldots,k$, $0\leqq\xi_1\leqq\cdots\leqq\xi_k\leqq 1$ and $i_s^{N'}/N'\to\eta_s$, $0\leqq\eta_1\leqq\cdots\leqq\eta_{k+1}\leqq 1$. Moreover, from (4.1) we have that for

$$P_k(x)=\sum_{j=0}^{k}(-1)^j\int_{\xi_j}^{\xi_{j+1}}K(x,y)dy,$$

$$P_k(\eta_j)=(-1)^{j+1}\|P_k\|_\infty\ j=1,\ldots,k+1.$$

Lemma 4.1 insures that $\|P_k\|_\infty>0$ and thus $0\leqq\eta_1<\cdots<\eta_{k+1}\leqq 1$. We conclude that P_k has at least k distinct zeros. Consequently, again by Lemma 4.1, $0<\xi_1<\cdots<\xi_k<1$.

According to Proposition 2.1 $\|A\mathbf{x}_k^0(N)\|_\infty\leqq\|A\mathbf{z}(N)\|_\infty$, where $\mathbf{z}(N)$ is any N-vector which alternates k times. Letting $N\to\infty$ and using (4.2) we obtain

$$\varrho_k\leqq\frac{1}{2}\max_{0\leqq x\leqq 1}\omega\left(K(x,\cdot);\ \frac{1}{k}\right)+\frac{1}{k}\|K\|_\infty$$

and thus $\lim_{k\to\infty}\varrho_k=0$.

The function P_k constructed above is unique. To see this let P be any other function with the same properties as P_k. Thus P has $k+1$ points of equioscillation $0\leqq e_1<\cdots<e_{k+1}\leqq 1$ and k alternations. By Theorem 2.1 and an appropriate limiting argument $(N\to\infty)$, $\|P\|_\infty\geqq\|P_k\|_\infty$. Hence $S^+(E(e_1),\ldots,E(e_{k+1}))\geqq k$, where $E=P-P_k$. But $E(x)=\int_0^1 K(x,y)h(y)dy$, where

$$S^-(h)=\sup\{S^-(h(x_1),\ldots,h(x_m))/x_1<\cdots<x_m\}\leqq k-1.$$

This fact contradicts Theorem 2.1 (again with the obvious limiting argumen as $N\to\infty$). Thus P_k is unique and it remains only to prove that $\varrho_k<\varrho_{k-}$

The fact that $\varrho_k \leqq \varrho_{k-1}$ follows by the same limiting argument $(N \to \infty)$ employed above and Lemma 3.1. Now if $\varrho_k = \varrho_{k-1}$, then for $E_k = P_k - P_{k-1}$ $S^-(E_k(\eta_1), \ldots, E_k(\eta_{k+1})) \geqq k$ but, as before $S^-(E_k) \leqq k-1$. This contradiction implies that $\varrho_k < \varrho_{k-1}$ and completes the proof of the theorem.

Remark 4.1. In addition to the fact that P_k is uniquely determined by the conditions of Theorem 4.1, it is the unique solution of the minimum problem,

$$\min_{0<y_1<\ldots<y_k<1} \max_{0 \leqq x \leqq 1} \left| \sum_{j=0}^{k} (-1)^j \int_{y_j}^{y_{j+1}} K(x,y)dy \right|.$$

The proof of this fact is the same as the uniqueness proof given in Theorem 4.1. Since if

$P(x) = \sum_{j=0}^{k} (-1)^j \int_{y_j}^{y_{j+1}} K(x,y)dy$ and $\|P\|_\infty \leqq \|P_k\|_\infty$ then for $E = P_k - P$ we have $S^+(E) = \sup \{S^+(E(x_1), \ldots, E(x_s)) : x_1 < \cdots < x_s\} \geqq k$, while as before from Lemma 2.1, $S^-(E) \leqq k-1$.

Thus we see from the proof of Theorem 4.1 that using the results of Section 3 and Lemma 4.1 it becomes an easy matter to extend our previous results on matrices to integral operators.

In particular we have

Theorem 4.2. *Let K be strictly totally positive kernel. Then for every $k \geqq 1$ and ϱ, $\varrho_k < \varrho < \varrho_{k-1}$ there exist two unique functions*

$$P_k^s(x; \varrho) = \sum_{j=0}^{k} (-1)^j \int_{\xi_j^s(\varrho)}^{\xi_{j+1}^s(\varrho)} K(x,y)dy, \; s = 1,2$$

$0 = \xi_0^s(\varrho) < \xi_1^s(\varrho) < \cdots < \xi_k^s(\varrho) < \xi_{k+1}^s(\varrho) = 1$ *which equioscillate k times about ϱ with opposite orientation, i. e.,* $P_k^s(\eta_i^s(\varrho); \varrho) = (-1)^{i+s} \| P_k^s(\cdot; \varrho)\|_\infty = (-1)^{i+s}(\varrho)$

$i = 1, \ldots, k$, $0 \leqq \eta_1^s(\varrho) < \eta_2^s(\varrho) < \cdots < \eta_k^s(\varrho) \leqq 1$, $s = 1,2$.

