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Positive and Minimal Projections in Function Spaces

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1. INTRODUCTION

If L is a linear lattice and E is a linear subspace of L, it is natural to ask whether there is a positive projection of L onto E (a projection P is *positive*, or *monotone*, if $x \geqslant 0$ implies $Px \geqslant 0$). This is always the case, for example, when L is $L^p(\mu)$ and E is a closed linear sublattice [4, Chap. 3]. However, much less is known about the situation when L is the function space $C(X)$ (X compact, Hausdorff) with supremum norm, though for certain subspaces Korovkin's theorem implies that there is no positive projection.

In Section 2, we give necessary and sufficient conditions for there to be a positive projection of a normed linear lattice L onto an n-dimensional subspace L_n. As a corollary, we see that if M is a closed sublattice of a Banach lattice L and there is a positive projection of M onto L_n, then there is a positive projection of L onto L_n. In particular, every finite-dimensional sublattice of L admits a positive projection. When L is $C(X)$, our characterization reduces to the following: L_n admits a positive projection if and only if there exist positive functions $b_1,..., b_n$ in L_n and points $x_1,..., x_n$ of X such that $b_i(x_j) = \delta_{ij}$.

In Section 3, we study the companion problem for finite-codimensional subspaces of $C(X)$. We prove, in fact, that if X has no isolated points, then such subspaces never admit positive projections.

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In Section 4, we are concerned with projections of $C(X \times Y)$ onto certain natural subspaces. Here we consider minimal as well as positive projections. More precisely, let M be the subspace consisting of all functions of the form $\phi(x,y) = f(x) + g(y)$. Also, for fixed $x^* \in X$, $y^* \in Y$, let $C_0(X \times Y)$ be the set of functions in $C(X \times Y)$ that vanish at (x^*, y^*), and let $M_0 = M \cap C_0(X \times Y)$. Then there exists no positive projection of $C(X \times Y)$ onto M, and exactly one positive projection P^* of $C_0(X \times Y)$ onto M_0. Furthermore, if X and Y are infinite, then for any projection P of $C(X \times Y)$ onto M, $\|P\| \geqslant 3$, while for any projection P of $C_0(X \times Y)$ onto M_0, $\|P\| \geqslant \|P^*\| = 2$. These results easily generalize to the product of k spaces X^i. Also, the method of proof establishes exact estimates (in the first case) for the norms when some or all of the spaces X^i are finite.

2. Finite-Dimensional Subspaces

Our first result applies to general normed linear lattices. A *linear lattice* (or *Riesz space*) is a linear space (over the real field) with a lattice ordering $\geqslant$ such that

$$x \geqslant 0, \quad y \geqslant 0 \quad \text{implies} \quad x + y \geqslant 0,$$

$$x \geqslant 0, \quad \lambda \in \mathbb{R}^+ \quad \text{implies} \quad \lambda x \geqslant 0.$$

We use the usual notation: $\sup\{x,y\} = x \vee y$, $\inf\{x,y\} = x \wedge y$, $|x| = x \vee (-x)$. A *normed linear lattice* is a normed linear space equipped with a lattice ordering such that $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$. If the space is also complete with respect to the norm, it is called a *Banach lattice*.

Let L_n denote an n-dimensional linear subspace of a normed linear lattice L. Suppose that there is a positive projection P of L onto L_n. It is then elementary (and well-known) that the ordering of L_n is a lattice ordering: in fact, we have

$$\sup_{L_n}(x,y) = P(x \vee y)$$

for $x, y \in L_n$. (This does not mean that L_n is a sublattice of L since $x \vee y$ need not belong to L_n). The positive cone in L_n is closed, hence Archimedean. By a standard result on finite-dimensional linear lattices [4, p. 70], L_n has a basis $\{b_1, \ldots, b_n\}$ such that $\sum_1^n \lambda_i b_i \geqslant 0$ if and only if $\lambda_i \geqslant 0$ for all i. We deduce the following characterization of finite-dimensional subspaces that admit positive projections:

THEOREM 1. *Let L_n be an n-dimensional subspace of a normed linear lattice L. Then the following statements are equivalent*:

(i) *there is a positive projection of* L *onto* L_n,

(ii) *there exist positive elements* b_i *of* L_n *and positive linear functionals* f_i *on* L $(i = 1,..., n)$ *such that* $f_i(b_j) = \delta_{ij}$.

Proof. If (ii) holds, then a positive projection P is simply given by $Px = \sum_1^n f_i(x)\, b_i$.

Conversely, suppose that there is a positive projection P. Then L_n has a basis $\{b_1,..., b_n\}$ as above. For $x = \sum_1^n \lambda_i b_i \in L_n$, let $g_i(x) = \lambda_i$. The g_i are positive linear functionals defined on L_n, and $g_i(b_j) = \delta_{ij}$. The required functionals on L are given by $f_i = g_i \circ P$. ∎

COROLLARY. *Let* L *be a Banach lattice,* M *a closed linear sublattice. Let* L_n *be an n-dimensional subspace of* M. *If there is a positive projection of* M *onto* L_n, *then there is a positive projection of* L *onto* L_n.

