A problem of approximation using multivariate polynomials



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# A problem of approximation using multivariate polynomials

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**Contents**



Let $g$ be a fixed polynomial on $\mathbb{R}^d$. We consider the problem of characterizing the span of $\{f(h(\cdot))\}$ for all $f \in C(\mathbb{R})$ as $h$ varies over all shifts and/or dilations of $g$.

## §1. Introduction

The direct contributions of Kolmogorov to approximation theory are few in terms of the number of papers: they can probably be counted on the fingers. However, these contributions have been both profound and influential. In the survey article [9], Tikhomirov reviews six papers of Kolmogorov related to problems in approximation theory. These include a paper on the error bounds in approximating functions in the Sobolev space $\widetilde{W}^{(r)}_\infty$ of periodic functions on $[-\pi, \pi]$ by their partial Fourier sums; the seminal paper on $n$-widths; a famous paper on exact inequalities for intermediate derivatives on the full real line (in the uniform norm), called today the Landau–Kolmogorov inequality; an article which in retrospect is related to an $n$-width problem on octahedra; a paper characterizing best uniform approximations from finite-dimensional subspaces in the space of continuous complex-valued functions on a compact set, today called the Kolmogorov criterion; and the paper where the concept of $\varepsilon$-entropy was introduced.

Conspicuous by its absence is the series of articles related to Hilbert's 13th problem on superposition of functions. Some people are of the opinion that this result does not constitute a part of approximation theory. And they are technically correct. It is the antithesis of the essence of approximation theory. This theorem is concerned with the exact representation of functions, rather than with their approximation. However, this does not imply that it is unimportant to the theory. Sometimes, as in this case, it is just the opposite.

Hilbert's 13th problem is entitled *Impossibility of the solution of the general equation of the 7th degree by means of functions of only two variables.* It seems to have been motivated by problems connected with nomography, a subject which, for various justifiable reasons, is of little or no interest today (the reader may consult Evesham [2] for a discussion and history of nomography). Hilbert conjectured, in his 13th problem, that *the equation of the seventh degree*

$$f^7 + xf^3 + yf^2 + zf + 1 = 0$$

*cannot be solved with the help of any continuous functions of only two variables.* This particular problem was soon generalized in different ways. As Vitushkin aptly put it ([10] p. 256), "Various mathematicians have understood the 13th problem differently and have attributed to it results of a different character". For a history and development of this problem we refer the reader to Lorentz [6] and Vitushkin [10].

In a series of papers in the late 50s, Kolmogorov proved the following startling result, which is considered to be the resolution, in the negative, of Hilbert's 13th problem.

**Theorem 1.1.** *There exist continuous functions of one variable $h_{ij}$ ($i = 1, \dots, \dots, 2d+1$, $j = 1, \dots, d$) such that every continuous function $f$ of $d$ variables on $[0,1]^d$ can be represented in the form*

$$f(x_1, \dots, x_d) = \sum_{i=1}^{2d+1} g_i\Big(\sum_{j=1}^{d} h_{ij}(x_j)\Big), \tag{1.1}$$

*where the $g_i$ are continuous functions of one variable which depend on $f$.*

Since then there have been numerous generalizations of this theorem in various directions. Attempts to understand the nature of this theorem have also led to interesting concepts related to the complexity of functions. Nonetheless the theorem itself has had few, if any, direct applications. This seems rather surprising, since it does state that we can, in some sense, consider multivariate functions as just compositions of univariate functions.

Perhaps one reason for the paucity of applications is related to the following result, which is due to Vitushkin and Henkin (separately and together). See [11] for a survey of their results. It says that one cannot demand that the $h_{ij}$ be $C^1$ functions. Since the answer is in the negative, we only formulate it for functions of two variables in $[0,1]^2$.

**Theorem 1.2.** *Let $h_i, i = 1, \dots, m$, be a set of fixed continuously differentiable functions defined on $[0,1]^2$. Then the set of functions*

$$\Big\{\sum_{i=1}^{m} g_i(h_i(\cdot,\cdot)) : g_i \in C(\mathbb{R})\Big\}$$

*is nowhere dense in the space of all functions continuous in $[0,1]^2$ with the topology of uniform convergence.*

If the representation (1.1), or something similar, had turned out to exist with smooth, calculable $h_{ij}$, then multivariate approximation theory might well look different today. We could then calculate the $h_{ij}$ once and for all, and thus reduce many multivariate problems to univariate problems. This is idle speculation. The $h_{ij}$ are neither smooth nor calculable. Our search of the literature found only one paper, namely Frisch et al [3], which attempted to use the Kolmogorov result directly to approximate multivariate functions.

However, as we have mentioned, there is a fundamental difference between the representation and the approximation of functions. In practice we should give up the expectation of exact representation. It is our hope that it might be possible, using smooth, calculable functions and the idea of superposition (composition), to develop good methods of approximating multivariate functions. This is the theme which we wish to initiate in this paper.

In other words, we will consider specific classes $\Phi$ of functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ and study the linear space

$$\operatorname{span}\{g(\varphi(\cdot)) : \varphi \in \Phi, g \in C(\mathbb{R})\}.$$

If $\Phi$ is composed of finitely many smooth ($C^1$) functions, then this set is, by Theorem 1.2, nowhere dense in the space of all functions continuous on any compact set in $\mathbb{R}^d$ (with non-empty interior) with the topology of uniform convergence. However, if $\Phi$ is composed of an infinite set of functions, then this is no longer necessarily true. In this paper we will look at this density problem and some related algebraic problems for three sets of $\Phi$ each constructed simply by shifts and/or dilations of a fixed polynomial.

## §2. Statements of the problems

Let $g$ be any real-valued polynomial of $d$ variables. Consider all shifts or all dilations, or both, of this fixed polynomial. That is, let

$$G_1 = \operatorname{span}\{ g(\cdot - b_1, \ldots, \cdot - b_d) : \mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{R}^d \},$$
$$G_2 = \operatorname{span}\{ g(a_1\cdot, \ldots, a_d\cdot) : \mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{R}^d \}$$

and

$$G_3 = \operatorname{span}\{ g(a_1 \cdot -b_1, \ldots, a_d \cdot -b_d) : \mathbf{a}, \mathbf{b} \in \mathbb{R}^d \}.$$

Now, for $i = 1, 2, 3$, let

$$\mathcal{G}_i = \operatorname{span}\{ f(h(\cdot)) : h \in G_i,\ f \in C(\mathbb{R}) \}.$$

The question we will consider is that of the density of each of the $\mathcal{G}_i$ in the space of continuous real-valued functions defined on $\mathbb{R}^d$, endowed with the topology of uniform convergence on compact sets. (We could have fixed a compact subset of $\mathbb{R}^d$ (with non-empty interior), and only considered uniform convergence thereon. In the case of $\mathcal{G}_1$ and $\mathcal{G}_3$, the two settings are essentially equivalent. They are equivalent for $\mathcal{G}_2$ if the compact set contains the origin in its interior.) One simple feature of this topology is that if we put

$$\mathcal{P}_1 = \operatorname{span}\{ (g(\cdot - b_1, \ldots, \cdot - b_d))^k : \mathbf{b} \in \mathbb{R}^d,\ k \in \mathbb{Z}_+ \},$$
$$\mathcal{P}_2 = \operatorname{span}\{ (g(a_1\cdot, \ldots, a_d\cdot))^k : \mathbf{a} \in \mathbb{R}^d,\ k \in \mathbb{Z}_+ \}$$

and

$$\mathcal{P}_3 = \operatorname{span}\left\{ \, (g(a_1 \cdot -b_1, \ldots, a_d \cdot -b_d))^k : \mathbf{a}, \mathbf{b} \in \mathbb{R}^d, \ k \in \mathbb{Z}_+ \, \right\},$$

then from Weierstrass' theorem it follows that $\overline{\mathcal{P}}_i = \overline{\mathcal{G}}_i$, $i = 1,2,3$. Another aspect of this topology is that polynomials are dense thereon. Each $\mathcal{P}_i$ is a linear subspace of all polynomials on $\mathbb{R}^d$. Thus if $\mathcal{P}_i = \Pi$ (where $\Pi$ denotes the set of all polynomials on $\mathbb{R}^d$), then we certainly have $\overline{\mathcal{P}}_i = C(\mathbb{R}^d)$ in our topology. However, it may not be necessary that $\mathcal{P}_i = \Pi$ in order to obtain density.

