ON n-WIDTHS OF PERIODIC FUNCTIONS

By

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§1. Introduction

Let X be a normed linear space and $\mathcal{K}$ a subset of X. The n-width of $\mathcal{K}$, relative to X, in the sense of Kolmogorov, is given by

$$d_n(\mathcal{K}; X) = \inf_{X_n} \sup_{a\in\mathcal{K}} \inf_{x\in X_n} \| a - x \|_X$$

where the infimum is taken over all n-dimensional subspaces X_n of X. If there exists an n-dimensional subspace X_n^* for which

$$d_n(\mathcal{K}; X) = \sup_{a\in\mathcal{K}} \inf_{x\in X_n^*} \| a - x \|_X,$$

then X_n^* is said to be optimal.

A typical choice for $\mathcal{K}$ is the image of a unit ball of some normed linear space Y (which may be different from X) under a compact linear mapping K of Y into X. Thus, $\mathcal{K} = \{Ky : \| y \|_Y \leqq 1\}$. When $X = Y$ is a Hilbert space, then it is possible to obtain an expression for $d_n(\mathcal{K}; X)$ and identify optimal subspaces. These facts originated with the example given by Kolmogorov in his paper [2] in which the concept of n-widths was introduced. In this case, $d_n(\mathcal{K}; X)$ is related to the s-numbers of K.

Our concern here is with computing the n-widths and determining optimal subspaces for various classes of periodic functions. In [7] and [8], C. A. Micchelli and the author were concerned with the problem of obtaining n-widths and optimal subspaces for $X = L^q$, $Y = L^p$ (where $p = \infty$, or $q = 1$), and where K had the above form. The results were, in the main, restrictions to the case where the mapping K had certain total positivity properties. These properties are in a certain sense natural and give rise to particularly elegant solutions of the n-width problem.

An example of the above mentioned theory provides a solution to the n-widths of the Sobolev spaces $W_p^{(r)}[0,1]$ in $L^q[0,1]$ (for $p = \infty$ or $q = 1$), a result which had been previously obtained in the case $p = q = \infty$ by V. M. Tichomirov [14] and which was, to some extent, the motivation behind much of [7].

If, in place of considering $W_p^{(r)}[0,1]$, we consider $\tilde{W}_p^{(r)}[0,2\pi]$, the restriction of $W_p^{(r)}[0,2\pi]$ to 2π-periodic functions, then the situation is somewhat altered. Any $f \in \tilde{W}_p^{(r)}[0,2\pi]$ may be written in the form

$$f(x) = a_0 + \frac{1}{\pi}\int_0^{2\pi} \bar{B}_r(x-t) f^{(r)}(t)\,dt$$

where $\bar{B}_r(x)$ is a given known kernel, $\|f^{(r)}\|_p < \infty$, and $\int_0^{2\pi} f^{(r)}(t)dt = 0$. Unfortunately, various problems now arise, not the least of which is that $\bar{B}_r(x-t)$ is not totally positive. Nevertheless this class is important and natural, and even has a certain symmetry lacking in $W_p^{(r)}$. Thus it has been extensively studied. If we set

$$\tilde{B}_p^{(r)} = \{f \colon f \in \tilde{W}_p^{(r)}, \|f^{(r)}\|_p \leqq 1\},$$

then Tichomirov [14] was able to determine $d_{2n-1}(\tilde{B}_\infty^{(r)}; \tilde{C})$ and $d_{2n}(\tilde{B}_\infty^{(r)}; \tilde{C})$ where $\tilde{C} = \tilde{C}[0,2\pi]$ is the class of 2π-periodic continuous functions, while $d_{2n-1}(\tilde{B}_1^{(r)}; L^1)$ was studied by Makovoz [4] and Subbotin [12], and Makovoz [5] also computed $d_{2n-1}(\tilde{B}_\infty^{(r)}; L^1)$.

The purpose of the present work is to determine the underlying structure of the functions of $\tilde{W}_p^{(r)}$ which allow us, in certain cases, to determine their n-widths and identify various optimal subspaces. In this search we were led to a consideration of the n-widths of

$$\mathcal{K}_p = \left\{f \colon f(x) = \int_0^{2\pi} \phi(x-y) h(y)\,dy,\ \|h\|_p \leqq 1\right\}$$

in $L^q[0,2\pi]$, where $\phi(x)$ is a given 2π-periodic function with certain cyclic variation diminishing (CVD) properties. With the aid of these tools, we are able to refine and extend the known results, and hopefully provide a better intuitive understanding of the underlying theory.

The organization of the paper runs as follows. Section 2 contains a discussion of the definition and properties of CVD kernels. In addition, we also prove that if

$$T_{n-1} = \text{span}\{1, \sin x, \cos x, \cdots, \sin(n-1)x, \cos(n-1)x\},$$

i.e., trigonometric polynomials of degree $\leqq n-1$, and if ϕ is a CVD kernel, then for any $t_{n-1} \in T_{n-1}$, the function $\phi - t_{n-1}$ has at most $2n$ sign changes on any interval of length 2π. This sign change property is referred to by various authors as property

A_n or N_n, and has immediate application in the problem of characterizing the extent to which the class $\mathcal{K}_\infty$ is approximable in L^∞ from T_{n-1}.

Section 3 is concerned with the characterization of optimal subspaces for $d_{2n-1}(\mathcal{K}_\infty; L^\infty)$ and $d_{2n}(\mathcal{K}_\infty; L^\infty)$. We prove that T_{n-1} is an optimal $2n$ (and hence $2n-1$) dimensional subspace, and that $S_{2n}^* = \operatorname{span}\{\phi(x - \pi k/n)\}_{k=1}^{2n}$ is also an optimal $2n$-dimensional subspace. In fact, let us define the linear n-width of $\mathcal{K}_p$ in L^q by

$$\delta_n(\mathcal{K}_p; L^q) = \inf_{X_n} \inf_{P_n : L^q \to X_n} \sup_{\|h\|_p \leq 1} \left\| \int_0^{2\pi} \phi(\cdot - y)h(y)dy - P_n h(\cdot) \right\|_q,$$

where the first infimum is taken over all n-dimensional subspaces X_n and the second infimum is taken over all linear maps P_n from L^q to X_n. Then we in fact show that $\delta_n(\mathcal{K}_\infty; L^\infty) = d_n(\mathcal{K}_\infty; L^\infty)$ for all n, i.e., linear approximation methods suffice.

Section 4 is concerned with these same problems if we replace $\mathcal{K}_\infty$ by

$$\mathcal{K}_\infty(H) = \left\{ f : f(x) = \int_0^{2\pi} \phi(x-y)h(y)dy, \quad |h(y)| \leq H(y) \right\}$$

where H is some given positive continuous function. The results of Section 3 concerning the optimality of T_{n-1} and the fact that $d_{2n-1}(\mathcal{K}_\infty; L^\infty) = d_{2n}(\mathcal{K}_\infty; L^\infty)$ are no longer valid. We are however able to compute $d_{2n}(\mathcal{K}_\infty(H); L^\infty)$ and obtain an optimal $2n$-dimensional subspace.

In Section 5, we show that S_{2n}^* is an optimal $2n$-dimensional subspace for $d_{2n}(\mathcal{K}_\infty; L^p)$ for all $1 \leq p \leq \infty$, and compute the $2n$-width. In Section 6 we show that this same result holds for $d_{2n}(\mathcal{K}_p; L^1)$ and also, by duality, consider $d_{2n-1}(\mathcal{K}_1; L^1)$. We return to a consideration of the n-widths of the Sobolev spaces in Section 7. All the above results carry over in this case and we also draw upon a result of Taikov [13] to obtain the $(2n-1)$-widths $d_{2n-1}(\tilde{B}_p^{(r)}; L^1)$, $1 \leq p \leq \infty$.

§2. Cyclic variation diminishing kernels

Our concern is with cyclic variation diminishing (CVD) kernels ϕ. The definition, various properties, and examples of CVD kernels may be found in Mairhuber, Schoenberg, and Williamson [3], and Karlin [1]. Perhaps the best known CVD kernel is the de la Vallée Poussin kernel also studied in Pólya and Schoenberg [11]. On the not unreasonable assumption that many readers are not totally familiar with this concept, we sketch the definition and some of the properties.

Various definitions of sign changes and zeros ($S^-, S^+, Z, Z^*, \cdots$) of functions and vectors exist, and are intimately connected with the definitions of total positivity, sign consistency, variation diminishing, etc., $\cdots$. Our concern here is with periodic functions where the ordering induces a slightly altered definition.

If $\boldsymbol{x} = (x_1, \cdots, x_n)$ is a real non-trivial vector, then $S^-(\boldsymbol{x})$ indicates the number of sign changes in the sequence $x_1, \cdots, x_n$, with zero terms discarded. The number $S_c^-(\boldsymbol{x})$ of cyclic variations of sign of $\boldsymbol{x}$ is given by

$$S_c^-(x) = \max_i S^-(x_i, x_{i+1}, \cdots, x_n, x_1, \cdots, x_i) = S^-(x_k, x_{k+1}, \cdots, x_n, x_1, \cdots, x_k)$$

where k is any integer for which $x_k \neq 0$. Obviously, $S_c^-(\boldsymbol{x})$ is invariant under cyclic permutations, and $S_c^-(\boldsymbol{x})$ is always an even number.