Proof. Theorem 1 gives us positive linear functionals f_i defined on M. It is well-known that every positive functional on a Banach lattice is continuous ([4, p. 84]; if there were positive elements x_k with $\|x_k\| \leqslant 2^{-k}$ and $f(x_k) \geqslant k$, then no definition would be possible for $f(\sum x_k)$). Further, every continuous positive functional defined on M has a positive extension defined on L [4, p. 86]. ∎

In particular, every finite-dimensional linear sublattice admits a positive projection. As we shall see, this is far from being the case for infinite-dimensional sublattices of $C(X)$, though it is true in $L^p(\mu)$, $1 \leqslant p < \infty$ [4, p. 212].

Remark. It is sufficient in Theorem 1 if L, instead of having a lattice ordering, has an Archimedean ordering satisfying the Riesz decomposition property, that is, if $x_1, x_2 \geqslant 0$ and $0 \leqslant y \leqslant x_1 + x_2$, then $y = y_1 + y_2$, where $0 \leqslant y_i \leqslant x_i$, $i = 1, 2$. Finite-dimensional spaces with this property are order-isomorphic to $\mathbb{R}^n$ with the usual order.

The next result shows that in the case $L = C(X)$, we can take the functionals in Theorem 1 to be point-evaluations.

THEOREM 2. *Let* X *be a compact, Hausdorff space and let* L_n *be an n-dimensional subspace of* $C(X)$. *Then the following statements are equivalent*:

(i) *there is a positive projection of* $C(X)$ (*or of a closed linear sublattice of* $C(X)$) *onto* L_n,

(ii) *there exist non-negative fuctions* $b_1,..., b_n$ *in* L_n *and points* $x_1,..., x_n$ *in* X *such that* $b_i(x_j) = \delta_{ij}$.

Proof. Let the functionals f_i be as in Theorem 1, and let $S(f_i)$ denote the support of f_i. For $j \geqslant 2$, we have $f_1(b_j) = 0$, hence $b_j(x) = 0$ for all x in $S(f_1)$.

Since $f_1(b_1) = 1$, there exists x_1 in $S(f_1)$ with $b_1(x_1) > 0$. Replace b_1 by a positive scalar multiple to obtain $b_1(x_1) = 1$. The points $x_2, ..., x_n$ are found similarly. ∎

Remarks. (1) In the same way, one sees that if for a certain set of non-negative functions $b_1, ..., b_n$ in L_n, there exist *unique* points x_i such that $b_i(x_i) > 0$ and $b_i(x_j) = 0$ for $j \neq i$, then there is a unique positive projection onto L_n.

(2) For any positive linear mapping T of $C(X)$ into itself, we have $\|T\| = \|Te\|$, where e is the function with constant value 1. Hence if P is a positive projection onto a subspace containing e, then $\|P\| = 1$ (and if e is in the subspace, $\|P\| = 1$ implies P is positive). For the positive projection $Pu = \sum_1^n u(x_i)\, b_i$ given by Theorem 2, we have $\|P\| = \|b_1 + \cdots + b_n\|$. If the subspace does not contain e, this may well be greater than 1, even when a non-positive projection of norm 1 exists. To obtain a simple example, let X be a 3-point set, so that $C(X)$ is $\mathbb{R}^3$. Let L_2 be the subspace consisting of elements (x, y, z) satisfying $x = y + 2z$. Then there is an unique positive projection given by $P(x, y, z) = (y + 2z, y, z)$. (This corresponds to $b_1 = (1, 1, 0)$, $b_2 = (2, 0, 1)$.) Clearly, $\|P\| = 3$. However, there is a non-positive projection with norm 1, namely, $Q(x, y, z) = (x, y, (x - y)/2)$.

On the other hand, if the subspace does contain e, then the problem of finding the minimal norm projection is equivalent, in a certain sense, to that of finding the "least negative" projection. By this we mean the following. It is easy to show that for $f \geqslant 0$,

$$\frac{1}{\|f\|} \min(Pf)(x) \geqslant \frac{1}{2}(1 - \|P\|),$$

and equality is attained when we take the infimum over all $f \geqslant 0$.

It was shown by Morris and Cheney [2, Theorem 9] that if $n \geqslant 3$ and L_n is an n-dimensional Chebyshev subspace of $C[a, b]$ containing the constant functions, then every projection onto L_n has norm greater than 1. Consequently there is no positive projection onto L_n. Using our Theorem 2, we can prove the following stronger statement.

COROLLARY. *Let L_n be an n-dimensional subspace of $C[a, b]$. Assume that L_n contains an m-dimensional Chebyshev subspace X_m, where $m \geqslant 3$. Then there is no positive projection onto L_n.*

Proof. Since X_m is a Chebyshev subspace, for each x_0 in (a, b) there exists an f_0 in X_m such that $f_0 \geqslant 0$ on $[a, b]$ and $f_0(x) = 0$ only for $x = x_0$.