In the last section of this paper we discuss $\mathcal{P}_2$ and $\mathcal{P}_3$. The analysis in both these cases is elementary. We show that $\mathcal{P}_3 = \Pi$ if and only if

$$\frac{\partial g}{\partial x_j} \neq 0, \qquad j = 1, \ldots, d.$$

The necessity of this condition is rather obvious, and thus both the question of reproducing all polynomials and that of density are one and the same. For the space $\mathcal{P}_2$ we prove that $\mathcal{P}_2 = \Pi$ if and only if

$$\frac{\partial g}{\partial x_j}(0) \neq 0, \qquad j = 1, \ldots, d.$$

That is, the necessary and sufficient condition for $\mathcal{P}_2 = \Pi$ is that $g$ contains linear terms in each of the variables $x_j, j = 1, \ldots, d$. That this is not equivalent to the density of $\mathcal{P}_2$ in $C(\mathbb{R}^d)$ can be easily seen even in the case where $d = 1$. Conditions for density are not known when $d \geqslant 2$.

The main object of this paper is the study of $\mathcal{P}_1$, which is much more interesting than either $\mathcal{P}_2$ or $\mathcal{P}_3$. The space $\mathcal{P}_1$ was studied in our paper [7]. A simple necessary condition for $\mathcal{P}_1$ to contain all of $\Pi$ or to be dense in $C(\mathbb{R}^d)$ is that $G_1$ must separate points. Since $g$ is a polynomial, this may be shown to be equivalent to the linear independence of the first partial derivatives of $g$. That is, $G_1$ separates points if and only if the polynomials $\left\{ \frac{\partial g}{\partial x_j} \right\}_{j=1}^{d}$ are linearly independent. This easily verified condition is in fact equivalent to $\mathcal{P}_1 = \Pi$ if $d = 2$. This was the main result of [7]. Moreover the equivalence is not valid for $d \geqslant 4$. In the next section we prove that this equivalence does hold for $d = 3$. We will outline two proofs of this fact. One proof uses differential geometry and algebra. The other method of proof, using algebraic geometry, is a consequence of the main results in a paper of Gordan and Noether [4], which 'corrected' an earlier paper of Hesse [5]. Thinking over these results provides us with a better understanding of the case $d = 4$, although we still do not have a complete characterization of all the polynomials $g$ for which $\mathcal{P}_1 = \Pi$ in $\mathbb{R}^4$. Nor do we know, in general, if it is necessary for the density of $\mathcal{P}_1$ in $C(\mathbb{R}^d)$ that it equal $\Pi$. (For $d = 2$ and $d = 3$ they are one and the same.)

We also wish to point out that while we will be considering all shifts and/or dilations, this is not necessary. We can consider in any of the above problems a significantly restricted set of shifts and dilations (for example, on any lattice in $\mathbb{R}^d$) and get exactly the same results. This follows from the next two propositions.

**Proposition 2.1.** *For every polynomial g, the set*

$$\operatorname{span}\{\, g(\cdot - b_1, \dots, \cdot - b_d) : \mathbf{b} \in \mathbb{R}^d \,\} = \operatorname{span}\{\, g(\cdot - b_1, \dots, \cdot - b_d) : \mathbf{b} \in \mathcal{B} \,\}$$

*is not contained in any algebraic variety if* $\mathcal{B} \subseteq \mathbb{R}^d$ *(that is, no non-trivial polynomial vanishes on* $\mathcal{B}$*).*

**Proposition 2.2.** *For every polynomial g, the set*

$$\operatorname{span}\{\, g(a_1\cdot, \dots, a_d\cdot) : \mathbf{a} \in \mathbb{R}^d \,\} = \operatorname{span}\{\, g(a_1\cdot, \dots, a_d\cdot) : \mathbf{a} \in \mathcal{A} \,\}$$

*is not contained in any algebraic variety if* $\mathcal{A} \subseteq \mathbb{R}^d$.

The proof of Proposition 2.1 may be found in [7], while Proposition 2.2 is proved similarly.

Before embarking on the analysis of the various cases, we recall some standard multivariate notation. For $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{Z}_+^d$, we put

$$|\mathbf{j}| = j_1 + \cdots + j_d, \qquad \mathbf{j}! = j_1! \dots j_d!$$

and

$$\mathbf{x}^j = x_1^{j_1} \dots x_d^{j_d}, \qquad D^{\mathbf{j}} = \frac{\partial^{|\mathbf{j}|}}{\partial x_1^{j_1} \dots \partial x_d^{j_d}}.$$

If $q$ is a polynomial, $q(\mathbf{x}) = \sum_{\mathbf{j}} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$, then by $q(D)$ we mean the differential operator

$$\sum_{\mathbf{j}} c_{\mathbf{j}} D^{\mathbf{j}}.$$

We end this section by noting a relationship between shifts and differentiation, and between dilation and the monomials for polynomials. These two lemmas are both useful and important. They are easily proved.

**Theorem 2.3.** *Let g be any polynomial. Then*

$$\operatorname{span}\{\, g(\cdot - b_1, \dots, \cdot - b_d) : \mathbf{b} \in \mathbb{R}^d \,\} = \operatorname{span}\{\, D^{\mathbf{j}} g : \forall \mathbf{j} \in \mathbb{Z}_+^d \,\}.$$

**Lemma 2.4.** *Let g be any polynomial of the form*

$$g(\mathbf{x}) = \sum_{\mathbf{j}} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}.$$

*Then*

$$\operatorname{span}\{\, g(a_1\cdot, \dots, a_d\cdot) : \mathbf{a} \in \mathbb{R}^d \,\} = \operatorname{span}\{\, \mathbf{x}^{\mathbf{j}} : c_{\mathbf{j}} \neq 0 \,\}.$$

## §3. The space $\mathcal{P}_1$ and $d = 3$

For ease of exposition we first introduce some additional notation. For each polynomial $f$ we denote $\dfrac{\partial f}{\partial x_i}$ by $f_i$. We also put

$$\mathcal{P}(f) = \operatorname{span}\{\, (f(\cdot - b_1, \dots, \cdot - b_d))^k : \mathbf{b} \in \mathbb{R}^d,\ k \in \mathbb{Z}_+ \,\}.$$

(That is, we have changed the notation somewhat in that we have dropped the subscript 1 and shown the dependence on the polynomial.) For $\mathbf{p} = (p_1, \dots, p_d) \in \mathbb{N}^d$ (each $p_i$ is a positive integer) and $\mathbf{j} \in \mathbb{Z}_+^d$ we put $\mathbf{p} \cdot \mathbf{j} = \sum_{i=1}^d p_i j_i$. If

$$f(\mathbf{x}) = \sum_{\mathbf{p}\cdot\mathbf{j}=n} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}},$$

then the $\mathbf{p}$-*degree of* $f$ is $\max\{\mathbf{p} \cdot \mathbf{j} : c_{\mathbf{j}} \neq 0\}$. A polynomial $f$ is said to be $\mathbf{p}$-*homogeneous of* $\mathbf{p}$-*degree* $n$ if

$$f(\mathbf{x}) = \sum_{\mathbf{p}\cdot\mathbf{j}=n} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}.$$

We start with the statement of two simple facts which we will repeatedly use.

**Lemma 3.1.** *Assume that*

$$g = h + f,$$

*where $h$ is a* $\mathbf{p}$*-homogeneous polynomial of* $\mathbf{p}$*-degree $n$, and $f$ is a polynomial of* $\mathbf{p}$*-degree $< n$. If $\mathcal{P}(h) = \Pi$, then $\mathcal{P}(g) = \Pi$.*

Lemma 3.1 follows from a simple induction argument.

**Lemma 3.2.** *Assume that $g$ is a polynomial. The properties of $\mathcal{P}(g) = \Pi$ and $\overline{\mathcal{P}(g)} = C(\mathbb{R}^d)$ are unaffected by a linear (non-singular) change of variables.*

We will use Lemma 3.2 together with the following more fundamental result.