Proof. The proof follows from a limiting argument on Theorem 3.1. In the limit a full set of oscillations must exist since $\varrho > 0$ and a full set of alternations because of the condition $\varrho < \varrho_{k-1}$ and Lemma 4.1.

The uniqueness is proved as in the matrix case. If $P(x) = \Sigma_{j=0}^k (-1)$ $\int_{y_j}^{y_{j+1}} K(x,y)dy$ has k equioscillations with positive orientation then $S^+(P - P_k^1(\cdot; \varrho)) \geqq k$ while as before $S^-(P - P_k^1(\cdot; \varrho)) \leqq k-1$. Hence $P(x) = P_k^1(x; \varrho)$ for all $x \varepsilon [0,1]$.

Just as in the matrix case $P_k^1(x; \varrho_k) = P_k^2(x; \varrho_k)$, and $\xi_i^1(\varrho_k) = \xi_i^2(\varrho_k)$, $i = 1$ $\ldots, k$ (see Remark 4.1) and thus we can drop the superscript notation for $\varrho = \varrho_k$.

Theorem 4.3.

$$\eta_i(\varrho_k) < \eta_i(\varrho_{k-1}) < \eta_{i+1}(\varrho_k), \; i = 1, \ldots, k,$$

$$\xi_i(\varrho_k) < \xi_i(\varrho_{k-1}) < \xi_{i+1}(\varrho_k), \; i = 1, \ldots, k-1.$$

Moreover, for $\varrho_k < \varrho < \varrho_{k-1}$, $i = 1, \ldots, k$, $\xi_i^1(\varrho)$, $\eta_i^1(\varrho)$, are strictly increasing functions of ϱ traversing the intervals $(\xi_i(\varrho_k), \xi_i(\varrho_{k-1}))$, $(\eta_i(\varrho_k), \eta_i(\varrho_{k-1}))$, $(\xi_k(\varrho_{k-1}) = 1)$, *respectively. Similarly* $\eta_i^2(\varrho)$, $\xi_i^2(\varrho)$ *are strictly decreasing functions of* ϱ *traversing the intervals* $(\eta_i(\varrho_{k-1}), \eta_{i+1}(\varrho_k))$, $(\xi_{i-1}(\varrho_{k-1}), \xi_i(\varrho_k))$, $(\xi_0(\varrho_{k-1})=0)$.

The proof is similar to the matrix case and thus we omit the argument. Instead we turn to the extremal problems solved by the functions constructed in Theorems 4.1, 4.2.

For this purpose, we consider the probem

$$\max \{\int_0^1 f(x)h(x)dx : h \in \mathfrak{R}(\varrho)\}, \tag{4.3}$$

where $\mathfrak{R}(\varrho)=\{h: h \in L^\infty[0,1], \|h\|_\infty \leq 1, \|Kh\|_\infty \leq \varrho\}$, $(Kh)(x)=\int_0^1 K(x,y)h(y)dy$ and, in addition, the dual minimum problem

$$\min_\mu(\|f-K^T\mu\|_1+\varrho\|\mu\|). \tag{4.4}$$

Here $(K^T\mu)(x)=\int_0^1 K(y,x)d\mu(y)$, $d\mu(y)$ a signed measure with total variaftion $\|\mu\|$ and $\|\cdot\|_1$ the usual L^1-norm on [0,1]. See [9], [14] for the interpretation of this duality relation in the theory of optimal estimation. Clearly,

$$\int_0^1 f(x)h(x)dx=\int_0^1 (f(x)-(K^T\mu)(x)h(x)dx+\int_0^1 (Kh)(x)d\mu(x) \tag{4.5}$$

and thus

$$\int_0^1 f(x)h(x)dx \leq \|f-K^T\mu\|_1+\varrho\|\mu\| \tag{4.6}$$

or $h \in \mathfrak{R}(\varrho)$.