Assume that there is a positive projection of $C[a, b]$ onto L_n. Let $b_1, ..., b_n$ and $x_1, ..., x_n$ be as in the statement of Theorem 2. Choose x_0 in $(a, b) \backslash \{x_1, ..., x_n\}$, and let f_0 be as above. Express f_0 in the form $\sum_1^n \lambda_i b_i$.

Since $f_0(x_i) > 0$, we have $\lambda_i > 0$ for $i = 1, \ldots, n$. However, $f_0(x_0) = 0$, so $b_i(x_0) = 0$ for each i. This holds for all x_0 as above, which implies that each b_i is identically zero, a contradiction. ∎

Actually, an even stronger statement is true. We say that a subspace E of $C(X)$ has the "Korovkin property" if the identity is the only positive operator of $C(X)$ into itself that agrees with the identity on E. (This differs slightly from the usual definition, which refers to a sequence of positive operators.) The Korovkin property implies, of course, that there is no positive projection onto E.

In the situation of the above corollary, X_m (and hence L_n) has the Korovkin property. For f in $C[a, b]$ and each x, set

$$\bar{f}(x) = \inf\{ g(x) : g \in X_m, g \geqslant f \},$$
$$\underline{f}(x) = \sup\{ g(x) : g \in X_m, g \leqslant f \}.$$

It is easily proved that if X_m is an m-dimensional Chebyshev subspace, with $m \geqslant 3$, then $\bar{f}(x) = \underline{f}(x) = f(x)$ for each x in (a, b). Let T be any positive operator of $C[a, b]$ into itself which is the identity on X_m, and take any f in $C[a, b]$. Then it follows from the above equalities that $(Tf)(x) = f(x)$ for all x in (a, b), and hence that $Tf = f$ (cf. Berens and Lorentz [1] and Šaškin [3]).

While the Korovkin property implies the non-existence of a positive projection, the converse is certainly not true. For example, if E has the Korovkin property, then so has any subspace containing E. No such inclusion property holds for the existence or non-existence of positive projections. Two specific examples will be given after Proposition 3.

When L_n is considered as a subspace of $L^p[a, b]$, a statement similar to the above corollary holds even for $m = 2$. In fact, we have as a corollary of Theorem 1:

COROLLARY. *Let L_n be an n-dimensional subspace of $C[a, b]$. Assume that L_n contains an m-dimensional Chebyshev subspace X_m, where $m \geqslant 2$. Then there is no positive projection of $L^p[a, b]$ onto L_n (where $1 \leqslant p < \infty$).*

Proof. By the previous corollary, we need only consider the case $m = 2$. Since X_2 is a Chebyshev subspace, there exist g in X_2 such that $g(a) = 0$ and $g(x) > 0$ for $a < x \leqslant b$, and h in X_2 such that $h(a) > 0$. Let $b_1, \ldots, b_n$ and $f_1, \ldots, f_n$ be as in the statement of Theorem 1. Then g can be expressed as $\sum_1^n \mu_i b_i$, with each $\mu_i \geqslant 0$. Since $g - \varepsilon h$ takes negative values for each $\varepsilon > 0$, we must have $\mu_k = 0$ for some k. It follows that $f_k(g) = 0$. But this is impossible, since f_k is a non-zero, non-negative element of $L^{p'}[a, b]$ and g is a non-negative, continuous function that vanishes only at a. ∎

Let π_n denote the space of algebraic polynomials of degree $\leqslant n$. It follows from the elementary form of Korovkin's theorem (or from the above) that there is no positive projection of $C[a, b]$ onto π_n for $n \geqslant 2$. We give here a simple direct proof of a slightly stronger statement, together with a variant which is not obtainable from Korovkin-type theorems. Let $C_0[a, b]$ denote the set of functions in $C[a, b]$ that are 0 at a, and let $\pi_n^0 = \pi_n \cap C_0[a, b]$.

PROPOSITION 3. (i) *If $n \geqslant 2$, then there is no positive projection of π_{n+1} onto π_n.*

(ii) *If $n \geqslant 3$, then there is no positive projection of π_{n+1}^0 onto π_n^0.*

Proof. It is sufficient to consider $[a, b] = [0, 1]$. Write $r_k(x) = x^k$. For both (i) and (ii), suppose that there is a positive projection P, and let $P(r_{n+1}) = u$. Now $0 \leqslant r_{n+1} \leqslant r_n$, so $0 \leqslant u \leqslant r_n$. It is elementary that this, together with the fact that u is in π_n, implies that $u = \alpha r_n$ for some α in $[0, 1]$. Now $x^{n+1} \geqslant nx^2 - (n-1)x$ for x in $[0, 1]$. Hence $u(x) \geqslant nx^2 - (n-1)x$. In particular, $u(1) \geqslant 1$, so $\alpha \geqslant 1$.

For (i), let

$$h_k(x) = k\left(x - \frac{1}{4}\right)^2 + \frac{1}{2.4^n}.$$

Then h_k is in π_2, and for a suitable k we have $h_k \geqslant r_{n+1}$ (h_k is a "narrow" quadratic having twice the value of r_{n+1} at $\frac{1}{4}$). Hence $h_k \geqslant u$. Evaluation at $\frac{1}{4}$ gives $\alpha \leqslant \frac{1}{2}$, a contradiction.