**Proposition 3.3.** *Assume that $g$ is a polynomial, and that the polynomials $\{g_i\}_{i=1}^d$ are linearly independent. Then after a linear (non-singular) change of variables we can decompose $g$ in the form*

$$g = h + f,$$

*where $h$ is a* $\mathbf{p}$*-homogeneous polynomial of* $\mathbf{p}$*-degree $n$, $f$ is a polynomial of* $\mathbf{p}$*-degree $< n$, and the $\{h_i\}_{i=1}^d$ are linearly independent.*

*Proof.* We prove the result by a construction using an inductive argument. We assume that

$$g = h + f,$$

where

$$h(\mathbf{x}) = \sum_{\mathbf{p}\cdot\mathbf{j}=n} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}, \qquad f(\mathbf{x}) = \sum_{\mathbf{p}\cdot\mathbf{j}<n} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}},$$

and

(1) the $\{h_i\}_{i=1}^m$ are linearly independent,
(2) $h_i = 0$, $i = m+1, \dots, d$,
(3) $p_i = p \geqslant p_j$ for all $j = 1, \dots, m$ and $i = m+1, \dots, d$.

To start with we simply let $\mathbf{p} = (1, \dots, 1)$ and take $h$ to be the highest order homogeneous terms of $g$. By a linear change of variables we can easily obtain (1) and (2).

We shall assume that $m < d$, for otherwise there is nothing to prove, and construct a new $h$ satisfying (1), (2) and (3) with a strictly larger $m$.

Let $v/u$ be the irreducible fraction such that

$$\frac{v}{u} = \min\left\{\frac{n - \sum_{i=1}^{m} p_i j_i}{\sum_{i=m+1}^{d} p_i j_i} : c_{\mathbf{j}} \neq 0, \sum_{i=m+1}^{d} p_i j_i > 0\right\}. \tag{3.1}$$

Note that $\sum_{i=1}^{d} p_i j_i \leqslant n$ for all $\mathbf{j}$ with $c_{\mathbf{j}} \neq 0$, and if equality holds, then $p_i = 0$, $i = m+1, \dots, d$. From this it follows that $v > u$.

Let

$$p'_i = \begin{cases} up_i, & i = 1, \dots, m, \\ vp_i, & i = m+1, \dots, d, \end{cases}$$

and

$$n' = un.$$

Thus $p'_i = vp > p'_j$ for all $j = 1, \dots, m$ and $i = m+1, \dots, d$, and (3.1) holds with this new $p'_i$ and any $m' \geqslant m$. Furthermore

$$\mathbf{p}' \cdot \mathbf{j} = \sum_{i=1}^{d} p'_i j_i \leqslant n',$$

for all $\mathbf{j}$ with $c_{\mathbf{j}} \neq 0$.

Let $J^* = \{\mathbf{j} : c_{\mathbf{j}} \neq 0, \mathbf{p}' \cdot \mathbf{j} = n'\}$. Then $J^* = J \cup J'$, where $J = \{\mathbf{j} : c_{\mathbf{j}} \neq 0, \mathbf{p} \cdot \mathbf{j} = n\}$ and $j' = \{\mathbf{j} : \mathbf{j}$ attains the minimum in (3.1)$\}$. We put

$$h^*(\mathbf{x}) = \sum_{\mathbf{p}' \cdot \mathbf{j} = n'} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$$

and

$$f^*(\mathbf{x}) = g(\mathbf{x}) - h^*(\mathbf{x}) = \sum_{\mathbf{p}' \cdot \mathbf{j} < n'} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}.$$

Note that $h^* = h + h'$, where

$$h(\mathbf{x}) = \sum_{\mathbf{p} \cdot \mathbf{j} = n} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$$

and

$$h'(\mathbf{x}) = \sum_{\mathbf{j} \in J'} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}.$$

Not all the $\{h_i^*\}_{i=m+1}^{d}$ are identically zero. By a linear change of variables involving only the variables $x_{m+1}, \dots, x_d$, we may assume that there exists an $m'$,

$m+1 \leqslant m' \leqslant d$, such that (after again denoting the new variables by $x_{m+1}, \dots, x_d$) the $\{h_i^*\}_{i=m+1}^{m'}$ are linearly independent and $h_i^* = 0$, $i = m'+1, \dots, d$ (if $m' < d$).

Since the linear change of variables involves only variables with equal 'weights' $p_i' = vp$, it follows that by introducing these new variables we have not altered the fact that $h^*$ is a $\mathbf{p}'$-homogeneous polynomial of $\mathbf{p}'$-degree $n'$ or that $f^*$ is a polynomial of $\mathbf{p}'$-degree $< n'$.

It remains to prove that the $\{h_i^*\}_{i=1}^{m'}$ are linearly independent. Assume that

$$\sum_{i=1}^{m'} a_i h_i^* = 0. \tag{3.2}$$

Each $h_i^*$ is a $\mathbf{p}'$-homogeneous polynomial of $\mathbf{p}'$-degree $n' - p_i'$. But if $p_j' < p_i'$ for all $j = 1, \dots, m$ and $i = m+1, \dots, m'$, it follows from (3.2) that

$$\sum_{i=1}^{m} a_i h_i^* = 0,$$

and thus

$$\sum_{i=m+1}^{m'} a_i h_i^* = 0.$$

Since the $\{h_i^*\}_{i=m+1}^{m'}$ were constructed to be linearly independent, we have $a_{m+1} = \cdots = a_{m'} = 0$. For $i = 1, \dots, m$,

$$h_i^* = h_i + h_i',$$

where $h_i^*$ is a polynomial of $\mathbf{p}$-degree $n - p_i$ with leading $\mathbf{p}$-homogeneous term $h_i$. Since the $\{h_i\}_{i=1}^{m}$ are linearly independent, it follows that the $\{h_i^*\}_{i=1}^{m}$ are also linearly independent. Thus $a_1 = \cdots = a_m = 0$. This proves the proposition. □

In [7] we proved the following result.

**Proposition 3.4.** *Let $h$ be a $\mathbf{p}$-homogeneous polynomial. Then $\mathcal{P}(h) \neq \Pi$ if and only if there exists a non-trivial $\mathbf{p}$-homogeneous polynomial $q$ satisfying*

$$q(D)(h)^k = 0, \qquad k = 0, 1, 2, \dots .$$

We obtain as a consequence the following result.

**Corollary 3.5.** *If $h$ is a $\mathbf{p}$-homogeneous polynomial, then $\mathcal{P}(h) \neq \Pi$ implies that $\overline{\mathcal{P}(h)} \neq C(\mathbb{R}^d)$.*

In other words, the property of density is in this case equivalent to that of $\mathcal{P}(h)$ containing all polynomials.

*Proof.* If $h$ is a $\mathbf{p}$-homogeneous polynomial and $\mathcal{P}(h) \neq \Pi$, it follows from Proposition 3.4 that

$$q(D)(h)^k = 0, \qquad k = 0, 1, 2, \dots,$$

for some non-trivial polynomial $q$. Note that $q(D)D^{\mathbf{j}}(h)^k = D^{\mathbf{j}}q(D)(h)^k = 0$ for all $\mathbf{j} \in \mathbb{Z}_+^d$ and $k \in \mathbb{Z}_+$. Thus $q(D)$ annihilates

$$\mathcal{P}(h) = \operatorname{span}\{ D^{\mathbf{j}}(h)^k : \mathbf{j} \in \mathbb{Z}_+^d,\ k \in \mathbb{Z}_+ \}$$

(see Lemma 2.3). Now $q(D)$ is a partial differential operator with constant coefficients. Let $q^*(D)$ denote its adjoint. Choose any polynomial $g$ such that $q(D)g = 1$. Such a polynomial exists. Let $\varphi \in C_0^\infty$ be a function with support in a compact set $K$ such that $\int_K \varphi \neq 0$. If $\mathcal{P}(h)$ is dense in $C(K)$, there then exists a sequence of polynomials $p_n \in \mathcal{P}(h)$ which uniformly approximates $g$ on $K$. Thus

$$\lim_{n\to\infty} \int_K (q^*(D)\varphi)p_n = \int_K (q^*(D)\varphi)g.$$

However,

$$\int_K (q^*(D)\varphi)p_n = \int_K \varphi(q(D)p_n) = 0,$$

while

$$\int_K (q^*(D)\varphi)g = \int_K \varphi(q(D)g) = \int_K \varphi \neq 0,$$

a contradiction. □

The condition

$$q(D)(h)^k = 0, \qquad k = 0, 1, 2, \ldots,$$

is difficult to verify or use. We will utilize one part of an equivalent form of this condition. This equivalent form is to be found in [7]. The statement therein is slightly more restrictive, but the proof suffices for the following.