Let $f(x)$ be a piecewise continuous, real, periodic 2π function. We define $S_c^-(f) = \sup S_c^-(f(x_1), \cdots, f(x_m))$, where the supremum is extended over all $x_1 < \cdots < x_m < x_1 + 2\pi$, m arbitrary.

Let $\phi(x)$ be a continuous, real, periodic, 2π function on $[0, 2\pi]$. Our concern is with the transformation

$$f(x) = (\phi h)(x) = \int_0^{2\pi} \phi(x-y)h(y)dy. \tag{2.1}$$

Definition 2.1. The transformation (2.1) is said to be cyclic variation diminishing of order $2n$ (CVD_{2n}) if $S_c^-(f) \leqq S_c^-(h)$ for all h for which $S_c^-(h) \leqq 2n$.

In this case we shall also say that ϕ is a cyclic variation diminishing kernel of order $2n$, or ϕ is CVD_{2n}.

To properly understand concepts related to that of a CVD_{2n} kernel, we need the following definition.

Definition 2.2. The kernel $\phi(x)$ is said to be sign consistent of order l, SC_l, if

$$\varepsilon_l \phi\begin{pmatrix} x_1, \cdots, x_l \\ y_1, \cdots, y_l \end{pmatrix} = \varepsilon_l \det(\phi(x_i - y_j))_{i,j=1}^l \geqq 0 \tag{2.2}$$

whenever $0 \leqq x_1 < \cdots < x_l < 2\pi$, $0 \leqq y_1 < \cdots < y_l < 2\pi$, and $\varepsilon_l = \pm 1$, fixed.

$\phi(x)$ is said to be strictly sign consistent of order l (SSC_l) if strict inequality holds in (2.2).

If, as above, $\phi(x)$ is a periodic, continuous function, then it is easily seen that if the rank of $\phi(x)$ is at least l, then $\phi(x)$ cannot be SC_l for even l.

Definition 2.3. The kernel $\phi(x)$ is said to be a cyclic Pólya frequency kernel of order $2n+1$ (CPF_{2n+1}) if $\phi(x)$ is SC_{2l+1} with $\varepsilon_{2l+1}=1$, $l=0,1,\cdots,n$.

The relationship between CPF_{2n+1} and CVD_{2n} kernels is partially contained in the following theorem.

Theorem 2.1 ([1], [3]). *Assume that $\phi(x)$ is a continuous, real, periodic 2π function and that $\phi(x)$ has rank at least $2n+2$, i.e., there exists $0 \leqq y_1 < \cdots < y_{2n+2} < 2\pi$ such that $\dim(\operatorname{span}\{\phi(x-y_i)\}_{i=1}^{2n+2}) = 2n+2$. Then $\phi(x)$ is CVD_{2n} iff $\varepsilon\phi(x)$ is CPF_{2n+1} for some $\varepsilon = \pm 1$, fixed.*

Although the following result is not explicitly used in this paper, it is perhaps one of the more interesting properties of CVD_{2n} kernels.

Associated with ϕ is its Fourier series

$$\phi(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}, \qquad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x) e^{-inx}\,dx.$$

Since ϕ is real, $a_{-n} = \bar{a}_n$. There is a direct relationship between the Fourier coefficients $\{a_n\}_{n=-\infty}^{\infty}$ and the CVD_{2n} property of $\phi(x)$.

Theorem 2.2 ([1], [3]). *Assume that $\phi(x)$ is as in Theorem 2.1, and ϕ is CVD_{2a}. Then,*

$$a_0 \geqq |a_1| \geqq \cdots \geqq |a_n| \geqq |a_k|, \qquad k = n+1, n+2, \cdots.$$

While it is true that for any CVD_{2n} kernel, $S_c^-(\phi h) \leqq S_c^-(h)$ provided $S_c^-(h) \leqq 2n$, it is in fact not necessary to consider all h in order to determine whether a given kernel is CVD_{2n}.

Theorem 2.3 ([1], [3]). *The kernel $\phi(x)$ is CVD_{2n} iff*

$$S_c^-(\phi h) \leqq S_c^-(h)$$

for any trigonometric polynomial h for which $S_c^-(h) \leqq 2n$.

Connected with the class of CVD_{2n} kernels are the class of functions A_n.

Definition 2.4. A continuous function f of period 2π is said to belong to A_n if

$$S_c^-(f - t_{n-1}) \leqq 2n$$

for every $t_{n-1} \in T_{n-1}$.

The following property is essential in the consideration of the best approximation of ϕ by trigonometric polynomials of degree $\leqq n-1$.

Theorem 2.4. *Let $\phi(x)$ be a* CVD_{2n} *kernel of rank at least $2n-1$. Then $\phi \in A_n$.*

Proof. If $t_{n-1} \in T_{n-1}$ then

$$\int_0^{2\pi} \phi(x-y)t_{n-1}(y)dy = \int_0^{2\pi} \phi(y)t_{n-1}(x-y)dy$$

is also a trigonometric polynomial of degree $\leqq n-1$, i.e., is also in T_{n-1}. Furthermore, since the rank of $\phi(x)$ is at least $2n-1$, and $\phi(x)$ is SC_{2n-1}, the mapping induced by the kernel ϕ, as a mapping from T_{n-1} to T_{n-1}, is 1-1 and onto.

Let

$$f_M(x) = \begin{cases} M & 0 \leqq x \leqq \dfrac{1}{M}, \\ \\ 0 & x \notin \left[0, \dfrac{1}{M}\right]. \end{cases}$$

Given $t_{n-1} \in T_{n-1}$, there exists a $\tilde{t}_{n-1} \in T_{n-1}$ for which $\int_0^{2\pi} \phi(x-y)\tilde{t}_{n-1}(y)dy = t_{n-1}(x)$. Given $\tilde{t}_{n-1}$, there exists an M_0 such that for all $M \geqq M_0$, $S_c^-(f_M - \tilde{t}_{n-1}) \leqq 2n$. (Recall that $S_c^-(t_{n-1}) \leqq 2n-2$ for any $t_{n-1} \in T_{n-1}$.) Thus, since ϕ is CVD_{2n}, $S_c^-(\phi f_M - \phi\tilde{t}_{n-1}) \leqq 2n$. Since $\phi\tilde{t}_{n-1} = t_{n-1}$, and $\lim_{M\to\infty} \phi f_M(x) = \lim_{M\to\infty} M\int_0^{1/M} \phi(x-y)dy = \phi(x)$, it follows that $S_c^-(\phi - t_{n-1}) \leqq 2n$, i.e., $\phi \in A_n$.

§3. n-widths of $\mathcal{K}_\infty$ in L^∞

Let $\phi(x)$ be a CVD_{2n} kernel as in Section 2, with the additional assumption that $\{\phi(x-y_i)\}_{i=1}^k$ are linearly independent functions for any $k = 1, \cdots, 2n+1$, and any $0 \leqq y_1 < \cdots < y_k < 2\pi$. Set

$$\mathcal{K}_\infty = \left\{ f : f(x) = \int_0^{2\pi} \phi(x-y)h(y)dy,\ \|h\|_\infty \leqq 1 \right\}, \tag{3.1}$$

and let

$$d_n(\mathcal{K}_\infty; L^\infty) = d_n(\mathcal{K}_\infty) = \inf_{X_n} \sup_{f \in \mathcal{K}_\infty} \inf_{g \in X_n} \| f - g \|_\infty,$$

where the infimum is taken over all n-dimensional subspaces of $L^\infty[0, 2\pi]$, and

$$\delta_n(\mathcal{K}_\infty) = \inf_{P_n : L^\infty \to X_n} \sup_{\|h\|_\infty \leq 1} \| \phi h - P_n h \|_\infty,$$

where the infimum is taken over all linear maps P_n which map $L^\infty[0, 2\pi]$ onto some n-dimensional subspace of $L^\infty[0, 2\pi]$. Let $h_n^*(x) = (-1)^i$, $i\pi/n \leqq x < (i+1)\pi/n$, $i = 0, 1, \cdots, 2n-1$, and $f_n^*(x) = \int_0^{2\pi} \phi(x-y) h_n^*(y) dy$. The following theorem is the main result of this section.