For (ii), modify this slightly, as follows. Let

$$g_k(x) = k\left(x - \frac{1}{4}\right)^2 + \frac{1}{2.4^{n-1}}.$$

Choose k such that $x^n \leqslant g_k(x)$, so that $x^{n+1} \leqslant x g_k(x)$. The function $x g_k(x)$ is in π_3^0, so we obtain $\alpha x^n \leqslant x g_k(x)$. This gives $\alpha \leqslant \frac{1}{2}$, as before. ∎

Remarks. (1) There is a positive projection of $C[0, 1]$ onto π_1, and in fact onto any two-dimensional subspace containing the constant functions. For if f is a non-constant function, then we can define $f_1 = \alpha f + \beta$ such that $0 \leqslant f_1 \leqslant 1$ and f_1 attains the values 0, 1. Then f_1 and $1 - f_1$ satisfy the conditions of Theorem 2. The positive projection onto π_1 is unique.

(2) The subspace π_n^0 does not have the Korovkin property in $C[0, 1]$. This is shown by the mapping $(Tf)(x) = f(x) + f(0)$.

An additional example, containing the constant functions, is as follows. Let L_3 denote the subspace of $C[-1, 1]$ spanned by the functions $1, x^2, x^4$. The mapping $(Tf)(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x)$ shows that L_3 does not have the

Korovkin property. By an application of Theorem 2 (and zero counting), or by the method of Proposition 3, it is easily seen that there is no positive projection onto L_3.

EXAMPLE. The subspace π_2^0, consisting of polynomials of the form $ax + bx^2$, displays several interesting features. Note first that $ax + bx^2 \geqslant 0$ on $[0, 1]$ if and only if a, $a + b \geqslant 0$. Consequently, π_2^0 has a basis $\{b_1, b_2\}$ such that $\lambda_1 b_1 + \lambda_2 b_2 \geqslant 0$ if and only if $\lambda_1 \geqslant 0$ and $\lambda_2 \geqslant 0$: let $b_1(x) = x - x^2$, $b_2(x) = x^2$. We prove that there is no positive projection of $C[0, 1]$ onto π_2^0, showing that the existence of such a basis is not in itself sufficient. Let g_1, g_2 be the positive functionals on π_2^0 such that $g_i(b_j) = \delta_{ij}$. We show that there is no positive extension of g_1 defined on $C[0, 1]$ (or even on π_2); this implies the non-existence of a positive projection. Suppose, in fact, that f_1 were such an extension, and let $\alpha > 0$. Since

$$0 \leqslant (x - \alpha)^2 = x^2 - 2\alpha x + \alpha^2,$$

we have

$$0 \leqslant -2\alpha + \alpha^2 f_1(e),$$

so $f_1(e) \geqslant 2/\alpha$ for all $\alpha > 0$, which is impossible.

Since $C_0[0, 1]$ is a sublattice of $C[0, 1]$, it follows from the corollary of Theorem 1 that there is no positive projection of $C_0[0, 1]$ onto π_2^0. However, it is not hard to show that for each $n \geqslant 3$, there is a unique positive projection P_n of π_n^0 onto π_2^0: if $f(x) = a_1 x + \cdots + a_n x^n$, then $(P_n f)(x) = a_1 x + (a_2 + \cdots + a_n) x^2$. By considering $f_n(x) = 1 - (1 - x)^n$, one sees that $\|P_n\| \to \infty$ as $n \to \infty$.

3. Finite-Codimensional Subspaces of $C(X)$

In this section, we prove:

THEOREM 4. *Let X be a compact, Hausdorff space with no isolated points. Then there is no positive projection of $C(X)$ onto any proper, finite-codimensional subspace.*

The proof will be achieved by a series of lemmas. Suppose that P is such a projection, and let E be its (finite-dimensional) kernel. We shall work with E rather the range of P. Note first that if $f \in E$ and $0 \leqslant g \leqslant f$, then $Pg = 0$, so $g \in E$ (that is, E is "order-convex"). We deduce:

LEMMA 1. *E contains no non-zero, non-negative function.*

Proof. Suppose that E contains a non-zero, non-negative function f. We may assume that $f \geqslant 1$ on some open set G. Since X has no isolated points, G is infinite. Let (x_n) be a sequence of distinct points in G. For each n, there is a function f_n in $C(X)$ such that $0 \leqslant f_n \leqslant 1, f_n(x_n) = 1, f_n(x_i) = 0$ for $i < n$ and $f_n(x) = 0$ for all x in $X \backslash G$. Then $0 \leqslant f_n \leqslant f$, so $f_n \in E$, and it is clear that the sequence (f_n) is linearly independent, contradicting the fact that E is finite-dimensional. ▮

LEMMA 2. *Let* $E_1 = \{f \in E: \|f\| = 1\}$, *and for* f *in* $C(X)$, *let* $i(f) = \inf\{f(x): x \in X\}$. *Then there exists* $c > 0$ *such that* $i(f) \leqslant -c$ *for all* $f \in E_1$.