**Proposition 3.6.** *Let $h$ be any polynomial and*

$$q(\mathbf{x}) = \sum_{\mathbf{p}\cdot\mathbf{j}\leqslant t} d_{\mathbf{j}}\mathbf{x}^{\mathbf{j}}.$$

*We put $m = \max\{|\mathbf{j}| : d_{\mathbf{j}} \neq 0\}$. Then*

$$q(D)(h)^k = 0, \qquad k = 0, 1, 2, \ldots,$$

*if and only if*

$$\sum_{\mathbf{p}\cdot\mathbf{j}\leqslant t} d_{\mathbf{j}}\varphi_i(h;\mathbf{j};\cdot) = 0, \qquad i = 1, \ldots, m,$$

*where $\varphi_i(h;\mathbf{j};\cdot)$ is a polynomial which depends on $h$, $\mathbf{j}$ and $i$ (but is independent of $k$ and $m$),*

$$\varphi_{|\mathbf{j}|}(h;\mathbf{j};\cdot) = (h_1)^{j_1}\ldots(h_d)^{j_d},$$

*and $\varphi_i(h;\mathbf{j};\cdot) = 0$ for $i > |\mathbf{j}|$.*

As a consequence we have the following result.

**Proposition 3.7.** *If $h$ is a* $\mathbf{p}$*-homogeneous polynomial and $\mathcal{P}(h) \neq \Pi$, then there exists a homogeneous and* $\mathbf{p}$*-homogeneous polynomial $q^*$ for which*

$$q^*(h_1, \dots, h_d) = 0.$$

*Proof.* From Propositions 3.4 and 3.6 we have the existence of a polynomial

$$q(\mathbf{x}) = \sum_{\mathbf{p}\cdot\mathbf{j}=t} d_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$$

such that

$$\sum_{\mathbf{p}\cdot\mathbf{j}=t} d_{\mathbf{j}} \varphi_i(h; \mathbf{j}; \cdot) = 0, \qquad i = 1, \dots, m, \tag{3.3}$$

where $\varphi_i$ and $m$ are as defined in Proposition 3.6. For $|\mathbf{j}| < m$ we have $\varphi_m(h; \mathbf{j}; \cdot) = 0$, and for $|\mathbf{j}| = m$ we have $\varphi_m(h; \mathbf{j}; \cdot) = (h_1)^{j_1} \cdots (h_d)^{j_d}$. Putting $i = m$ in (3.3), we obtain

$$\sum_{\substack{\mathbf{p}\cdot\mathbf{j}=t \\ |\mathbf{j}|=m}} d_{\mathbf{j}} (h_1)^{j_1} \dots (h_d)^{j_d} = 0.$$

We define

$$q^*(\mathbf{x}) = \sum_{\substack{\mathbf{p}\cdot\mathbf{j}=t \\ |\mathbf{j}|=m}} d_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}.$$

Then $q^*$ is both $\mathbf{p}$-homogeneous and homogeneous. □

We will use Proposition 3.7 and a particular consequence thereof. We recall the following. Let $\mathbf{x} = (x_1, \dots, x_d)$ and

$$h(\mathbf{x}) = \sum_{|\mathbf{j}| \leqslant n} d_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}.$$

For $\mathbf{y} = (y_1, \dots, y_{d+1})$ we define

$$\widetilde{h}(\mathbf{y}) = \sum_{|\mathbf{j}| \leqslant n} d_{\mathbf{j}} y_1^{j_1} \dots y_d^{j_d} y_{d+1}^{n-|\mathbf{j}|}$$

to be the *homogenization* of the polynomial $h$.

**Corollary 3.8.** *Assume that $h$ is a* $\mathbf{p}$*-homogeneous polynomial of* $\mathbf{p}$*-degree $n$ in $\mathbb{R}^d$ ($\mathbf{p} \neq (1, \dots, 1)$), $\mathcal{P}(h)$ separates points, and $\mathcal{P}(h) \neq \Pi$. Then $\widetilde{h}$, the homogenization of $h$, is such that $\mathcal{P}(\widetilde{h})$ separates points, and there exists a homogeneous polynomial $q^*$ for which*

$$q^*(\widetilde{h}_1, \dots, \widetilde{h}_d) = 0.$$

*Proof.* As stated previously, $\mathcal{P}(h)$ separates points if and only if the first partial derivatives of $h$ are linearly independent. We prove that the first partial derivatives of $\widetilde{h}$ are linearly independent. Let

$$\sum_{i=1}^{d+1} a_i \widetilde{h}_i = 0.$$

We must have $a_{d+1} = 0$. This follows from the fact that $\widetilde{h}_{d+1}(x_1, \ldots, x_d, 1)$ is a **p**-homogeneous polynomial of **p**-degree $n$, while $\widetilde{h}_i(x_1, \ldots, x_d, 1) = h_i(x_1, \ldots, x_d)$ is a **p**-homogeneous polynomial of **p**-degree $n - p_i < n$ for each $i = 1, \ldots, d$. Since $a_{d+1} = 0$, we see, putting $x_{d+1} = 1$, that

$$\sum_{i=1}^{d} a_i h_i = 0.$$

Since the $\{h_i\}_{i=1}^d$ are linearly independent, we see that $a_1 = \cdots = a_d = 0$.

Since $h$ is a **p**-homogenous polynomial and $\mathcal{P}(h) \neq \Pi$, it follows from Proposition 3.7 that there is a homogenoeus polynomial $q^*$ such that

$$q^*(h_1, \ldots, h_d) = 0.$$

Consider the polynomial $q^*(\widetilde{h}_1, \ldots, \widetilde{h}_d)$. Since both $q^*$ and the $\widetilde{h}_i, i = 1, \ldots, d+1$, are homogeneous polynomials, it follows that $q^*(\widetilde{h}_1, \ldots, \widetilde{h}_d)$ is a homogeneous polynomial in $(x_1, \ldots, x_{d+1})$. If $x_{d+1} = 1$, then $\widetilde{h}_i = h_i, i = 1, \ldots, d$. Therefore, $q^*(\widetilde{h}_1, \ldots, \widetilde{h}_d)$ is a homogeneous polynomial which vanishes on the hyperplane $x_{d+1} = 1$. This implies that it vanishes identically. □

**Theorem 3.9.** *If $g$ is a polynomial in $\mathbb{R}^3$, then $\mathcal{P}(g) = \Pi$ if and only if $\mathcal{P}(g)$ separates points.*

*Proof.* The 'only if' part of Theorem 3.9 is obvious. Therefore, it suffices to prove that if $\mathcal{P}(g) \neq \Pi$, then the partial derivatives $g_1, g_2, g_3$ are linearly dependent. By Lemma 3.1 and Proposition 3.3 it suffices to prove Theorem 3.9 for **p**-homogeneous polynomials. Assume that $h$ is a **p**-homogeneous polynomial and $\mathcal{P}(h) \neq \Pi$.

The Hessian of a polynomial (or function) $f$ on $\mathbb{R}^d$ is defined as the matrix

$$\left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1}^{d}.$$

Let

$$H(f) = \det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1}^{d}.$$

We claim that if $q^*(h_1, \ldots, h_d) = 0$ for some polynomial $h$ on $\mathbb{R}^d$ and for some non-trivial polynomial $q^*$, then $H(h)$ vanishes identically. This can be proved in a number of ways. For example, the Hessian of $h$ is just the Jacobian of the functions $h_1, \dots, h_d$. If the determinant of the Hessian does not vanish identically, then it follows that the range of the map $(h_1, \dots, h_d) : \mathbb{R}^d \to \mathbb{R}^d$ contains an open ball. However, since $q^*(h_1, \dots, h_d) = 0$, it follows that the range of this map is a subset of the zero set of $q^*$, which cannot contain any open set. This contradiction implies that the determinant of the Hessian of $h$ vanishes identically.

Homogeneous polynomials for which the determinants of their Hessians vanish identically were investigated by Hesse [5] and by Gordan and Noether [4].

**Theorem 3.10** (Gordan and Noether). *Let $r$ be a homogeneous polynomial in $\mathbb{C}^d$ such that the determinant of its Hessian vanishes identically. If $d \leqslant 4$, then the first partial derivatives of $r$ are linearly dependent.*

*First proof of Theorem* 3.9. Assume that $h$ is a homogeneous polynomial. By Proposition 3.7 there exists a homogeneous polynomial $q^*$ such that $q^*(h_1, h_2, h_3) = 0$. Thus $H(h) = 0$. An application of Theorem 3.10 implies that $h_1, h_2, h_3$ are linearly dependent, which proves Theorem 3.9.