Now, we assume f has the property that it is in the (strict) convexity cone of the functions $K(x_1,\cdot),\dots,K(x_m,\cdot)$ for all $m \geq 1$ and $0 \leq x_1 < \cdots < x_m \leq 1$. Thus we require that

$$\begin{vmatrix} f(y_1) & f(y_2) & \dots & f(y_{m+1}) \\ K(x_1,y_1) & K(x_1,y_2) & \dots & K(x_1,y_{m+1}) \\ \vdots & \vdots & & \vdots \\ K(x_m,y_1) & K(x_m,y_2) & \dots & K(x_m,y_{m+1}) \end{vmatrix} > 0,$$

for all $0 \leq y_1 < \cdots < y_{m+1} \leq 1$.

For $\varrho_k \leq \varrho < \varrho_{k-1}$ we define the function

$$S(y)=\sum_{i=1}^k a_i K(\eta_i^1(\varrho), y)=(K^T\mu_\varrho^1)(y)$$

by requiring that $f(\xi_i^1(\varrho))=S(\xi_i^1(\varrho))$, $i=1,\dots,k$. Hence

$$f(y)\quad S(y)=\begin{vmatrix} f(y) & f(\xi_1^1(\varrho)) & \dots & f(\xi_k^1(\varrho)) \\ K(\eta_1^1(\varrho),y) & K(\eta_1^1(\varrho),\xi_1^1(\varrho)) & \dots & K(\eta_1^1(\varrho),\xi_k^1(\varrho)) \\ \vdots & \vdots & & \vdots \\ K(\eta_k^1(\varrho),y) & K(\eta_k^1(\varrho),\xi_1^1(\varrho)) & \dots & K(\eta_k^1(\varrho),\xi_k^1(\varrho)) \end{vmatrix} \left[K\begin{pmatrix} \eta_1^1(\varrho),\dots,\eta_k^1(\varrho) \\ \xi_1^1(\varrho),\dots,\xi_k^1(\varrho) \end{pmatrix} \right]^{-1}$$

and it follows that $(-1)^i(f(y)-S(y))>0$, if $\xi_i^1(\varrho)<y<\xi_{i+1}^1(\varrho)$, $i=0,1\ldots, k$, where $\xi_0^1(\varrho)=0$, $\xi_{k+1}^1(\varrho)=1$, and $a_i(-1)^{i+}>0$, $i=1,\ldots, k$. Thus, according to (4.5), for $h(x;\varrho)=(-1)^i, \xi_i^1(\varrho)<x<\xi_{i+1}^1(\varrho)$,

$$\int_0^1 f(x)h(x;\varrho)dx=\int_0^1 (f(x)-S(x))h(x;\varrho)dx+\int_0^1 P_k(x;\varrho)d\mu_\varrho(x)$$

$$=\int_0^1 |f(x)-K^T\mu_\varrho(x)|\,dx+\varrho\,\|\mu_\varrho\|.$$

From this equation and (4.6) we see that $h(x;\varrho)$ solves (4.3) while $d\mu_\varrho(x)$ solves (4.4).

The uniqueness of $h(x;\varrho)$, is argued as follows: If $h(x)$ is another solution of (4.3) then

$$\int_0^1 (f(x)-K^T\mu_\varrho(x))h(x)dx=\|f-K^T\mu_\varrho\|_1$$

and hence $h(x)=\operatorname{sgn}(f(x)-K^T\mu_\varrho(x))=h(x;\varrho)$. If $d\mu(x)$ is another solution of (4.4) then

$$\int_0^1 (f(x)-K^T\mu(x))h(x;\varrho)dx=\|f-K^T\mu\|_1$$

and

$$\int_0^1 P_k(x;\varrho)d\mu(x)=\varrho\,\|\mu\|.$$

Hence, if $P_k(x;\varrho)$ has extrema only on $\eta_1(\varrho),\ldots,\eta_k(\varrho)$, $\dot\varrho_k<\varrho<\varrho_{k-1}$, then $d\mu(x)$ is also unique.

Finally, we remark that the extremal problem of the type (4.3) may be extended to the class of functions

$$\mathfrak{K}_r(\varrho)=\{h:g=\sum_{i=1}^r a_ik_i+Kh,\ \|g\|_\infty\leqq\varrho,\ h\varepsilon L^\infty[0,1],\ \|h\|_\infty\leqq 1,\ (a_1,\ldots,a_r)\epsilon R^r\},$$

where $k_1(x),\ldots,k_r(x)$, $K(x,y)$ (jointly) satisfy a strict total positivity assumption. The details in this case are lengthier.

The exact n-widths for the class $\mathfrak{K}_r(\varrho)$ are studied in [8], [11], [12], [13].

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Received September 1, 1977