Proof. This is clear, since i is continuous, E_1 is compact and, by Lemma 1, $i(f) < 0$ for $f \in E_1$. ▮

LEMMA 3. *There exist* $x_1, \dots, x_k$ *in* X *such that if* $f \in E$ *and* $f(x_i) \geqslant 0$ *for each* i, *then* $f = 0$.

Proof. Let c be as in Lemma 2. Since E_1 is a compact subset of $C(X)$, it is equicontinuous. Therefore for each x in X, there is a neighbourhood $U(x)$ such that if $y \in U(x)$, then $|f(y) - f(x)| \leqslant c/2$ for all f in E_1. The space X can be covered by a finite choice of such neighbourhoods, say $U(x_1), \dots, U(x_k)$. If f is in E_1, then $f(x) \leqslant -c$ for some x, so $f(x_i) \leqslant -c/2$ for some i. ▮

LEMMA 4. *Write* $Q = I - P$. *If* f *is in* $C(X)$ *and* $f(x_i) = 0$ *for all* i, *then* $Qf = 0$.

Proof. It is sufficient to prove this for non-negative f (then consider f^+ and f^-). Suppose that $f \geqslant 0$ and $f(x_i) = 0$ for each i. Then $Pf = f - Qf \geqslant 0$, so $(Qf)(x_i) \leqslant 0$ for all i. Since Qf is in E, Lemma 3 gives $Qf = 0$. ▮

Proof of Theorem 4. Choose some $f \geqslant 0$ with $Qf \neq 0$. By Lemma 1, Qf has both positive and negative values. Hence $(Qf)(y) > 0$ for some y different from $x_1, \dots, x_k$. Let h be a non-negative function taking the value 1 at each x_i and 0 at y. Let $g = fh$. Then $g(x_i) = f(x_i)$ for all i, so by Lemma 4, $Qg = Qf$. In particular, $(Qg)(y) > 0$. But $g(y) = 0$, so $(Pg)(y) < 0$. This contradicts the positivity of P since $g \geqslant 0$. ▮

Remarks. (1) Let A be a closed, proper subset of X and let $C(X, A)$ denote the set of functions in $C(X)$ that vanish on A. The same reasoning shows that there is no positive projection of $C(X, A)$ onto any proper, finite-codimensional subspace of itself. It is well-known that every closed, order-convex linear sublattice (i.e., closed *lattice ideal*) of $C(X)$ is of the form $C(X, A)$.

(2) Let $C(X, x_0)$ denote the set of functions in $C(X)$ that vanish at x_0. If x_0 is not an isolated point (whether or not other isolated points exist), then there is no positive projection of $C(X)$ onto $C(X, x_0)$. This follows easily from the fact that any projection onto $C(X, x_0)$ has the form $Pf = f - f(x_0)\, g$, where g is a function with $g(x_0) = 1$.

4. Certain Subspaces of $C(X \times Y)$

We start with an elementary result.

PROPOSITION 5. *Let X be a compact, Hausdorff space and Y any topological space. Let L_n be an n-dimensional subspace of $C(X)$ spanned by $f_1,\ldots,f_n$. Let B^i $(i = 1,\ldots, n)$ be subspaces of $C(Y)$, each containing the constant functions. Let A_n be the set of functions of the form $\sum_1^n f_i(x)\, g_i(y)$, where $g_i \in B^i$ for each i. If there is no positive projection of $C(X)$ onto L_n, then there is no positive projection of $C(X \times Y)$ onto A_n.*

Proof. Choose any $y^* \in Y$, and let $(Q\phi)(x) = \phi(x, y^*)$ for ϕ in $C(X \times Y)$. If P were a positive projection of $C(X \times Y)$ onto A_n, then $QP|_{C(X)}$ would be a positive projection of $C(X)$ onto L_n. ∎

A natural application of this is the extension of Proposition 3 to polynomials in two or more variables. For $x, y \in I = [0, 1]$, set

$$\tilde{\pi}_n^2 = \left\{ \sum a_{ij} x^i y^j : i + j \leqslant n \right\},$$

$$\pi_{n,m}^2 = \left\{ \sum a_{ij} x^i y^j : i \leqslant n, j \leqslant m \right\}.$$

Then for $n \geqslant 2$, there is no positive projection of $C(I^2)$ onto $\tilde{\pi}_n^2$ or $\pi_{n,m}^2$. (In fact, examination of the proofs of Propositions 3 and 5 shows easily that there is no positive projection of $\tilde{\pi}_{n+1}^2$ onto $\tilde{\pi}_n^2$, or of $\pi_{n+1,m}^2$ onto $\pi_{n,m}^2$.) There is a unique positive projection of $C(I^2)$ onto $\pi_{1,1}^2$, given by interpolation at the corners of I^2. It follows from Theorem 2 that there is no positive projection of $C(I^2)$ onto $\tilde{\pi}_1^2$. The method of proof of Theorem 6 actually shows that there is no positive projection of $\pi_{1,1}^2$ onto $\tilde{\pi}_1^2$.