Theorem 3.1. *With the above assumptions and notation,*

$$d_{2n-1}(\mathcal{K}_\infty) = d_{2n}(\mathcal{K}_\infty) = \delta_{2n-1}(\mathcal{K}_\infty) = \delta_{2n}(\mathcal{K}_\infty) = \| f_n^* \|_\infty. \tag{3.2}$$

Furthermore,

(1) $T_{n-1} = \operatorname{span}\{1, \sin x, \cos x, \cdots, \sin(n-1)x, \cos(n-1)x\}$ *is an optimal subspace for* $d_{2n-1}(\mathcal{K}_\infty)$ *(and* $d_{2n}(\mathcal{K}_\infty)$*), and* $P_{2n-1}h(x) = \int_0^{2\pi} t_{n-1}^*(x-y) h(y) dy$ *is an optimal linear map for* $\delta_{2n-1}(\mathcal{K}_\infty)$*, where* $t_{n-1}^*(x)$ *is the unique best* L^1*-approximation to* $\phi(x)$ *on* $[0, 2\pi]$.

(2) $S_{2n}^* = \operatorname{span}\{\phi(x - k\pi/n)\}_{k=1}^{2n}$ *is also an optimal subspace for* $d_{2n}(\mathcal{K}_\infty)$*, and interpolation to* $f \in \mathcal{K}_\infty$ *from* S_{2n}^* *at* $\beta + \pi k/n$, $k = 1, \cdots, 2n$, *the zeros of* $f_n^*(x)$ *is an optimal linear map for* $\delta_{2n}(\mathcal{K}_\infty)$.

For the sake of convenience, the proof of the above theorem is divided into parts. Since $\delta_m(\mathcal{K}_\infty) \geqq d_m(\mathcal{K}_\infty)$ and $d_m(\mathcal{K}_\infty) \geqq d_{m+1}(\mathcal{K}_\infty)$ for any m, it is only necessary to prove that $d_{2n}(\mathcal{K}_\infty) \geqq \| f_n^* \|_\infty$, and to show that the linear deviation of $\mathcal{K}_\infty$ from T_{n-1} and S_{2n}^* is no more than $\| f_n^* \|_\infty$.

We first prove an ancillary lemma. For ease of exposition, we also introduce the following notation. Let $\Xi_{2n} = \{\xi : \xi = (\xi_1, \cdots, \xi_{2n}), 0 \leqq \xi_1 \leqq \cdots \leqq \xi_{2n} \leqq 2\pi\}$, i.e., the closed $2n$-dimensional simplex. For each $\xi \in \Xi_{2n}$, we define $h_\xi(x) = (-1)^{i+1}$, $\xi_{i-1} \leqq x < \xi_i$, $i = 1, \cdots, 2n$, where $\xi_0 = \xi_{2n} - 2\pi$, on an interval of length 2π, and then extend it periodically. Note that $S_c^-(h_\xi) \leqq 2n$ for every $\xi \in \Xi_{2n}$.

Lemma 3.1. *Let* $\xi, \eta \in \Xi_{2n}$. *Then,*

$$S_c^-(h_\xi \pm h_\eta) \leqq \min\{S_c^-(h_\xi), S_c^-(h_\eta)\} \leqq 2n. \tag{3.3}$$

Moreover, if $\xi_k = \eta_{k+2l}$ *for some* k *and some* l, *then*

$$S_c^-(h_\xi - h_\eta) \leqq 2(n-1). \tag{3.4}$$

Proof. Let $\xi = (\xi_1, \cdots, \xi_{2n})$. Since $h_\xi(x) = (-1)^{i+1}$, $\xi_{i-1} \leqq x < \xi_i$, $i = 1, \cdots, 2n$, and $|h_\eta(x)| \leqq 1$, then $(-1)^{i+1}(h_\xi \pm h_\eta)(x) \geqq 0$, $\xi_{i-1} \leqq x < \xi_i$, $i = 1, \cdots, 2n$, which implies, $S_c^-(h_\xi \pm h_\eta) \leqq S_c^-(h_\xi)$. Similarly, $S_c^-(h_\xi \pm h_\eta) \leqq S_c^-(h_\eta)$ from which (3.3) follows.

To prove (3.4), let us assume, without loss of generality, that $\eta_{k-1+2l} \leqq \xi_{k-1}$. Thus for $x \in [\xi_{k-1}, \xi_k]$, $(h_\xi - h_\eta)(x) \equiv 0$. Furthermore, as above, $(-1)^{i+1}(h_\xi - h_\eta)(x) \geqq 0$, $\xi_{i-1} \leqq x < \xi_i$, $i = 1, \cdots, 2n$. The result easily follows.

Proposition 3.1. *Let* P_{2n-1} *be as defined in the statement of Theorem* 3.1. *Then*

$$\sup_{\|h\|_\infty \leqq 1} \|\phi h - P_{2n-1}h\|_\infty = \|f_n^*\|_\infty. \tag{3.5}$$

Proof.

$$\begin{aligned}
\sup_{\|h\|_\infty \leqq 1} \|\phi h - P_{2n-1}h\|_\infty &= \sup_{\|h\|_\infty \leqq 1} \left\| \int_0^{2\pi} [\phi(x-y) - t_{n-1}^*(x-y)]h(y)dy \right\|_\infty \\
&= \max_x \int_0^{2\pi} |\phi(x-y) - t_{n-1}^*(x-y)|dy \\
&= \int_0^{2\pi} |\phi(y) - t_{n-1}^*(y)|dy \\
&= \|\phi - t_{n-1}^*\|_1.
\end{aligned}$$

Thus,

$$\sup_{\|h\|_\infty \leqq 1} \|\phi h - P_{n-1}h\|_\infty = \inf_{t_{n-1} \in T_{n-1}} \|\phi - t_{n-1}\|_{L^1[0,2\pi]}.$$

Since $\phi(y) - t_{n-1}(y)$ can vanish at at most a finite number of points, a necessary and sufficient condition for $t_{n-1}^*(y)$ to be a best L^1-approximation to $\phi(y)$ on $[0, 2\pi]$ is that

$$\int_0^{2\pi} \operatorname{sgn}((\phi - t_{n-1}^*)(y)) t_{n-1}(y)dy = 0 \tag{3.6}$$

for every $t_{n-1} \in T_{n-1}$. (From Theorem 2.4, $S_c^-(\phi - t_{n-1}^*) \leqq 2n$.)

We wish to prove that $(\phi - t^*_{n-1})(y)$ changes sign at $2n$ equally spaced points, i.e., that there exists an α such that sgn $((\phi - t^*_{n-1})(y)) = (-1)^i$, $\alpha + \pi i/n < y < \alpha + \pi(i+1)/n$, $i = 0, 1, \cdots, 2n-1$. Let $h^*_n(y; \alpha) = h^*_n(y - \alpha)$. Since $S^-_c(\phi - t^*_{n-1}) \leqq 2n$, there exists a $\xi \in \Xi_{2n}$ such that sgn $((\phi - t^*_{n-1})(y)) = \pm h_\xi(y)$.

From (3.6), $\int_0^{2\pi} h_\xi(y) t_{n-1}(y) dy = 0$ for all $t_{n-1} \in T_{n-1}$. It is easily verified that $\int_0^{2\pi} h^*_n(y; \alpha) t_{n-1}(y) dy = 0$ for all $t_{n-1} \in T_{n-1}$ and for every choice of α. Thus,

$$\int_0^{2\pi} [h^*_n(y; \alpha) \pm h_\xi(y)] t_{n-1}(y) dy = 0 \tag{3.7}$$

for all $t_{n-1} \in T_{n-1}$, and for each α. Since T_{n-1} has a $2n-1$ dimensional basis which is a Tchebycheff (T) system on $[0, 2\pi)$, a standard argument proves that $S^-_c(h^*_n(\cdot; \alpha) \pm h_\xi) \geqq 2n$ or $h^*_n(y; \alpha) \pm h_\xi(y) \equiv 0$. (If not, one may construct a $t_{n-1} \in T_{n-1}$ which agrees in sign with $h^*_n(y, \alpha) \pm h_\xi(y)$, and thus contradicts (3.7).) From Lemma 3.1, we see that there exists an α_0 and a choice of $\pm$ for which $S^-_c(h^*_n(\cdot; \alpha_0) \pm h_\xi) \leqq 2(n-1)$. Thus, $h^*_n(y; \alpha_0) \pm h_\xi(y) \equiv 0$. The uniqueness of $t^*_{n-1}(y)$ also easily follows.

Thus,

$$\begin{aligned}
\sup_{\|h\|_\infty \leqq 1} \| \phi h - P_{n-1} h \|_\infty &= \int_0^{2\pi} |\phi(y) - t^*_{n-1}(y)| dy \\
&= \pm \int_0^{2\pi} (\phi(y) - t^*_{n-1}(y)) h^*_n(y; \alpha_0) dy \\
&= \pm \int_0^{2\pi} \phi(y) h^*_n(y - \alpha_0) dy \\
&= \mp f^*_n(\alpha_0) \\
&\leqq \| f^*_n \|_\infty .
\end{aligned}$$

Since $P_{n-1} h^*_n = 0$ and $f^*_n = \phi h^*_n$, the proposition is proven.

Remark 3.1. An alternate equivalent proof of Proposition 3.1 exists. Rather than considering the best L^1-approximation to ϕ from T_{n-1}, we construct an explicit approximant $\tilde{t}_{n-1}$ which interpolates ϕ at $2n$ equally spaced points and such that $\phi - \tilde{t}_{n-1}$ changes sign there, and only there. The existence of such a function $\tilde{t}_{n-1}$ implies Proposition 3.1. The approach taken here is an indirect construction of the function $\tilde{t}_{n-1}$.