Suppose that $h$ is a $\mathbf{p}$-homogeneous polynomial, with $\mathbf{p} \neq (1,1,1)$, that is, it is not homogeneous. If $\mathcal{P}(h) \neq \Pi$ and $\mathcal{P}(h)$ separates points, that is, $h_1, h_2, h_3$ are linearly independent, then by Corollary 3.8 the homogenization $\widetilde{h}$ of $h$ is a homogeneous polynomial of four variables for which $\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3, \widetilde{h}_4$ are linearly independent, and

$$q^*(\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3) = 0$$

for some homogeneous polynomial $q^*$. Thus $H(\widetilde{h}) = 0$ and from Theorem 3.10 it follows that $\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3, \widetilde{h}_4$ are linearly dependent. This contradiction implies that $h_1, h_2, h_3$ are linearly dependent, proving Theorem 3.9. □

The proof of the Gordan–Noether theorem is beautiful and uses mainly elementary algebra and algebraic geometry. But it is rather complicated. This is particularly true in the case $d = 4$. We now outline a different proof of Theorem 3.9.

*Second proof of Theorem* 3.9. We first consider the case where $h$ is a homogeneous polynomial of degree $n$ in $\mathbb{R}^3$. Let $S$ be a surface in $\mathbb{R}^3$ defined by $h(x_1, x_2, x_3) = c \neq 0$. We choose $c$ for which $S$ is not empty. By Euler's formula, $\sum_{i=1}^{3} x_i h_i = nh$. Hence at every point of $S$ some $h_i \neq 0$, and $S$ is a smoooth, complete surface. Consider a point $\mathbf{s} = (s_1, s_2, s_3)$ on $S$. Suppose that $h_3(\mathbf{s}) \neq 0$. By the implicit function theorem there exists a unique smooth function $\varphi$ of two variables, defined in a neighbourhood of $(s_1, s_2)$, such that $\varphi(s_1, s_2) = s_3$ and $h(x_1, x_2, \varphi(x_1, x_2)) = c$. We can compute the partial derivatives of $\varphi$ at $\mathbf{x}$ in terms of the partial derivatives of $h$. Using Euler's formula, we obtain

$$\sum_{i=1}^{3} x_i \frac{\partial h_j}{\partial x_i} = (n-1) h_j,$$

and find that

$$K = \frac{\partial^2 \varphi}{\partial x_1^2} \frac{\partial^2 \varphi}{\partial x_2^2} - \left( \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \right)^2$$

is proportional to the determinant of the Hessian of $h$ at $\mathbf{x}$. The Gaussian curvature of $S$ at $\mathbf{x}$ is proportional to $K$ (see Spivak [8], p. 95). If $H(h)$ vanishes identically, then the Gaussian curvature of $S$ vanishes identically. By Auslander ([1], Theorem VI.3.3) a connected, smooth, complete surface $S'$ in $\mathbb{R}^3$ with zero Gaussian curvature is a cylinder, that is, after a linear change of variables $S'$ has a parametrization $(t, u) \to (x_1(t), x_2(t), u)$. Let $S'$ be a connected component of $S$. By the above theorem we may assume that $S'$ has a local parametrization $(t, u) \to (x_1(t), x_2(t), u)$ and therefore $h(x_1(t), x_2(t), u) = c$ in some neighbourhood. Taking the derivative with respect to $u$, we get $h_3(x_1, x_2, x_3) = 0$ on an open subset of $S$. Thus $h_3$ must be zero on an irreducible algebraic component of $S$, and must be divisible by its irreducible equation. But every factor of $h_3$ is homogeneous and every factor of $(h-c)$ has a constant term. It follows that $h_3 = 0$. In particular, the partial derivatives of $h$ are linearly dependent. (This proof, although quite natural, is not very satisfactory since it uses analytic methods to prove an algebraic fact about polynomials. The proof of the Gordan–Noether theorem in the case $d = 3$ is long, but not difficult.)

Let us now turn to the case where $h$ is a $\mathbf{p}$-homogeneous polynomial of $\mathbf{p}$-degree $n$ in $\mathbb{R}^3$, with $\mathbf{p} = (p_1, p_2, p_3) \neq (1, 1, 1)$. We shall assume that $p_1 \leqslant p_2 \leqslant p_3$, $(p_1 < p_3)$. Suppose that $\mathcal{P}(h) \neq \Pi$. By Proposition 3.7 there exists a homogeneous and $\mathbf{p}$-homogeneous polynomial

$$q^*(\mathbf{u}) = \sum_{\substack{\mathbf{p}\cdot\mathbf{j}=t\\ |\mathbf{j}|=m}} d_{\mathbf{j}}\mathbf{u}^{\mathbf{j}}$$

such that $q^*(h_1, h_2, h_3) = 0$. The two equations $p_1 j_1 + p_2 j_2 + p_3 j_3 = t$, $j_1 + j_2 + j_3 = m$ have a solution depending on one parameter $s$. If $a_1$ denotes the smallest power of $x_1$ that appears in $q^*$, then for any $\mathbf{j} = (j_1, j_2, j_3)$ for which $d_{\mathbf{j}} \neq 0$ we have $j_1 = a_1 + s(p_3 - p_2)$, $j_2 = a_2 + s(p_1 - p_3)$, $j_3 = a_3 + s(p_2 - p_1)$, where $a_1, a_2, a_3$ are non-negative integers, and $s \geqslant 0$ is a rational number. If $d = \gcd\{p_3 - p_2, p_2 - p_1\}$ and if $p_3 - p_2 = dk$, $p_2 - p_1 = dl$, then $p_3 - p_1 = d(k + l)$, and

$$q^*(\mathbf{u}) = \mathbf{u}^{\mathbf{a}} \sum_{i=0}^{r} c_i \left(u_1^k u_3^l u_2^{-k-l}\right)^i,$$

where $\mathbf{a} = (a_1, a_2, a_3)$. The polynomial $\sum_{i=0}^{r} c_i z^i$ splits into linear factors. Therefore

$$q^*(\mathbf{u}) = u_1^{a_1} u_2^{a_2 - r(k+l)} u_3^{a_3} c_r \prod_{i=1}^{r} \left(u_1^k u_3^l - \alpha_i u_2^{k+l}\right).$$

If $q^*(h_1, h_2, h_3) = 0$ we have either $h_i = 0$ for some $i$ (in which case we have finished) or $h_1^k h_3^l = \alpha h_2^{k+l}$ for some $\alpha \neq 0$ ($\alpha$ is real since all the $h_i$ are real). If $p_1 = p_2$ or $p_2 = p_3$, then $l = 0$ or $k = 0$, respectively. In either case it follows from the previous equation that the $h_i$ are linearly dependent. We therefore assume that $p_1 < p_2 < p_3$.

Let $D = \gcd\{h_1, h_2, h_3\}$, $h_1 = AD$, $h_2 = BD$, and $h_3 = CD$. Then $A^k C^l = \alpha B^{k+l}$ and $\gcd\{A, C\} = 1$. For any polynomial $f$ we have the set $f_i = \dfrac{\partial f}{\partial x_i}$.

We put $f_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$. Let $\deg_i f$ denote the degree of $f$ as a polynomial in $x_i$, that is, the largest power of $x_i$. If $f_i \neq 0$, we have $\deg_i f_i = \deg_i f - 1$. For any irreducible polynomial $P$, let $e(P, f)$ denote the exponent of $P$ in $f$, that is, $e(P, f) = \max\{i : P^i | f\}$. Now,

a) $e(P, 0) = \infty$,
b) if $P_i \neq 0$, and $e(P, f) > 0$, then $e(P, f_i) = e(P, f) - 1$,
c) if $P_i = 0$, then $e(P, f_i) \geqslant e(P, f)$.

Let $e(P, A) > 0$ and assume that $P_1 \neq 0$. We put $e(P, h_1) = s > 0$. Since $h_1^k h_3^l = \alpha h_2^{k+l}$, we deduce that $e(P, h_2) = t > 0$. From (b) and (c) we deduce that $e(P, h_{12}) \geqslant s - 1$ and $e(P, h_{21}) = t - 1$. Hence $t \geqslant s$. Therefore $e(P, B) \geqslant e(P, A) > 0$. Since $\gcd\{A, C\} = 1$, we have $e(P, C) = 0$. These facts contradict $A^k C^l = \alpha B^{k+l}$. Thus $\deg_1 A = 0$.