Let X, Y be compact, Hausdorff spaces neither containing only one point. Let M be the subspace of $C(X \times Y)$ consisting of functions of the form $\phi(x, y) = f(x) + g(y)$. Also, for a chosen $x^* \in X$, $y^* \in Y$, let $C_0(X \times Y)$ denote the set of functions in $C(X \times Y)$ which vanish at (x^*, y^*), and let $M_0 = M \cap C_0(X \times Y)$. We shall consider both positive and minimal projections of $C(X \times Y)$ onto M, and of $C_0(X \times Y)$ onto M_0. Our first result totally characterizes the positive projections.

THEOREM 6. *There is no positive projection of $C(X \times Y)$ onto M. There is exactly one positive projection P^* of $C_0(X \times Y)$ onto M_0, given by $(P^*\phi)(x,y) = \phi(x,y^*) + \phi(x^*,y)$.*

Proof. Suppose that P is a positive projection of $C(X \times Y)$ onto M. Let $f \in C(X)$, $g \in C(Y)$ be arbitrary functions such that $0 \leqslant f, g \leqslant 1$, and for which there exist points $x_0, x_1 \in X$ and $y_0, y_1 \in Y$ satisfying $f(x_0) = g(y_0) = 0$, $f(x_1) = g(y_1) = 1$. Set $u(x,y) = f(x)g(y)$, and let $(Pu)(x,y) = h(x) + k(y)$, with $h(x_0) = 0$. Then $(Pu)(x_0, y) = k(y) \geqslant 0$ for all y in Y. Now, $f(x) \geqslant u(x,y)$, so $f(x) \geqslant h(x) + k(y)$ for all x, y. Taking $x = x_0$, we see that $k = 0$. It follows by similar reasoning that $h = 0$. By construction, $[1 - f(x)][1 - g(y)] \geqslant 0$. Application of P gives $1 - f(x) - g(y) \geqslant 0$ for all x, y. Set $x = x_1$ and $y = y_1$ to obtain a contradiction.

Obviously P^* is a positive projection of $C_0(X \times Y)$ onto M_0. Let P be any positive projection of $C_0(X \times Y)$ onto M_0. For $\phi \in C_0(X \times Y)$, define $\psi(x,y) = \phi(x,y) - \phi(x,y^*) - \phi(x^*,y)$. The result follows if we prove that $P\psi = 0$. Now, ψ enjoys the property that $\psi(x,y^*) = \psi(x^*,y) = 0$ for all x, y. Set $(P\psi)(x,y) = h(x) + k(y)$, where $h(x^*) = k(y^*) = 0$. Define $\psi^+(x) = \max\{\psi(x,y): y \in Y\}$, and $\psi^-(x) = \min\{\psi(x,y): y \in Y\}$. Thus $\psi^+(x) \geqslant \psi(x,y) \geqslant \psi^-(x)$ for all x, y. Since $\psi^+, \psi^- \in M_0$, it follows that $0 = \psi^+(x^*) \geqslant h(x^*) + k(y) \geqslant \psi^-(x^*) = 0$ for all $y \in Y$. Thus $k = 0$. Similarly $h = 0$. This proves the theorem. ∎

Remark. The first part of the proof also shows that there is no positive projection of $C(X \times Y)$ onto M_0. This is interesting in the light of the results of Sections 2 and 3.

For any $(x^*, y^*) \in X \times Y$ the map defined by $(P\phi)(x,y) = \phi(x,y^*) + \phi(x^*,y) - \phi(x^*,y^*)$ is a projection of $C(X \times Y)$ onto M of norm 3. We prove that this is minimal if both X and Y contain an infinite number of points.

THEOREM 7. *Let X and Y be infinite, compact, Hausdorff spaces. Let P be any projection of $C(X \times Y)$ onto M. Then $\|P\| \geqslant 3$.*

Proof. Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be any two sets of n distinct points in X and Y, respectively. Let $f_i \in C(X)$, $i = 1,\ldots, n$, satisfy

$$f_i(x_j) = \delta_{ij}, \qquad i,j = 1,\ldots, n,$$

$$f_i(x) \geqslant 0, \qquad i = 1,\ldots, n, \quad x \in X,$$

$$\sum_{i=1}^n f_i(x) = 1, \qquad x \in X.$$

Similarly define $g_i \in C(Y)$, $i = 1,\ldots, n$, with respect to the points $\{y_i\}_{i=1}^n$.

Set $\phi^{rs}(x,y)=f_r(x)\,g_s(y)$, $r,s=1,\dots,n$. It easily follows that

$$\phi^{rs}(x_i,y_j)=\delta_{ri}\delta_{sj}, \qquad r,s,i,j=1,\dots,n, \tag{1}$$

$$\sum_{r=1}^{n}\phi^{rs}(x,y)=g_s(y), \qquad s=1,\dots,n, \tag{2}$$

$$\sum_{s=1}^{n}\phi^{rs}(x,y)=f_r(x), \qquad r=1,\dots,n. \tag{3}$$

Now, set $(P\phi^{rs})(x_i,y_j)=a_{ij}^{rs}$.