Since $f^*_n(x) = \int_0^{2\pi} \phi(x - y) h^*_n(y) dy$, it follows that $S^-_c(f^*_n) \leqq 2n$. Moreover, $f^*_n(x + \pi/n) = -f^*_n(x)$ so that f^*_n has $2n$ equally spaced zeros, which are at the $2n$

sign changes of $f_n^*(x)$. Let $\beta + \pi k/n$, $k = 1, \cdots, 2n$, denote these $2n$ zeros. The following lemma may be found in [6] and [8]. For completeness, we include its proof.

Lemma 3.2. *For ϕ and β as above,*

$$\phi\begin{pmatrix} \beta + \frac{\pi}{n}, \beta + \frac{2\pi}{n}, \cdots, \beta + 2\pi \\ \frac{\pi}{n}, \quad \frac{2\pi}{n}, \cdots, 2\pi \end{pmatrix} \neq 0. \tag{3.8}$$

Proof. The proof is by negation. Assume that the above determinant is zero. Then there exist $2n$ constants (not all zero) $\alpha_1, \cdots, \alpha_{2n}$ such that $u(x) = \sum_{i=1}^{2n} \alpha_i \phi(x - \pi i/n)$ vanishes at $\beta + \pi j/n$, $j = 1, \cdots, 2n$. Since $\{\phi(x - \pi i/n)\}_{i=1}^{2n}$ are linearly independent, then there exists a $z \in [0, 2\pi)$ for which $u(z) \neq 0$. Thus, since $f_n^*(\beta + \pi j/n) = 0$, $j = 1, \cdots, 2n$, there exists a $c \neq 0$ such that $f_n^*(x) - cu(x)$ vanishes at $x = \beta + \pi j/n$, $j = 1, \cdots, 2n$, and $x = z$. Since $\{\phi(\beta + \pi j/n - y)\}_{j=1}^{2n} \cup \{\phi(z - y)\}$ is a WT-system of dimension $2n + 1$ on $[0, 2\pi)$, there exist coefficients $\{\delta_i\}_{i=1}^{2n+1}$ (not all zero) such that

$$v(y) = \sum_{j=1}^{2n} \delta_j \phi\left(\beta + \frac{\pi j}{n} - y\right) + \delta_{2n+1}\phi(z - y)$$

changes sign at $\pi i/n$, $i = 1, \cdots, 2n$, and nowhere else. Now

$$\begin{aligned}
0 &= \sum_{j=1}^{2n} \delta_j\left[f_n^*\left(\beta + \frac{\pi j}{n}\right) - cu\left(\beta + \frac{\pi j}{n}\right)\right] + \delta_{2n+1}[f_n^*(z) - cu(z)] \\
&= \sum_{j=1}^{2n} \delta_j\left[\int_0^{2\pi} \phi\left(\beta + \frac{\pi j}{n} - y\right)h_n^*(y)dy\right] + \delta_{2n+1}\left[\int_0^{2\pi} \phi(z - y)h_n^*(y)dy\right] \\
&\quad - c\sum_{j=1}^{2n} \delta_j\left[\sum_{i=1}^{2n} \alpha_i\phi\left(\beta + \frac{\pi j}{n} - \frac{\pi i}{n}\right)\right] - c\delta_{2n+1}\left[\sum_{i=1}^{2n} \alpha_i\phi\left(z - \frac{\pi i}{n}\right)\right] \\
&= \int_0^{2\pi} v(y)h_n^*(y)dy - c\sum_{i=1}^{2n} \alpha_i v\left(\frac{\pi i}{n}\right) \\
&= \pm\int_0^{2\pi} |v(y)|dy \neq 0.
\end{aligned}$$

This contradiction implies (3.8).

Let Q_{2n} denote the linear map which is interpolation to $f \in \mathcal{K}_\infty$ at $\beta + \pi k/n$, $k = 1, \cdots, 2n$, from S_{2n}^*. This map, by Lemma 3.2, is well defined.

Proposition 3.2. *Let Q_{2n} be as defined. Then*

$$\sup_{\|h\|_\infty \leq 1} \| \phi h - Q_{2n} h \|_\infty = \| f_n^* \|_\infty. \tag{3.9}$$

Proof. We shall make use of the SC_{2n+1} property of $\phi(x)$, and the fact that

$$\int_0^{2\pi} \phi(x-y)h(y)dy - Q_{2n}h(x) = \int_0^{2\pi} \frac{\phi\begin{pmatrix} x, \beta+\frac{\pi}{n}, \beta+\frac{2\pi}{n}, \cdots, \beta+2\pi \\ y, \quad \frac{\pi}{n}, \quad \frac{2\pi}{n}, \cdots, \quad 2\pi \end{pmatrix}}{\phi\begin{pmatrix} \beta+\frac{\pi}{n}, \beta+\frac{2\pi}{n}, \cdots, \beta+2\pi \\ \frac{\pi}{n}, \quad \frac{2\pi}{n}, \cdots, \quad 2\pi \end{pmatrix}} h(y)dy.$$

Since ϕ is SC_{2n+1},

$$\left[\operatorname{sgn} \phi \begin{pmatrix} x, \beta+\frac{\pi}{n}, \beta+\frac{2\pi}{n}, \cdots, \beta+2\pi \\ y, \quad \frac{\pi}{n}, \quad \frac{2\pi}{n}, \cdots, \quad 2\pi \end{pmatrix}\right] h_n^*(y)\varepsilon(x) \geqq 0, \qquad \text{for all } y \in [0, 2\pi),$$

where $\varepsilon(x)$ is either 1 or -1 depending on x. Thus,

$$\sup_{\|h\|_\infty \leq 1} \| \phi h - Q_{2n} h \|_\infty = \sup_{\|h\|_\infty \leq 1} \max_x \left| \int_0^{2\pi} \phi(x-y)h(y)dy - Q_{2n}h(x) \right|$$

$$= \max_x \left| \int_0^{2\pi} \frac{\phi\begin{pmatrix} x, \beta+\frac{\pi}{n}, \beta+\frac{2\pi}{n}, \cdots, \beta+2\pi \\ y, \quad \frac{\pi}{n}, \quad \frac{2\pi}{n}, \cdots, \quad 2\pi \end{pmatrix}}{\phi\begin{pmatrix} \beta+\frac{\pi}{n}, \beta+\frac{2\pi}{n}, \cdots, \beta+2\pi \\ \frac{\pi}{n}, \quad \frac{2\pi}{n}, \cdots, \quad 2\pi \end{pmatrix}} h_n^*(y)dy \right|$$

$$= \max_x \left| \int_0^{2\pi} \phi(x-y)h_n^*(y)dy \right| = \| f_n^* \|_\infty.$$

We have here used the fact that $\int_0^{2\pi} \phi(\beta + \pi j/n - y)h_n^*(y)dy = 0$, $j = 1, \cdots, 2n$.

To complete the proof of Theorem 3.1, we need the following result.

Proposition 3.3. $d_{2n}(\mathcal{K}_\infty) \geqq \|f_n^*\|_\infty$.

The proof of Proposition 3.3 makes critical use of the following lemma.

Lemma 3.3. *For each* $\xi \in \Xi_{2n}$, *set* $f_\xi(x) = \int_0^{2\pi} \phi(x-y) h_\xi(y) dy$. *Then,*

$$\min_{\xi \in \Xi_{2n}} \|f_\xi\|_\infty = \|f_n^*\|_\infty. \tag{3.10}$$

Proof. Since $f_n^*(x + \pi/n) = -f_n^*(x)$, $f_n^*(x)$ achieves its norm in $L^\infty[0, 2\pi]$, $2n$ times, alternately. That is, there exists a γ such that

$$(-1)^i f_n^*\left(\gamma + \frac{\pi i}{n}\right) = \|f_n^*\|_\infty, \qquad i = 1, \cdots, 2n.$$

Assume $\xi \in \Xi_{2n}$, and $\|f_\xi\|_\infty < \|f_n^*\|_\infty$. Thus $f_n^*(x+\alpha) \pm f_\xi(x)$ has at least $2n$ sign changes in $[0, 2\pi)$, i.e., $S_c^-(f_n^*(x+\alpha) \pm f_\xi(x)) \geqq 2n$, for each α and each choice of $\pm$.

Now,

$$f_n^*(x+\alpha) \pm f_\xi(x) = \int_0^{2\pi} \phi(x-y)[h_n^*(y-\alpha) \pm h_\xi(y)]dy.$$

Since ϕ is CVD_{2n}, it therefore follows that $S_c^-(h_n^*(y-\alpha) \pm h_\xi(y)) \geqq 2n$ for each α and each choice of $\pm$. We now obtain a contradiction of Lemma 3.1 as in Proposition 3.1.