If $\deg_1 C \neq 0$, then $\deg_1 h_{31} = \deg_1 h_3 - 1 = \deg_1 C + \deg_1 D - 1$, while $\deg_1 h_{13} \leqslant \deg_1 h_1 = \deg_1 D$. Thus $\deg_1 C \leqslant 1$. If $\deg_1 C = 1$, then from the fact that $\deg_1 A = 0$ and $A^k C^l = \alpha B^{k+l}$ it follows that $0 < \deg_1 B < 1$, a contradiction. Thus $\deg_1 C = 0$ and $\deg_1 B = 0$.

In the same way we can prove that $A, B$ and $C$ are independent of $x_3$. The polynomials $A, B$ and $C$, as factors of $\mathbf{p}$-homogeneous polynomials, are themselves $\mathbf{p}$-homogeneous. Thus $A, B$ and $C$ must each be a power of $x_2$. Let $A, B$ and $C$ be powers of $x_2$ of degree $m_1, m_2$ and $m_3$, respectively. Since $h$ is a $\mathbf{p}$-homogeneous polynomial of degree $n$, each of the polynomials $h_i$ is a $\mathbf{p}$-homogeneous polynomial of degree $n - p_i$. Let $m$ denote the $\mathbf{p}$-degree of $D$. Thus $n - p_1 = m_1 p_2 + m$, $n - p_2 = m_2 p_2 + m$ and $n - p_3 = m_3 p_2 + m$. But this implies that $p_1$ (and $p_3$) divide $p_2$. Since, by assumption, $0 < p_1 < p_2$, we have arrived at a contradiction. This implies that the $h_i$ are linearly dependent, and proves Theorem 3.9. □

Another application of the previous results is the following.

**Proposition 3.11.** *Let $g$ be a polynomial of total degree 2. If $\mathcal{P}(g)$ separates points, then $\mathcal{P}(g) = \Pi$.*

*Proof.* We apply Lemma 3.1 and Proposition 3.3 to obtain $h$. If $h$ is homogeneous, we write

$$h(x_1, \ldots, x_d) = \sum_{i \neq j} a_{ij} x_i x_j + \sum_i \frac{a_{ii}}{2} x_i^2 .$$

The first partial derivatives of $h$ are linearly indepedent if and only if the matrix $\{a_{ij}\}_{i,j=1}^d$ is non-singular. The determinant of the Hessian of $h$, $H(h)$, is simply the determinant of this same matrix. This proves the result.

Assume that $h$ is $\mathbf{p}$-homogeneous, $\mathbf{p} \neq (1, 1, \ldots, 1)$. If the first partial derivatives of $h$ are linearly independent, then, after a linear change of variables, $h$ is of the form

$$h(x_1, \ldots, x_d) = \sum_{\substack{i \neq j;\\ i,j \leqslant d-1}} a_{ij} x_i x_j + \sum_{i=1}^{d-1} \frac{a_{ii}}{2} x_i^2 + b x_d ,$$

where $b \neq 0$, and the matrix $\{a_{ij}\}_{i,j=1}^{d-1}$ is non-singular. A simple calculation shows that $H(\widetilde{h})$, where $\widetilde{h}$ is the homogenization of $h$, is the determinant of the matrix of the $a_{ij}$, raised to the power $-b^2$. The result follows by using Corollary 3.8. □

## §4. The space $\mathcal{P}_1$ for $d \geqslant 4$ and some open problems

In [7] we stumbled across an example of a polynomial $h^*$ in $\mathbb{R}^4$ that is $\mathbf{p}$-homogeneous, and for which $\mathcal{P}(h^*) \neq \Pi$ and $\mathcal{P}(h^*)$ separates points. That is, Theorem 3.9 is not valid for $d \geqslant 4$. The example given was

$$h^*(x_1, x_2, x_3, x_4) = x_1 x_4^2 + x_2 + x_3 x_4. \tag{4.1}$$

We can check that

$$q(D)(h^*)^k = 0, \qquad k = 0, 1, 2, \dots,$$

for $q(u_1, u_2, u_3, u_4) = u_1 u_2 - u_3^2$.

Gordan and Noether showed in [4] that there is a whole class of homogeneous polynomials in $\mathbb{C}^5$ whose first partial derivatives are linearly independent, but for which the determinant of their Hessian vanishes identically. This class (up to a linear change of variables) is a subset of the class of polynomials of the form

$$h(x_1, x_2, x_3, x_4, x_5) = f(x_1 P_1(x_4, x_5) + x_2 P_2(x_4, x_5) + x_3 P_3(x_4, x_5), x_4, x_5), \tag{4.2}$$

where each $P_i$ is a homogeneous polynomial of degree $m, i = 1, 2, 3$, and $f$ is an $(m+1, 1, 1)$-homogeneous polynomial (implying that $h$ is homogeneous). It is immediate that $H(h) = 0$ for each $h$ of the form (4.2), while some minor assumptions imply that the $\{h_i\}_{i=1}^5$ are linearly independent.

We proved (see the beginning of the proof of Theorem 3.9) that $\mathcal{P}(h) \neq \Pi$, for $h$ homogeneous, implies that $H(h) = 0$. We do not know if the converse holds. However, if $h$ is of the form (4.2), then this is indeed the case. Since $P_1, P_2$ and $P_3$ are three polynomials in two variables, they are algebraically dependent. That is,

$$\sum_{i+j+k \leqslant l} a_{ijk} (P_1)^i (P_2)^j (P_3)^k = 0$$

for some choice of non-trivial $\{a_{ijk}\}$. Since $P_1, P_2, P_3$ are homogeneous polynomials of the same degree, we can assume $i + j + k = l$ in the above formula. We put

$$q(u_1, u_2, u_3, u_4, u_5) = \sum_{i+j+k=l} a_{ijk} u_1^i u_2^j u_3^k. \tag{4.3}$$

Then it is readily verified that

$$q(D)(h)^k = 0, \qquad k = 0, 1, 2, \dots .$$

Thus $\mathcal{P}(h) \neq \Pi$ for every $h$ of the form (4.2).

For an $h$ of the form (4.2) we put

$$v(x_1, x_2, x_3, x_4) = h(x_1, x_2, x_3, x_4, 1).$$

Then

$$q(D)(v)^k = 0, \qquad k = 0, 1, 2, \ldots,$$

where $q$ is as in (4.3). Thus $\mathcal{P}(v) \neq \Pi$. The polynomial $v$ may or may not be $\mathbf{p}$-homogeneous. The polynomial $h^*$ of (4.1) should be considered from this perspective.

Based on the above results and those of §3, we should in our estimation be asking the following questions.

*Question* 1. What is the general form of the polynomial $g$ for which $\mathcal{P}(g) \neq \Pi$?

*Question* 2. Is it true that if $\mathcal{P}(g) \neq \Pi$, then there exists a polynomial $q$ such that $q(D)(g)^k = 0$ for all $k$?

*Question* 3. Is $\mathcal{P}(h) \neq \Pi$ equivalent to $H(h) = 0$ for every homogeneous polynomial $h$?

*Question* 4. Is it true that $\mathcal{P}(g) \neq \Pi$ implies $\overline{\mathcal{P}(g)} \neq C(\mathbb{R}^d)$?

We know that the answers to Questions 2 and 4 are in the affirmative if $g$ is a $\mathbf{p}$-homogeneous polynomial (and by the proof of Corollary 3.5 a positive response to Question 2 implies a positive response to Question 4). Furthermore, the answer to Question 3 is yes if $d \leqslant 4$, since $H(h) = 0$ in this case implies the linear dependence of the $\{h_i\}$. Finally we know a complete solution to Question 1 in the case $d \leqslant 3$.