From (1), (2), and (3), we obtain

$$\sum_{r=1}^{n}a_{ij}^{rs}=\delta_{sj}, \qquad s,i,j=1,\dots,n, \tag{4}$$

$$\sum_{s=1}^{n}a_{ij}^{rs}=\delta_{ri}, \qquad r,i,j=1,\dots,n. \tag{5}$$

Furthermore, since P is a projection onto M, and functions in M satisfy $\phi(z_0,w_0)+\phi(z_1,w_1)=\phi(z_0,w_1)+\phi(z_1,w_0)$, we have

$$a_{ij}^{rs}=a_{i1}^{rs}+a_{1j}^{rs}-a_{11}^{rs}, \qquad r,s,i,j=1,\dots,n. \tag{6}$$

By (6),

$$\sum_{i,j=1}^{n}a_{ij}^{ij}=\sum_{i,j=1}^{n}[a_{i1}^{ij}+a_{1j}^{ij}-a_{11}^{ij}],$$

and applying (4) and (5) we obtain

$$\sum_{i,j=1}^{n}a_{ij}^{ij}=2n-1.$$

We show that $a_{ij}^{ij}\geqslant(3-\|P\|)/2$ for each i,j. It then follows that

$$2n-1\geqslant(3-\|P\|)\,n^2/2$$

or $\|P\|\geqslant 3-(4n-2)/n^2$. This is true for all n, so $\|P\|\geqslant 3$.

Consider $i,j=1$ (a similar proof holds for each choice of i,j). Since $0\leqslant(1-f_1(x))(1-g_1(y))\leqslant 1$, we have

$$|1-2f_1(x)-2g_1(y)+2f_1(x)\,g_1(y)|\leqslant 1$$

for all $x \in X$, $y \in Y$. Applying P and evaluating at $x = x_1$, $y = y_1$, gives

$$|1 - 2f_1(x_1) - 2g_1(y_1) + 2(P\phi^{11})(x_1, y_1)| \leqslant \|P\|$$

from which $|-3 + 2a_{11}^{11}| \leqslant \|P\|$. Thus $a_{11}^{11} \geqslant (3 - \|P\|)/2$. ∎

Remark. If X contains exactly n points and Y contains exactly m points, then the above argument shows that for any projection P of $C(X \times Y)$ onto M, $\|P\| \geqslant 3 - (2n + 2m - 2)/nm$. This lower bound is in fact attained by the choice

$$\begin{aligned} a_{ij}^{rs} &= -1/nm, && r \neq i, \quad s \neq j, \\ &= (n-1)/nm, && r = i, \quad s \neq j, \\ &= (m-1)/nm, && r \neq i, \quad s = j, \\ &= (n+m-1)/nm, && r = i, \quad s = j, \end{aligned}$$

where the $\{a_{ij}^{rs}\}_{r,i=1}^{n}\,{}_{s,j=1}^{m}$ are understood to be defined as in the proof of Theorem 7.

The projection P^* of $C_0(X \times Y)$ onto M_0 as given in Theorem 6 is of norm 2. Our next result, which is a variant of Theorem 7, shows that this is minimal.

THEOREM 8. *Let X and Y be infinite, compact, Hausdorff spaces. Let P be any projection of $C_0(X \times Y)$ onto M_0. Then $\|P\| \geqslant 2$.*

Proof. As in the proof of Theorem 7, let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be any two sets of n distinct points ($n \geqslant 2$) in X and Y, respectively, with $x_1 = x^*$ and $y_1 = y^*$. Let $\{f_i\}_{i=1}^n$ and $\{g_i\}_{i=1}^n$ be as in the proof of Theorem 7, and set $\phi^{rs}(x, y) = f_r(x)\, g_s(y)$ for $r, s = 1,\dots, n$; $(r, s) \neq (1. 1)$. For notational convenience, set $\phi^{11} = 0$. Thus $\phi^{rs} \in C_0(X \times Y)$ for all $r, s = 1,\dots, n$. Now

$$\phi^{rs}(x_i, y_j) = \delta_{ri}\delta_{sj}, \qquad r, s, i, j = 1,\dots, n, \quad (r, s) \neq (1, 1), \tag{7}$$

$$\sum_{r=1}^{n} \phi^{rs}(x, y) = g_s(y), \qquad s = 2,\dots, n, \tag{8}$$

$$\sum_{s=1}^{n} \phi^{rs}(x, y) = f_r(x), \qquad r = 2,\dots, n. \tag{9}$$

Set $(P\phi^{rs})(x_i, y_j) = a_{ij}^{rs}$. Since P maps $C_0(X \times Y)$ onto M_0, $a_{11}^{rs} = 0$, $r, s = 1,\dots, n$. From the definition of a_{ij}^{rs}, we have

$$a_{ij}^{rs} = a_{i1}^{rs} + a_{1j}^{rs}, \qquad a_{ij}^{11} = 0, \qquad a_{11}^{rs} = 0, \qquad r, s, i, j = 1,\dots, n, \tag{10}$$

and from (7), (8) and (9),

$$\sum_{r=1}^{n} a_{ij}^{rs} = \delta_{sj}, \qquad s = 2,\dots, n, \tag{11}$$

$$\sum_{s=1}^{n} a_{ij}^{rs} = \delta_{ri}, \qquad r = 2,\dots, n. \tag{12}$$

By (10), (11) and (12),

$$\begin{aligned}\sum_{i,j=1}^{n} a_{ij}^{ij} &= \sum_{i,j=1}^{n} [a_{i1}^{ij} + a_{1j}^{ij}] \\ &= \sum_{i=2}^{n} \sum_{j=1}^{n} a_{i1}^{ij} + \sum_{j=2}^{n} \sum_{i=1}^{n} a_{1j}^{ij} \\ &= 2(n-1).\end{aligned}$$