Proof of Proposition 3.3. To prove Proposition 3.3, we must prove that given any $2n$-dimensional subspace X_{2n}, there exists an $f \in \mathcal{K}_\infty$ for which $\min_{g \in X_{2n}} \|f - g\|_\infty \geqq \|f_n^*\|_\infty$.

Let $S_{2n} = \{t: t = (t_1, \cdots, t_{2n+1}), \Sigma_{i=1}^{2n+1} t_i^2 = 2\pi\}$. For each $t \in S_{2n}$, let $\xi(t) \in \Xi_{2n}$ be defined by $\xi_0(t) = 0$, and $\xi_i(t) = \Sigma_{j=1}^{i} t_j^2$, $i = 1, \cdots, 2n+1$. Thus $\xi_0(t) = 0 \leqq \xi_1(t) \leqq \cdots \leqq \xi_{2n}(t) \leqq \xi_{2n+1}(t) = 2\pi$. Set $h_{\xi(t)}(x) = \operatorname{sgn} t_i$, $\xi_{i-1}(t) \leqq x < \xi_i(t)$, $i = 1, \cdots, 2n+1$. Since $2n+1$ is odd, either $h_{\xi(t)}(x)$ or $-h_{\xi(t)}(x)$ is equal to some h_η where $\eta \in \Xi_{2n}$ for each $t \in S_{2n}$.

The idea of the proof of Proposition 3.3 is to consider, for each $t \in S_{2n}$, the best approximation to $\phi h_{\xi(t)}$ from X_{2n} and to show that there exists a $t^* \in S_{2n}$ for which the best approximation is the zero function. An application of Lemma 3.3 would then prove the proposition. A slight difficulty arises due to the fact that the best L^∞-approximation from X_{2n} is not necessarily unique. To circumvent this difficulty we consider the best L^p-approximation and let $p \uparrow \infty$.

The idea of this proof is to be found in [8]. Let $1 < p < \infty$, and $X_{2n} = \operatorname{span}\{g_i\}_{i=1}^{2n}$. For each $t \in S_{2n}$, let $\sum_{i=1}^{2n} a_i(t) g_i(x)$ be the best L^p-approximation to $\int_0^{2\pi} \phi(x-y) h_{\xi(t)}(y)dy$ from X_{2n}. The mapping $A(t) = (a_1(t), \cdots, a_{2n}(t))$ is a continuous odd mapping of S_{2n} into R^{2n}. By the Borsuk Antipodensatz (see e.g. [9]) there exists a $t^* \in S_{2n}$ for which $a_i(t^*) = 0$, $i = 1, \cdots, 2n$. Thus

$$\sup_{f \in \mathcal{K}_\infty} \inf_{g \in X_{2n}} \|f - g\|_p \geqq \min_{\xi \in \Xi_{2n}} \|f_\xi\|_p.$$

Letting $p \uparrow \infty$, and since the set $\{f_\xi : \xi \in \Xi_{2n}\}$ is compact, we obtain

$$\sup_{f \in \mathcal{K}_\infty} \inf_{g \in X_{2n}} \|f - g\|_\infty \geqq \min_{\xi \in \Xi_{2n}} \|f_\xi\|_\infty.$$

From Lemma 3.3, $\min_{\xi \in \Xi_{2n}} \|f_\xi\|_\infty = \|f_n^*\|_\infty$, and thus the proposition and the theorem are proven.

§4. n-widths of $\mathcal{K}_\infty(H)$ in L^∞

It seems to some extent strange, if we have at our disposal a $2n-1$ dimensional subspace T_{n-1} which is optimal for $d_{2n-1}(\mathcal{K}_\infty)$ and $d_{2n}(\mathcal{K}_\infty)$, that we should concern ourselves with an optimal $2n$-dimensional subspace $\{\phi(x - \pi k/n)\}_{k=1}^{2n}$. One of the reasons for this is historical and this is discussed to some extent in Section 7. Another reason is that this latter subspace is reasonable, is fairly easy to work with, and it is often advantageous to realize that it is optimal. Moreover, in this section, we see that perturbations of this subspace remain optimal under certain perturbations of $\mathcal{K}_\infty$, where d_{2n-1} and d_{2n} are no longer equal.

Let $H(y)$ be a continuous, positive function. (Both these conditions may be considerably weakened.) Let

$$\mathcal{K}_\infty(H) = \left\{ f : f(x) = \int_0^{2\pi} \phi(x-y)h(y)dy, |h(y)| \leqq H(y) \right\}.$$

In this section we prove the following theorem.

Theorem 4.1. *Let the assumptions and notation of Section 3 hold. Then,*

(1) *The minimum in*

$$\min_{\xi \in \Xi_{2n}} \left\| \int_0^{2\pi} \phi(x-y) h_\xi(y) H(y) dy \right\|_\infty \tag{4.1}$$

is attained by a $\xi^* \in \Xi_{2n}$ *for which* $\xi^* = (\xi_1^*, \cdots, \xi_{2n}^*)$, *where* $0 \leqq \xi_1^* < \xi_2^* < \cdots < \xi_{2n}^* < 2\pi$, *i.e., by a* ξ^* *in the interior of* Ξ_{2n}.

(2) $$d_{2n}(\mathcal{K}_\infty(H)) = \delta_{2n}(\mathcal{K}_\infty(H)) = \left\| \int_0^{2\pi} \phi(\cdot - y) h_\xi^*(y) H(y) dy \right\|_\infty .$$

(3) $S_{2n}^* = \operatorname{span}\{\phi(x - \xi_i^*)\}_{i=1}^{2n}$ *is an optimal subspace for* $d_{2n}(\mathcal{K}_\infty(H))$, *and interpolation from* S_{2n}^* *to any function in* $\mathcal{K}_\infty(H)$ *at the zero of* $f^*(x) = \int_0^{2\pi} \phi(x - y) h_\xi^*(y) H(y) dy$ *is an optimal linear method of approximation.*

(4) $d_{2n-1}(\mathcal{K}_\infty(H)) \geqq d_{2n}(\mathcal{K}_\infty(H))$, *with equality iff* $H(y) = H$, *i.e.,* $H(y)$ *is a constant.*

Let

$$\Xi_{2n}(\eta) = \{\xi : \xi \in \Xi_{2n}, \xi_i = \eta \quad \text{for some } i\}.$$

The proof of Theorem 4.1 relies upon the following propositions.

Proposition 4.1. *Given* $\eta \in [0, 2\pi)$, *there exists a* $\xi^*(\eta) \in \Xi_{2n}(\eta)$ *satisfying*

(a) $S_c^-(h_{\xi^*(\eta)}) = 2n$,

(b) *The function* $f_\eta^*(x) = \int_0^{2\pi} \phi(x - y) h_{\xi^*(\eta)}(y) H(y) dy$ *has* $2n$ *points of equioscillation.*

Proof. Fix η and let $S_{2n-1} = \{t : t = (t_1, \cdots, t_{2n}), \sum_{j=1}^{2n} t_j^2 = 2\pi\}$. Set $\xi_1(\eta; t) = \eta$, and $\xi_{i+1}(\eta; t) = \eta + \sum_{j=1}^{i} t_j^2$, $i = 1, \cdots, 2n$. Thus $\xi(\eta) \in \Xi_{2n}(\eta)$. As previously, let $h_{\xi(\eta;t)}(x) = \operatorname{sgn} t_i$, $\xi_i(\eta; t) \leqq x < \xi_{i+1}(\eta; t)$, $i = 1, \cdots, 2n$. Let Y_{2n-1} be any $2n - 1$ dimensional T-system on $[0, 2\pi]$. (We may, for example, choose $Y_{2n-1} = \operatorname{span}\{1, x, \cdots, x^{2n-2}\}$.) By the Borsuk Antipodensatz and the arguments of the previous section, we see that there exists a $t^* \in S_{2n-1}$, for which the best L^∞-approximation to

$$\int_0^{2\pi} \phi(x - y) h_{\xi(\eta; t^*)}(y) H(y) dy = f_\eta^*(x)$$

from Y_{2n-1} is zero. (In the situation considered herein it is not necessary, as in the previous section, to alter the L^∞ norm to a strictly convex norm since the best L^∞-approximation from Y_{2n-1} is unique.) Since Y_{2n-1} is a T-system on $[0, 2\pi]$, the error function necessarily equioscillates at $2n$ points. Thus $f_\eta^*(x)$ necessarily equioscillates at $2n$ points on $[0, 2\pi]$. Since $2n$ is even it follows that $f_\eta^*(x)$ necessarily equioscillates at $2n$ points in $[0, 2\pi)$, and part (b) of the proposition is proven. Let $\xi^*(\eta) = \xi(\eta; t^*)$. Since $f^*_\eta(x)$ equioscillates at $2n$ points on $[0, 2\pi)$, $S_c^-(f^*_\eta) \geqq 2n$. Part (a) of the proposition is a direct consequence of the CVD_{2n} property of ϕ and the fact that $S_c^-(h_{\xi^*(\eta)}) \leqq 2n$ since $\xi^*(\eta) \in \Xi_{2n}(\eta)$.