Gordon and Noether also claim to have determined all homogeneous polynomials in $\mathbb{C}^5$ for which the determinants of their Hessians vanish identically. They claim that all such polynomials are of the form (4.2). Unfortunately we were not convinced by part of their argument, although we have found no reason to doubt the veracity of their result. We will thus record their statement as a 'claim', *caveat emptor.*

**Claim 4.1** (Gordan and Noether [4]). *Let $h$ be a homogeneous polynomial in $\mathbb{C}^5$ such that the determinant of its Hessian vanishes identically and the $\{h_i\}_{i=1}^5$ are linearly independent. Then, after a linear change of variables, $h$ has the form*

$$h(x_1, x_2, x_3, x_4, x_5) = f(x_1 P_1(x_4, x_5) + x_2 P_2(x_4, x_5) + x_3 P_3(x_4, x_5), x_4, x_5),$$

*where each $P_i$ is a homogeneous polynomial of degree $m, i = 1, 2, 3$, and $f$ is an $(m+1, 1, 1)$-homogeneous polynomial.*

If we accept the statement of Claim 4.1 for $d = 5$, then we get an affirmative answer to Question 3 for $d = 5$. Assuming Claim 4.1 we can give a complete solution to Question 1 for $d = 4$ and $g$ a $\mathbf{p}$-homogeneous polynomial.

**Claim 4.2.** *Let $h(x_1, x_2, x_3, x_4)$ be a $\mathbf{p}$-homogeneous polynomial ($\mathbf{p} \neq (1, 1, 1, 1)$) such that $\mathcal{P}(h)$ separates points and $\mathcal{P}(h) \neq \Pi$. Then, after a renumbering of the variables,*

$$h(x_1, x_2, x_3, x_4) = f\big(x_1 + c_2 x_2 (x_4 + a x_3)^\alpha + c_3 x_3 (x_4 + a x_3)^\beta, x_4 + a x_3\big), \qquad (4.4)$$

*where $0 < \alpha < \beta$, $c_2 \neq 0$, $c_3 \neq 0$, and $f$ is a $(p_1, p_4)$-homogeneous polynomial, $p_1 = p_2 + \alpha p_4 = p_3 + \beta p_4$, where $p_3 = p_4$ if $a \neq 0$.*

*Proof.* Let $\widetilde{h}$ be the homogenization of $h$. By Corollary 3.8 $\mathcal{P}(\widetilde{h})$ separates points and there exists a homogeneous polynomial $q^*(u_1, u_2, u_3, u_4)$ such that $q^*(\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3, \widetilde{h}_4) = 0$. We may assume that $q^*$ is a polynomial of minimal degree with this property. Then $v_i = q_i^*(\widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3, \widetilde{h}_4) \neq 0$ or $q_i^* \equiv 0$. Let $\rho = \gcd(v_1, v_2, v_3, v_4)$ and let $v_i = \rho H^i$. Then $H = (H^1, H^2, H^3, H^4)$ defines a homogeneous map from $\mathbb{C}^5$ to $\mathbb{C}^4$. It is proved in [4] that $\dim \operatorname{Im} H \leqslant 3$ (projective dimension $\leqslant 2$).

It is claimed that if $\mathcal{P}(\widetilde{h})$ separates points, then $\dim \operatorname{Im} H \leqslant 2$, but this part of the argument is not clear to us. It is further proved that if $\dim \operatorname{Im} H \leqslant 2$, then $H^1, H^2, H^3, H^4$ are linearly dependent, say $H^4 = a_1 H^1 + a_2 H^2 + a_3 H^3$. Further argument in [4] shows that then

$$h(x_1, x_2, x_3, x_4) = f(x_1 P_1(y) + x_2 P_2(y) + x_3 P_3(y), y),$$

where

$$y = a_1 x_1 + a_2 x_2 + a_3 x_3 - x_4.$$

Since $\mathcal{P}(h)$ separates points, $P_1, P_2, P_3$ are linearly independent and in particular at least one of them must have positive degree. Therefore the homogeneous part of $h$ of maximal degree is divisible by $y$. From this it follows that $y$ itself must be $\mathbf{p}$-homogeneous. A short argument then implies that $x_1 P_1(y) + x_2 P_2(y) + x_3 P_3(y)$ is also $\mathbf{p}$-homogeneous. Hence each $P_i$ must be a power of $y$ and thus

$$h(x_1, x_2, x_3, x_4) = f\big(c_1 x_1 y^{n_1} + c_2 x_2 y^{n_2} + c_3 x_3 y^{n_3}, y\big).$$

Since the $P_i$ are linearly independent, $n_1, n_2, n_3$ must be distinct. Thus the 'weights' $p_1, p_2, p_3$ are also all different. Since $y = a_1 x_1 + a_2 x_2 + a_3 x_3 - x_4$ is $\mathbf{p}$-homogeneous, it now follows that, up to a renumbering, $a_1 = a_2 = 0$. Moreover, if $p_3 \neq p_4$, then also $a_3 = 0$. In this case $y = -x_4$. If $p_3 = p_4$, then it follows that $n_3 > \max\{n_1, n_2\}$.

We now factor out $c_i y^{n_i}$, for $n_i = \min\{n_1, n_2, n_3\}$, from the first argument of $f$. Up to a renumbering, $h$ can now be written in the form (4.4). □

*Remark* 4.3. If $h$ is a homogeneous polynomial in $\mathbb{C}^4$, then by Theorem 3.10 we have $\mathcal{P}(h) = \Pi$ if and only if $\mathcal{P}(h)$ separates points.

In a rather weak sense the answer to Question 2 is in the positive, and we can thus solve Question 1. However this result is far from satisfactory. We mean 'weak sense' since $q$ need not be a polynomial.

**Proposition 4.4.** *Let $g$ be a polynomial in $\mathbb{C}^d$. Then $\mathcal{P}(g) \neq \Pi$ if and only if there exists a power series*

$$q(\mathbf{u}) = \sum_{\mathbf{j} \in \mathbb{Z}_+^d} d_{\mathbf{j}} \mathbf{u}^{\mathbf{j}}$$

*such that*

$$q(D)(g)^k = 0, \qquad k = 0, 1, 2, \ldots. \tag{4.5}$$

*Remark* 4.5. Note that (4.5) is well defined, since

$$D^{\mathbf{j}}(g)^k = 0$$

for all but a finite number of $\mathbf{j} \in \mathbb{Z}_+^d$.

*Proof.* If (4.5) holds, then $\mathcal{P}(g) \neq \Pi$ whenever $q(D)f = 0$ for all $f \in \mathcal{P}(g)$, while, for example, $q(D)q' \neq 0$, where $q'$ is the lowest order homogeneous term of $q$.

Assume that

$$\mathcal{P}(g) = \operatorname{span}\{\, D^{\mathbf{j}}(g)^k : \mathbf{j} \in \mathbb{Z}_+^d, k \in \mathbb{Z}_+ \,\} \neq \Pi.$$

Then $\mathcal{P}(g) \cap \pi_k \neq \pi_k$ for some $k$. Thus there exist $d_{\mathbf{j}}, |\mathbf{j}| \leqslant k$, not all zero, such that

$$\sum_{\mathbf{j}} d_{\mathbf{j}} c_{\mathbf{j}} = 0$$

for all $f \in \mathcal{P}(g) \cap \pi_k$ of the form

$$f(\mathbf{x}) = \sum_{\mathbf{j}} c_{\mathbf{j}} \frac{\mathbf{x}^{\mathbf{j}}}{\mathbf{j}!}. \tag{4.6}$$

We now show how to construct $d_{\mathbf{j}}, |\mathbf{j}| = k+1$, such that

$$\sum_{\mathbf{j}} d_{\mathbf{j}} c_{\mathbf{j}} = 0$$

for all $f \in \mathcal{P}(g) \cap \pi_{k+1}$, written in the form (4.6), without altering the previously defined $d_{\mathbf{j}}, (|\mathbf{j}| \leqslant k)$. We then continue this process and define

$$q(\mathbf{u}) = \sum_{\mathbf{j} \in \mathbb{Z}_+^d} d_{\mathbf{j}} \mathbf{u}^{\mathbf{j}}.$$

Then $q(D)f = 0$ for all $f \in \mathcal{P}(g)$, which proves the proposition.