We show that $a_{ij}^{ij} \geqslant 2 - \|P\|$, for $i,j = 1,\dots, n$, $(i,j) \neq (1, 1)$. It then follows that

$$2(n-1) = \sum_{i,j=1}^{n} a_{ij}^{ij} \geqslant (n^2 - 1)(2 - \|P\|)$$

or $\|P\| \geqslant 2 - 2/(n+1)$. This is true for all n, so $\|P\| \geqslant 2$.

We divide the proof of this claim into two cases. First assume that $i,j > 1$. For convenience, consider $i = j = n$. Since $0 \leqslant f_n, g_n \leqslant 1$,

$$0 \leqslant f_n(x) + g_n(y) - f_n(x)\, g_n(y) \leqslant 1.$$

Because $f_n + g_n - f_n g_n \in C_0(X \times Y)$, we can apply P and evaluate at $x = x_n$, $y = y_n$ to obtain

$$|2 - a_{nn}^{nn}| \leqslant \|P\|.$$

So $a_{nn}^{nn} \geqslant 2 - \|P\|$.

Now assume that either i or j, but not both, is equal to 1. For convenience set $i = 1$, $j = n$. The function $g_n - (1 - f_1)(1 - g_n) \in C_0(X \times Y)$, and since $0 \leqslant f_1, g_n \leqslant 1$,

$$| g_n(y) - (1 - f_1(x))(1 - g_n(y))| \leqslant 1.$$

Applying P and evaluating at $x = x_1$, $y = y_n$ gives

$$|2 - a_{1n}^{1n}| \leqslant \|P\|.$$

So $a_{1n}^{1n} \geqslant 2 - \|P\|$. This completes the proof. ∎

Remark. Theorems 7 and 8 generalize as follows. Let $X^1, X^2, ..., X^k$ be infinite, compact, Hausdorff spaces. There exists no positive projection of $C(X^1 \times X^2 \times \cdots \times X^k)$ onto $M = C(X^1) + C(X^2) + \cdots + C(X^k)$, and any projection of $C(X^1 \times X^2 \times \cdots \times X^k)$ onto M is of norm at least $2k-1$. Let $x_i^* \in X^i$, $i = 1, ..., k$. Set $C_0(X^1 \times \cdots \times X^k) = \{\phi : \phi \in C(X^1 \times \cdots \times X^k), \phi(x_1^*, ..., x_k^*) = 0\}$ and $M_0 = M \cap C_0(X^1 \times \cdots \times X^k)$. There is a unique positive projection P^* of $C_0(X^1 \times \cdots \times X^k)$ onto M_0, given by $(P^*\phi)(x_1, ..., x_k) = \phi(x_1, x_2^*, ..., x_k^*) + \cdots + \phi(x_1^*, x_2^*, ..., x_k)$. For every projection P of $C_0(X^1 \times \cdots \times X^k)$ onto M_0, $\|P\| \geqslant \|P^*\| = k$.

If X^i contains exactly m_i points, $i = 1, ..., k$, then every projection P of $C(X^1 \times \cdots \times X^k)$ onto M satisfies

$$\|P\| \geqslant (2k-1) - (2k-2)\left[\sum_i 1/m_i - \sum_{i \neq j} 1/m_i m_j + \cdots + (-1)^{k-1}/m_1 m_2 \cdots m_k\right],$$

and this lower bound is attained (see the remark after Theorem 7).

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Note added in proof. Remark. Many results relating to positive projections are to be found in Donner [5]. In particular, it follows as a special case of his Theorem 4.7 that a finite-dimensional subspace L_n of a Banach lattice L admits a positive projection if (and only if) it (i) is a lattice in the induced ordering and (ii) any subset of L_n that has an upper bound in L has an upper bound in L_n. Our Theorem 1 provides a simple proof of this. One need only establish that any positive linear functional on L_n has a positive extension defined on L. This is an immediate consequence of the Hahn–Banach theorem and the fact that there is a K such that $\|(x^+)_{L_n}\| \leqslant K \|(x^+)_L\|$ for all $x \in L_n$, where $(x^+)_E = \sup_E(x, 0)$. To prove this, assume instead that there are elements x_n with $\|(x_n^+)_L\| \leqslant 2^{-n}$ and $\|(x_n^+)_{L_n}\| > n$. Let $y = \sum_1^\infty (x_n^+)_L$ and $A = \{x \in L_n : x \leqslant y\}$. Then A contains 0 and all x_n, which leads to a contradiction of (ii).

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