Proposition 4.2.

$$\min_{\xi \in \Xi_{2n}(\eta)} \left\| \int_0^{2\pi} \phi(x-y) h_\xi(y) H(y)\, dy \right\|_\infty = \left\| \int_0^{2\pi} \phi(x-y) h_{\xi^*(\eta)}(y) H(y)\, dy \right\|_\infty = \|f^*_\eta\|_\infty.$$

The proof of Proposition 4.2 is totally analogous to the proof of Lemma 3.3.

Proof of Theorem 4.1. Propositions 4.1 and 4.2 immediately imply Theorem 4.1, Part 1. Let η^* be such that $\min_\eta \|f^*_\eta\|_\infty$ is attained and let $f^*(x) = f^*_{\eta^*}(x)$, and $\xi^* = \xi^*(\eta^*)$. Thus,

$$\min_{\xi \in \Xi_{2n}} \left\| \int_0^{2\pi} \phi(x-y) h_\xi(y) H(y)\, dy \right\|_\infty = \left\| \int_0^{2\pi} \phi(x-y) h_{\xi^*}(y) H(y)\, dy \right\|_\infty = \|f^*\|_\infty.$$

Now, by the method of proof of Proposition 3.3, it is readily shown that

$$d_{2n}(\mathcal{K}_\infty(H)) \geqq \min_{\xi \in \Xi_{2n}} \left\| \int_0^{2\pi} \phi(x-y) h_\xi(y) H(y)\, dy \right\|_\infty = \|f^*\|_\infty,$$

and also that for each $\eta \in [0, 2\pi)$,

$$d_{2n-1}(\mathcal{K}_\infty(H)) \geqq \min_{\xi \in \Xi_{2n}(\eta)} \left\| \int_0^{2\pi} \phi(x-y) h_\xi(y) H(y)\, dy \right\|_\infty = \|f^*_\eta\|_\infty.$$

Therefore,

$$d_{2n-1}(\mathcal{K}_\infty(H)) \geqq \max_\eta \|f^*_\eta\|_\infty, \quad \text{while} \quad d_{2n}(\mathcal{K}_\infty(H)) \geqq \min_\eta \|f^*_\eta\|_\infty.$$

The remaining portions of Theorem 4.1 will therefore ensue if we can show that $\delta_{2n}(\mathcal{K}_\infty(H)) \leqq \|f^*\|_\infty$ and that interpolation to $f \in \mathcal{K}_\infty(H)$ from S^*_{2n} at the zeros of $f^*(x)$ is indeed optimal. The proof of this fact is totally analogous to the proof of the corresponding result in Section 3.

§5. $2n$-widths of $\mathcal{K}_\infty(H)$ in L^p

The preceding analysis need not only be restricted to a consideration of $d_n(\mathcal{K}_\infty; L^\infty)$ or $d_n(\mathcal{K}_\infty(H); L^\infty)$. As we shall soon see, we have effectively laid the groundwork for the consideration of $d_{2n}(\mathcal{K}_\infty(H); L^p)$ for every $p \in [1, \infty]$. In fact the proof of the following theorem has been almost completely given in the preceding sections.

Theorem 5.1. *Let the assumptions and notation of Section* 3 *hold. Let* $1 \leqq p < \infty$, *p fixed. Then,*

I. *The minimum in*

$$\min_{\xi \in \Xi_{2n}} \left\| \int_0^{2\pi} \phi(x-y) h_\xi(y) H(y) dy \right\|_p \tag{5.1}$$

is attained by a $\xi^* \in \Xi_{2n}$ *for which* $\xi^* = (\xi_1^*, \cdots, \xi_{2n}^*)$, *where* $0 \leqq \xi_1^* < \xi_2^* \cdots < \xi_{2n}^* < 2\pi$ *i.e., by a* ξ^* *in the interior of* Ξ_{2n}.

II. $d_{2n}(\mathcal{K}_\infty(H); L^p) = \delta_{2n}(\mathcal{K}_\infty(H); L^p) = \left\| \int_0^{2\pi} \phi(\cdot - y) h_{\xi^*}(y) H(y) dy \right\|_p.$

III. $S_{2n}^* = \operatorname{span}\{\phi(x - \xi_i^*)\}_{i=1}^{2n}$ *is an optimal subspace for* $d_{2n}(\mathcal{K}_\infty(H); L^p)$, *and interpolation from* S_{2n}^* *to any function in* $\mathcal{K}_\infty(H)$ *at the* $2n$ *zeros of* $f^*(x) = \int_0^{2\pi} \phi(x-y) h_{\xi^*}(y) H(y) dy$ *is an optimal linear method of approximation.*

IV. $d_{2n-1}(\mathcal{K}_\infty(H); L^p) \geqq d_{2n}(\mathcal{K}_\infty(H); L^p)$ *and the inequality is strict if* $H(y)$ *is not a constant.*

Proof. The proof of this theorem follows from the proof of Theorem 4.1 if we can prove part I of the theorem, i.e., ξ^* is in the interior of Ξ_{2n}, and also that $f^*(x)$ has $2n$ zeros.

A compactness argument shows that since Ξ_{2n} is closed, the minimum in (5.1) is attained. Assume that $\xi^* \in \Xi_{2n}$ attains this minimum, with $\xi^* = (\xi_1^*, \cdots, \xi_{2k}^*)$, where $0 \leqq \xi_1^* < \cdots < \xi_{2k}^* < 2\pi$, for some $k = 0, 1, \cdots, n$. A simple perturbation argument shows that since ξ^* is optimal,

$$\begin{aligned} &\int_0^{2\pi} \left| \int_0^{2\pi} \phi(x-y) h_{\xi^*}(y) H(y) dy \right|^{p-1} \\ &\operatorname{sgn}\left(\int_0^{2\pi} \phi(x-y) h_{\xi^*}(y) H(y) dy \right) \phi(x - \xi_i^*) H(\xi_i^*) dx = 0, \qquad i = 1, \cdots, 2k. \end{aligned} \tag{5.2}$$

It then follows from the fact that $\{\phi(x-\xi_i^*)\}_{i=1}^{2k}$ is a weak Tchebycheff system on $[0,2\pi)$ that $S_c^-(\int_0^{2\pi}\phi(x-y)h_{\xi^*}(y)H(y)dy)\geqq 2k$. Since $\phi(x-y)$ is CVD_{2n}, $n\geqq k$, and $S_c^-(h_{\xi^*})=2k$, we have $S_c^-(\int_0^{2\pi}\phi(x-y)h_{\xi^*}(y)H(y)dy)\leqq 2k$. Thus $S_c^-(\int_0^{2\pi}\phi(x-y)h_{\xi^*}(y)H(y)dy)=2k$, and it remains to prove that $k=n$.

Assume that $k<n$. For ε sufficiently small and for each $\xi\not\in\{\xi_1^*,\cdots,\xi_{2k}^*\}$, we introduce $\xi_\varepsilon^*\in\Xi_{2n}$ which aside from the knots $\xi_1^*,\cdots,\xi_{2k}^*$ has the additional two knots $\xi-\varepsilon$ and $\xi+\varepsilon$. Since ξ_ε^* is admissable in (5.1) and ξ^* with $\varepsilon=0$ is optimal, it follows as in (5.2) that

$$\int_0^{2\pi}\left|\int_0^{2\pi}\phi(x-y)h_\xi^*(y)H(y)dy\right|^{p-1} \tag{5.3}$$
$$\operatorname{sgn}\left(\int_0^{2\pi}\phi(x-y)h_\xi^*(y)H(y)dy\right)\phi(x-\xi)H(\xi)dx=0.$$

The fact that (5.3) holds for every $\xi\in[0,2\pi)$ immediately leads to a contradiction. Thus $k=n$ and Theorem 5.1 follows. Note that the claim of Theorem 5.1, IV, is rather weak. Even in the case where $H(y)$ is a constant function, nothing is known concerning $d_{2n-1}(\mathcal{K}_\infty;L^p)$.

However, if $H(y)$ is a constant (we shall assume $H(y)\equiv 1$) then more can be proven concerning the optimal subspace S_{2n}^*.

Theorem 5.2. *Let the previous assumptions obtain, $H(y)\equiv 1$, and $p\in[1,\infty)$. Then*

$$\xi^*=\left(\frac{\pi}{n},\frac{2\pi}{n},\cdots,\frac{2(n-1)\pi}{n},2\pi\right) \textit{ is a solution of the minimum problem } (5.1).$$

It thus follows from Theorem 5.1 that $S_{2n}^*=\operatorname{span}\{\phi(x-k\pi/n)\}_{k=1}^{2n}$ is an optimal subspace for $d_{2n}(\mathcal{K}_\infty;L^p)$, and the analogous result holds for $\delta_{2n}(\mathcal{K}_\infty;L^p)$.

The proof of Theorem 5.2 utilizes certain technical details. We see little point in entering into an analysis of these details. Instead, we provide a sketch thereof.