The set $\mathcal{P}(g) \cap \pi_{k+1}$ is finite-dimensional. Let $f_1, \ldots, f_s$ be a basis for $\mathcal{P}(g) \cap \pi_{k+1}$. Then

$$f_i = q_i + r_i, \qquad i = 1, \ldots, s,$$

where the $q_i$ are homogeneous of degree $k+1$, and the $r_i$ are polynomials of degree $\leqslant k$. Thus from (4.6)

$$q_i(\mathbf{x}) = \sum_{|\mathbf{j}|=k+1} c_{\mathbf{j}}^{(i)} \frac{\mathbf{x}^{\mathbf{j}}}{\mathbf{j}!}, \qquad r_i(\mathbf{x}) = \sum_{|\mathbf{j}| \leqslant k} c_{\mathbf{j}}^{(i)} \frac{\mathbf{x}^{\mathbf{j}}}{\mathbf{j}!}.$$

Consider the equations

$$\sum_{|\mathbf{j}|=k+1} d_{\mathbf{j}} c_{\mathbf{j}}^{(i)} = -\sum_{|\mathbf{j}| \leqslant k} d_{\mathbf{j}} c_{\mathbf{j}}^{(i)} = e_i, \qquad i = 1, \ldots, s.$$

Note that the right-hand side is well defined. We need to prove that the set of equations

$$\sum_{|\mathbf{j}|=k+1} d_{\mathbf{j}} c_{\mathbf{j}}^{(i)} = e_i, \qquad i = 1, \ldots, s,$$

has a solution. Assume not. This implies the existence of $y_i, i = 1, \ldots, s$, such that

$$\sum_{i=1}^{s} y_i c_{\mathbf{j}}^{(i)} = 0 \quad \text{for all } |\mathbf{j}| = k+1$$

and

$$\sum_{i=1}^{s} y_i e_i \neq 0.$$

However, $\sum_{i=1}^{s} y_i c_{\mathbf{j}}^{(i)} = 0$ for all $|\mathbf{j}| = k+1$ is equivalent to

$$\sum_{i=1}^{s} y_i q_i = 0.$$

Then

$$\sum_{i=1}^{s} y_i f_i = \sum_{i=1}^{s} y_i r_i \in \mathcal{P}(g) \cap \pi_k.$$

Now

$$\sum_{i=1}^{s} y_i r_i(\mathbf{x}) = \sum_{|\mathbf{j}| \leqslant k} \left( \sum_{i=1}^{s} y_i c_{\mathbf{j}}^{(i)} \right) \frac{\mathbf{x}^{\mathbf{j}}}{\mathbf{j}!}.$$

By the definition of the $d_{\mathbf{j}}, |\mathbf{j}| \leqslant k$,

$$\sum_{|\mathbf{j}| \leqslant k} d_{\mathbf{j}} \left( \sum_{i=1}^{s} y_i c_{\mathbf{j}}^{(i)} \right) = 0.$$

However,

$$0 \neq -\sum_{i=1}^{s} y_i e_i = \sum_{i=1}^{s} y_i \left( \sum_{|\mathbf{j}| \leqslant k} d_{\mathbf{j}} c_{\mathbf{j}}^{(i)} \right) = \sum_{|\mathbf{j}| \leqslant k} d_{\mathbf{j}} \left( \sum_{i=1}^{s} y_i c_{\mathbf{j}}^{(i)} \right),$$

which is a contradiction. □

*Remark* 4.6. If $\overline{\mathcal{P}(g)} \neq \overline{C(\mathbb{R}^d)}$, then there is a much simpler method of proving the proposition. Namely, let $F$ be a non-trivial linear functional which annihilates $\mathcal{P}(g)$; then

$$d_{\mathbf{j}} = F(\mathbf{x}^{\mathbf{j}}/\mathbf{j}!).$$

*Remark* 4.7. If $g$ is a homogeneous or quasi-homogeneous polynomial, and $\mathcal{P}(g) \neq \Pi$, then we can choose the $d_{\mathbf{j}}$ so that all but a finite number of them are zero. This is what was done in Proposition 3.4, and has many advantages.

## §5. $\mathcal{P}_2$ and $\mathcal{P}_3$

We refer the reader to §2 for the definitions of $\mathcal{P}_2$ and $\mathcal{P}_3$. The following two propositions characterize exactly when $\mathcal{P}_2$ and $\mathcal{P}_3$ are equal to $\Pi$.

**Proposition 5.1.** $\mathcal{P}_2 = \Pi$ *if and only if*

$$\frac{\partial g}{\partial x_j}(\mathbf{0}) \neq 0, \qquad j = 1, \ldots, d.$$

*Proof.* If $\dfrac{\partial g}{\partial x_j}(\mathbf{0}) = 0$ for some $j \in \{1, \ldots, d\}$, then it is easily seen that $x_j$ is not in $\mathcal{P}_2$. We therefore assume that

$$\frac{\partial g}{\partial x_j}(\mathbf{0}) \neq 0, \qquad j = 1, \ldots, d.$$

Thus

$$g(\mathbf{x}) = \alpha_0 + \sum_{i=1}^{d} \alpha_i x_i + \sum_{|\mathbf{j}| \geqslant 2} c_{\mathbf{j}} \mathbf{x}^{\mathbf{j}},$$

where $\alpha_i \neq 0$, $i = 1, \ldots, d$.

Assume that $\alpha_0 = 0$. We claim that in this case

$$\mathbf{x}^{\mathbf{m}} \in \operatorname{span}\left\{ (g(a_1 \cdot, \ldots, a_d \cdot))^{|\mathbf{m}|} : \mathbf{a} \in \mathbb{R}^d \right\}.$$

The lowest order homogeneous term of $g^{|\mathbf{m}|}$ is $\left(\sum_{i=1}^{d} \alpha_i x_i\right)^{|\mathbf{m}|}$. Now

$$\left(\sum_{i=1}^{d} \alpha_i x_i\right)^{|\mathbf{m}|} = \sum_{|\mathbf{j}| = |\mathbf{m}|} \beta_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$$

and $\beta_{\mathbf{j}} \neq 0$ for all such $\mathbf{j}$. Applying Lemma 2.4 we obtain our result.

If $\alpha_0 \neq 0$, we put

$$p = g - \alpha_0 \ \left( = g - \alpha_0 (g)^0 \right).$$

Then

$$\frac{\partial p}{\partial x_j}(\mathbf{0}) = \frac{\partial g}{\partial x_j}(\mathbf{0}) \neq 0, \qquad j = 1, \ldots, d.$$

From our previous result, each polynomial may be obtained as a linear combination of the $(p(a_1 \cdot, \ldots, a_d \cdot))^k$. Arguing in the reverse direction, we prove the proposition. □

An immediate consequence of Proposition 5.1 is the following proposition.

**Proposition 5.2.** $\mathcal{P}_3 = \Pi$ *if and only if*

$$\frac{\partial g}{\partial x_j} \neq 0, \qquad j = 1, \ldots, d.$$

*Proof.* If $\dfrac{\partial g}{\partial x_j} = 0$ for some $j \in \{1, \ldots, d\}$, then $g$ does not depend on $x_j$ and can neither contain all polynomials nor be dense in $C(\mathbb{R}^d)$.

Assume that

$$\frac{\partial g}{\partial x_j} \neq 0, \qquad j = 1, \ldots, d.$$

Thus there exists a point $\mathbf{b}^* = (b_1^*, \ldots, b_d^*)$ such that

$$\frac{\partial g}{\partial x_j}(\mathbf{b}^*) \neq 0, \qquad j = 1, \ldots, d.$$

Now,

$$\widetilde{\mathcal{P}}_3 = \operatorname{span}\left\{ \left(g(a_1(\cdot - b_1^*), \ldots, a_d(\cdot - b_d^*))\right)^k : \mathbf{a} \in \mathbb{R}^d \right\} \subseteq \mathcal{P}_3.$$

From Proposition 5.1 we deduce that $\widetilde{\mathcal{P}}_3 = \Pi$. Thus $\mathcal{P}_3 = \Pi$. □

As noted, the condition given in Proposition 5.2 is both necessary and sufficient for $\overline{\mathcal{P}}_3 = C(\mathbb{R}^d)$. This is not true of the condition given in Proposition 5.1. For example, when $d = 1$ it is known that span $\{x^{m_l}\}_{l=0}^{\infty}$ (distinct increasing $m_l \in \mathbb{Z}_+$) is dense in $C[a, b]$ for every finite interval $[a, b]$ if and only if $m_0 = 0$, and

$$\sum_{m_l \text{ odd}} 1/m_l = \sum_{\substack{m_l \text{ even} \\ l > 0}} 1/m_l = \infty,$$

We recall that we always have the constant function in $\mathcal{P}_2$ (since we permit taking the 0th power). It is an elementary exercise to prove, based on the above fact and Lemma 2.4, that for every polynomial $g$ that is not even we have

$$\overline{\mathcal{P}}_2 = C(\mathbb{R}).$$

We do not know how to characterize such $g$ when $d \geqslant 2$.

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