Uniqueness of Smoothest Interpolants

Allan Pinkus

Extremal problems play an essential role in approximation theory. One such elegant problem is the following. Assume we are given m fixed points $0 \le t_1 < \cdots < t_m \le 1$, m real data $e_1, \ldots, e_m$, and $n \in \mathbb{N}$. Our problem is to solve

$$\inf\{\|f^{(n)}\|_p : f \in C^{(n)}[0,1],\ f(t_i) = e_i,\ i = 1, \ldots, m\}, \tag{1}$$

where $\|\cdot\|_p$ is the usual $L^p[0,1]$ norm, $1 \le p \le \infty$, and to characterize f for which the above infimum is attained.

If $m \le n$, then this problem has a simple solution. There are algebraic polynomials of degree at most $n-1$ satisfying the interpolation data, and any of these polynomials obviously solve our problem.

If $m > n$ then, in general, the above infimum is not attained within $C^{(n)}[0,1]$. We are in the wrong space. For $1 < p \le \infty$ we should be in $W_p^{(n)}[0,1]$, the standard Sobolev space of real-valued functions on $[0,1]$ with $n-1$ absolutely continuous derivatives and nth derivative existing a.e. as a function in $L^p[0,1]$. It is here that our solution is to be found. For $p = 1$ the situation is slightly more complicated, but known (see e.g., Fisher, Jerome [8], de Boor [5]).

This extremal problem has played an important role in the development of spline theory. In the case $p = 2$ the unique solution of this extremal problem is a natural spline of degree $2n-1$ (with simple knots $t_1, \ldots, t_m$). This oft-quoted result, see de Boor [3], Schoenberg [18], was fundamental in the history of spline theory. For $p = \infty$ a solution (not necessarily unique) is a perfect spline of degree n with at most $m-n-1$ knots, see Karlin [9], de Boor [4]. For $p = 1$ there is a solution which is a spline of degree $n-1$ with at most $m-n$ knots.

In general, for $1 < p < \infty$, this problem has a unique solution. It is attained by the unique function f satisfying $f(t_i) = e_i$, $i = 1, \ldots, m$, whose nth derivative has the form

$$f^{(n)}(x) = |h(x)|^{q-1} \operatorname{sgn}(h(x)) \tag{2}$$

where $1/p + 1/q = 1$, and h is a spline of degree $n-1$ with the simple knots $t_1, \ldots, t_m$, which vanishes identically off $[t_1, t_m]$.

Let us now generalize this problem somewhat and bring it outside the realm of classic moment type problems, or minimum norm extension problems, where (1) may and should be considered. Assume that the data $e_1, \ldots, e_m$ is given as above, and we look at this same problem of finding the infimum of $\|f^{(n)}\|_p$ over $f \in W_p^{(n)}[0,1]$ satisfying $f(t_i) = e_i$, $i = 1, \ldots, m$, but we permit the points $t_1, \ldots, t_m$ to vary within $[0,1]$ while maintaining their order. That is, consider

(3)
$$\inf\{\|f^{(n)}\|_p : f \in W_p^{(n)}[0,1],\ f(t_i) = e_i,\ i = 1, \ldots, m,\ 0 \le t_1 < \cdots < t_m \le 1\},$$

for $1 \le p \le \infty$, and try to characterize those f for which the above infimum is attained.

Why consider this problem? Aside from its intrinsic interest, geometric motivation comes from the fact that curves may be parametrized differently. Thus we are, in a sense, asking for a best parametrization. A physical interpretation, for the natural generalization of this problem to $\mathbb{R}^d$, is that a solution of (3) represents a trajectory through prescribed points with least kinetic energy (the case $n = 2$). These matters are discussed somewhat in Marin [10] and Töpfer [19].

Marin [10] proves, in the one-dimensional setting, the existence, characterization and uniqueness of the solution of (3) in the case $p = 2$ and $n = 2$. Scherer, Smith [17] deal with the existence problem in $\mathbb{R}^d$, again restricted to $p = 2$, while Scherer [15] studies the periodic version thereof. In Rademacher, Scherer [14] the problem of existence and characterization in $\mathbb{R}^d$ is studied for all p.

In Pinkus [13] was considered (3) (in $\mathbb{R}$) for all $p \in [1, \infty]$. Before explaining some of the results obtained we can and will assume that

$$(4) \qquad (e_i - e_{i-1})(e_{i+1} - e_i) < 0, \quad i = 2, \ldots, m-1,$$

and $t_1 = 0$, $t_m = 1$. The condition (4) follows from continuity considerations. Under these assumptions we proved in Pinkus [13] the following.

For $1 < p < \infty$ the optimal f^* is of the form (2) and has the additional property that the optimal $\{t_i\}_{i=1}^m$ are the extreme points of f^*. That is, f^* is strictly monotone on $[t_i, t_{i+1}]$, $i = 1, \ldots, m-1$, or equivalently,

$$(5) \quad f^*(t_i) = e_i \quad i = 1, \ldots, m, \qquad \text{and} \qquad f^{*\prime}(t_i) = 0 \quad i = 2, \ldots, m-1.$$

Essentially the same result holds when $p = 1$ and $p = \infty$. For $p = 1$, f^* is a spline of degree $n-1$ with $m-n$ knots which satisfies (5).

For $p = \infty$ the solution is unique. It is the unique perfect spline of degree n with $m-n-1$ knots satisfying (5). A further generalization of this $p = \infty$ result is due to Bojanov [2]. In Draganova [7] is to be found a study of the periodic problem. Naidenov [12] gives an algorithm for the construction of this unique perfect spline. The case $m = n+1$ is especially interesting since the solution is the unique algebraic polynomial of degree $n+1$ satisfying (5). This particular problem has a history, see Davis [6], Mycielski and Paszkowski [11], and Bojanov [1].

Thus in the one-dimensional setting we have in (3), for $1 \le p \le \infty$, existence and characterization, and uniqueness for all n if $p = \infty$ (where, interestingly enough, problem (1) does not always have a unique solution). In addition Rademacher and Scherer [14], and Uluchev [20] have proven uniqueness for $n = 2$ and all $p \in (1, \infty)$, while Uluchev [20] also proved uniqueness in the case $n = 3$ with $p = 2$. Uniqueness in $\mathbb{R}^d$ for $p = 2$, all n, is studied in Scherer [16]. This is where things now stand. So the uniqueness problem even in the one-dimensional setting is far from settled, while relatively little is known in the multi-dimensional setting.

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Allan Pinkus
Department of Mathematics
Technion–Israel Institute of Technology
Haifa, 32000
Israel
pinkus@math.technion.ac.il