Our problem is one of zero counting. As defined in this paper, ϕ is CVD_{2n} which implies that $S_c^-(f)\leqq S_c^-(h)$ if $S_c^-(h)\leqq 2n$ and $f(x)=\int_0^{2\pi}\phi(x-y)h(y)dy$. Let f be a continuous periodic function of period 2π. Let $S_c^+(f)$ denote the number of zeros of f where sign changes are counted once and zeros which are not sign changes are counted twice (as if they were double zeros). If ϕ is CVD_{2n} and ϕ is SSC_{2l+1}, $l=0,1,\cdots,n$ (see Definition 2.2), then $S_c^+(f)\leqq S_c^-(h)$, if $S_c^-(h)\leqq 2n$, where f and h are related as above. It is this latter property which we wish to hold for ϕ, i.e., that $S_c^+(f)\leqq S_c^-(h)$ provided $S_c^-(h)\leqq 2n$. We shall assume henceforth that ϕ has the above property by uniformly approximating our kernel ϕ by kernels with the desired property. One method of approximation is by smoothing and it is the details of this process which we omit. For kernels of the above form we shall prove the following result.

Theorem 5.3. *Assume that* $\xi = (\xi_1, \cdots, \xi_{2n})$ *with* $0 \leqq \xi_1 < \xi_2 \cdots < \xi_{2n} < 2\pi$ *satisfies*

(5.4)
$$\int_0^{2\pi} \left| \int_0^{2\pi} \phi(x-y)h_\xi(y)dy \right|^{p-1}$$
$$\operatorname{sgn}\left(\int_0^{2\pi} \phi(x-y)h_\xi(y)dy\right)\phi(x-\xi_i)dx = 0, \qquad i = 1,\cdots,2n$$

for some $p \in [1,\infty)$. *Then* $\xi_{i+1} - \xi_i = \pi/n$, $i = 1,\cdots,2n$ (*where* $\xi_{2n+1} = \xi_1 + 2\pi$).

Theorem 5.3 states that on the basis of (5.2), the unique solution of (5.1) (up to translation) is the vector ξ with equally spaced knots. If ϕ is simply CVD_{2n} and does not possess the additional properties assumed above, then by approximating ϕ, we obtain the result, without uniqueness. Thus Theorem 5.2 is proven. Our proof of Theorems 5.2 and 5.3 follows the method of approach of Zensykbaev [15].

Proof of Theorem 5.3. Due to the translation invariance property of $\mathcal{K}_\infty$, we may assume that the ξ in the interior of Ξ_{2n} which satisfies (5.4) is such that $\xi_1 = 0 < \xi_2 < \cdots < \xi_{2n} < 2\pi$, and $\delta = \xi_2 - \xi_1 = \min_{i=1,\cdots,2n} \xi_{i+1} - \xi_i$, where $\xi_{2n+1} = 2\pi$.

Let $f(x) = \int_0^{2\pi} \phi(x-y)h_\xi(y)dy$, and $F(y) = \int_0^{2\pi} |f(x)|^{p-1}\operatorname{sgn}(f(x))\,\phi(x-y)dx$. Thus from (5.4), $F(\xi_i) = 0$, $i = 1,\cdots,2n$ and it is in fact (see the proof of Theorem 5.1) easily seen that these $2n$ zeros of F are sign changes and F has no additional zeros.

Let $g(x) = \int_0^{2\pi} \phi(x-y)h_\xi(y+\delta)dy$. From Lemma 3.1, $S_c^-(h_\xi(y) + h_\xi(y+\delta)) \leqq 2(n-1)$. Thus $S_c^-(f(x)+g(x)) \leqq 2(n-1)$. Since $\operatorname{sgn}(a+b) = \operatorname{sgn}(|a|^{p-1}\operatorname{sgn} a + |b|^{p-1}\operatorname{sgn} b)$ for a, b real and $1 < p < \infty$, it therefore follows that $S_c^-(|f(x)|^{p-1}\operatorname{sgn} f(x) + |g(x)|^{p-1}\operatorname{sgn} g(x)) \leqq 2(n-1)$ for any $1 \leqq p < \infty$. Thus, $S_c^+(F(y)+G(y)) \leqq 2(n-1)$, where $G(y) = \int_0^{2\pi}|g(x)|^{p-1}\operatorname{sgn}(g(x))\phi(x-y)dx$. A simple change of variable argument shows that $G(y) = F(y+\delta)$ whence $S_c^+(F(y)+F(y+\delta)) \leqq 2(n-1)$.

Now $F(y+\delta)$ vanishes at $\xi_{i+1} - \xi_2$, $i = 1,\cdots,2n$, and $\xi_i \leqq \xi_{i+1} - \xi_2 \leqq \xi_{i+1}$. Since $(-1)^i \varepsilon F(y) > 0$, $\xi_i < y < \xi_{i+1}$, $i = 1,\cdots,2n$, where $\xi_{2n+1} = 2\pi$ and $\varepsilon = \pm 1$, fixed, therefore $(-1)^i\varepsilon(F(\xi_i) + F(\xi_i + \delta)) \geqq 0$, $i = 1,\cdots,2n$. Hence $S_c^1(F(y)+F(y+\delta)) \geqq 2n$, a contradiction unless $F(y) \equiv -F(y+\delta)$. The theorem follows.

§6. n-widths of $\mathcal{K}_p$ in L^1

In this section we concern ourselves with

$$\mathscr{K}_p = \left\{f: f(x) = \int_0^{2\pi} \phi(x-y)h(y)dy,\ \|h\|_p \leq 1\right\},$$

and especially $\mathscr{K}_1$ and its closure in $L^1[0,2\pi]$, $\bar{\mathscr{K}}_1 = \{f: f(x) = \int_0^{2\pi} \phi(x-y)d\mu(y), \|\mu\| \leq 1\}$, where $\|\mu\| =$ total variation of μ on $[0,2\pi)$. We shall compute the n-widths of $\mathscr{K}_p$, $1 \leq p \leq \infty$ and $\bar{\mathscr{K}}_1$ as subsets of $L^1[0,2\pi]$.

Our method of proof is intimately connected with the duality between L^p and L^q, for $1/p + 1/q = 1$ and the effective computation of what is referred to as the n-width, in the sense of Gel'fand, of $\mathscr{K}_\infty$ as a subset of $L^q[0,2\pi]$, $1 \leq q \leq \infty$.

Before stating the result, we recall from Section 3 that $f_n^*(x) = \int_0^{2\pi} \phi(x-y)h_n^*(y)dy$, where $h_n^*(y)$ is the step function with $2n$ changes of sign from -1 to 1 at equally spaced points.

Theorem 6.1. *With the above notation and under the assumptions of Section* 3,

(I) $d_j(\mathscr{K}_1; L^1) = d_j(\bar{\mathscr{K}}_1; L^1) = \delta_j(\mathscr{K}_1; L^1) = \delta_j(\bar{\mathscr{K}}_1; L^1) = \|f_n^*\|_\infty$ *for* $j = 2n-1, 2n$.

(II) $d_{2n}(\mathscr{K}_p; L^1) = \delta_{2n}(\mathscr{K}_p; L^1) = \|f_n^*\|_q$, *for* $1 \leq p \leq \infty$, *where* $1/p + 1/q = 1$.

Furthermore,

(A) T_{n-1} *is an optimal subspace for* $d_{2n-1}(\mathscr{K}_1; L^1)$ (*and* $d_{2n-1}(\bar{\mathscr{K}}_1; L^1)$) *and* $P_{2n-1}h(x) = \int_0^{2\pi} t_{n-1}^*(x-y)h(y)dy$ *is an optimal linear map for* $\delta_{2n-1}(\mathscr{K}_1; L^1)$, *where* $t_{n-1}^*(x)$ *is the best* L^1*-approximation to* $\phi(x)$ *from* T_{n-1}.

(B) $S_{2n}^* = \text{span}\ \{\phi(x - \pi i/n)\}_{i=1}^{2n}$ *is an optimal subspace for* $d_{2n}(\mathscr{K}_p; L^1)$ *and interpolation from* S_{2n}^* *to* $f \in \mathscr{K}_p$ *at* $\beta + \pi i/n$, $i = 1, \cdots, 2n$, *the zeros of* $f_n^*(x)$, *is an optimal linear map for* $\delta_{2n}(\mathscr{K}_p; L^1)$.

Proof. The proof is an application of the extremal properties of $f_n^*(x)$ which were proven in Sections 3 and 5.

For $h \in L^1[0,2\pi]$, let $P_{2n-1}h(x) = \int_0^{2\pi} t_{n-1}^*(x-y)h(y)dy$. Thus,

$$\delta_{2n-1}(\mathscr{K}_1; L^1) \leq \sup_{\|h\|_1 \leq 1} \left\| \int_0^{2\pi} \phi(x-y)h(y)dy - \int_0^{2\pi} t_{n-1}^*(x-y)h(y)dy \right\|_1$$

$$= \sup_{\|h\|_1 \leq 1} \left\| \int_0^{2\pi} [\phi(x-y) - t_{n-1}^*(x-y)]h(y)dy \right\|_1$$

$$= \sup_{\|h\|_1 \leq 1} \sup_{\|g\|_\infty \leq 1} \int_0^{2\pi} \int_0^{2\pi} [\phi(x-y) - t_{n-1}^*(x-y)]h(y)g(x)dydx$$

$$\leq \sup_{\|g\|_\infty \leq 1} \left\| \int_0^{2\pi} [\phi(x-y) - t_{n-1}^*(x-y)]g(x)dx \right\|_\infty = \|f_n^*\|_\infty.$$

The last inequality follows from Section 3.

Similarly $\delta_{2n-1}(\tilde{\mathcal{K}}_1; L^1) \leqq \|f_n^*\|_\infty$.

We also wish to show that the interpolation as defined in (B) is also optimal. Let Q_{2n} denote the linear map from $L^p[0,2\pi]$ to S_{2n}^* which interpolates to $\int_0^{2\pi} \phi(x-y)h(y)dy$ at $\beta + \pi i/n$, $i = 1,\cdots,2n$. Thus,

$$\sup_{\|h\|_p \leqq 1} \left\| \int_0^{2\pi} \phi(x-y)h(y)dy - Q_{2n}h \right\|_1$$

$$= \sup_{\|h\|_p \leqq 1} \left\| \int_0^{2\pi} \frac{\phi\begin{pmatrix} x, \beta+\frac{\pi}{n}, \beta+\frac{2\pi}{n}, \cdots, \beta+2\pi \\ y, \quad \frac{\pi}{n}, \quad \frac{2\pi}{\pi}, \cdots, \quad 2\pi \end{pmatrix}}{\phi\begin{pmatrix} \beta+\frac{\pi}{n}, \beta+\frac{2\pi}{n}, \cdots, \beta+2\pi \\ \frac{\pi}{\nu}, \quad \frac{2\pi}{n}, \cdots, \quad 2\pi \end{pmatrix}} h(y)dy \right\|_1$$

$$= \sup_{\|h\|_p \leqq 1} \sup_{\|g\|_\infty \leqq 1} \int_0^{2\pi}\int_0^{2\pi} \frac{\phi\begin{pmatrix} x, \beta+\frac{\pi}{n}, \beta+\frac{2\pi}{n}, \cdots, \beta+2\pi \\ y, \quad \frac{\pi}{n}, \quad \frac{2\pi}{n}, \cdots, \quad 2\pi \end{pmatrix}}{\phi\begin{pmatrix} \beta+\frac{\pi}{n}, \beta+\frac{2\pi}{n}, \cdots, \beta+2\pi \\ \frac{\pi}{n}, \quad \frac{2\pi}{n}, \cdots, \quad 2\pi \end{pmatrix}} h(y)g(x)dydx$$

$$\leqq \sup_{\|g\|_\infty \leqq 1} \left\| \int_0^{2\pi} \frac{\phi\begin{pmatrix} x, \beta+\frac{\pi}{n}, \beta+\frac{2\pi}{n}, \cdots, \beta+2\pi \\ y, \quad \frac{\pi}{n}, \quad \frac{2\pi}{n}, \cdots, \quad 2\pi \end{pmatrix}}{\phi\begin{pmatrix} \beta+\frac{\pi}{n}, \beta+\frac{2\pi}{n}, \cdots, \beta+2\pi \\ \frac{\pi}{n}, \quad \frac{2\pi}{n}, \cdots, \quad 2\pi \end{pmatrix}} g(x)dx \right\|_q$$

where $1/p + 1/q = 1$. It is now a simple matter, as in Section 3, to prove that this quantity is bounded above by $\|f_n^*\|_q$.

Thus it remains to show that $\|f_n^*\|_q$ is a lower bound for $d_{2n}(\mathcal{K}_p; L^1)$. The proof is a fairly straightforward consequence of the extremal properties of $f_n^*(x)$ and the Hobby–Rice theorem. The Hobby–Rice theorem (which may be seen to be an application of the Borsuk Antipodensatz, see [10]) states that given any $2n$

functions $\{v_i\}_{i=1}^{2n}$ in $L^1[0,2\pi]$, there exists a $\xi \in \Xi_{2n}$ for which $(v_i, h_\xi) = 0$, $i = 1, \cdots, 2n$.

Let X_{2n} be any $2n$-dimensional subspace of $L^1[0,2\pi]$. By a standard duality theorem of best approximation

$$\sup_{\|h\|_p \leq 1} \inf_{g \in X_{2n}} \left\| \int_0^{2\pi} \phi(x-y)h(y)dy - g(x) \right\|_1$$

$$= \sup_{\|h\|_p \leq 1} \sup_{\substack{\|f\|_\infty \leq 1 \\ f \perp X_{2n}}} \int_0^{2\pi} \int_0^{2\pi} \phi(x-y)h(y)f(x)dydx$$

$$= \sup_{\substack{\|f\|_\infty \leq 1 \\ f \perp X_{2n}}} \left\| \int_0^{2\pi} \phi(x-y)f(x)dx \right\|_q.$$

By the Hobby–Rice theorem there exists a $\xi \in \Xi_{2n}$ for which $h_\xi \perp X_{2n}$. Thus,

$$\sup_{\|h\|_p \leq 1} \inf_{g \in X_{2n}} \left\| \int_0^{2\pi} \phi(x-y)h(y)dy - g(x) \right\|_1$$

$$\geqq \min_{\xi \in \Xi_{2n}} \left\| \int_0^{2\pi} \phi(x-y)h_\xi(x)dx \right\|_q$$

$$= \|f_n^*\|_q.$$

This last equality is a consequence of the extremal properties of f_n^*. Since the right hand side is independent of X_{2n}, $d_{2n}(\mathcal{K}_p; L^1) \geqq \|f_n^*\|_q$, and the theorem is proven.

Remark 6.1. Here and in Section 3 we have shown that for any ϕ within our class, T_{n-1} is an optimal $2n-1$ dimensional subspace for the $2n-1$ and $2n$ widths of $\mathcal{K}_\infty$ relative to L^∞, and $\mathcal{K}_1$ relative to L^1. This same fact is also true for the $2n-1$ and $2n$ widths of $\mathcal{K}_2$ relative to L^2. The proof of this fact may be found in Melkman and Micchelli [6] and is dependent upon the fact that T_{n-1} is also the span of the $2n-1$ eigenfunctions associated with the largest $2n-1$ eigenvalues of the positive semi-definite kernel $\phi^T\phi(x-y) = \int_0^{2\pi}\phi(z-x)\phi(z-y)dz$.

If, as in Section 2, $\phi(x)$ admits the Fourier series representation

$$\phi(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx},$$

then e^{inx} and e^{-inx} are eigenfunctions of $\phi^T\phi(x)$ with corresponding eigenvalues $4\pi^2|a_n|^2$, $n = 1, 2, \cdots$, and the constant function is an eigenfunction of multiplicity one with eigenvalue $4\pi^2 a_0^2$. It follows from Theorem 2.2 that T_{n-1} is the span of the $2n-1$ eigenfunctions with largest eigenvalues.

It is also proven in [6], that span $\{\phi(x-\pi k/n)\}_{k=1}^{2n}$ is an optimal subspace for $d_{2n}(\mathcal{K}_2; L^2)$, and $d_{2n-1}(\mathcal{K}_2; L^2) = d_{2n}(\mathcal{K}_2; L^2)$.

§7. Sobolev spaces

The preceding sections were concerned with the class of periodic functions of the form $f(x) = \int_0^{2\pi} \phi(x-y)h(y)dy$, where $\phi(x)$ is CVD_{2n}. An important and related class of functions is the Sobolev space

$$\tilde{W}_p^{(r)}[0,2\pi] = \tilde{W}_p^{(r)} = \{f\colon f^{(r-1)}\text{abs. cont.}, \|f^{(r)}\|_p < \infty, f \text{ periodic of period } 2\pi\}.$$

This class of periodic functions is related to our previous class $\mathcal{K}_p$ in that any $f \in \tilde{W}_p^{(r)}$ may be written in the form

$$f(x) = a_0 + \frac{1}{\pi}\int_0^{2\pi} \bar{B}_r(x-t) f^{(r)}(t)dt,$$

with the stipulation that $\|f^{(r)}\|_p < \infty$, and $\int_0^{2\pi} f^{(r)}(t)dt = 0$ (a_0 is an arbitrary constant). The function (kernel) $\bar{B}_r(x)$ is known both as the Bernouilli monospline and the Dirichlet kernel. One reason for these two names is that on the one hand

$$\bar{B}_r(x) = \sum_{k=1}^{\infty} \frac{\cos\left(kx - \frac{r\pi}{2}\right)}{k^r}, \qquad r = 1, 2, \cdots$$

while on the other hand it is not difficult to prove that $\bar{B}_r(x)$ is, for $x \in [0, 2\pi]$, a polynomial of degree r. Essentially (there is a problem of uniform convergence of the above infinite series if $r = 1$), if we set $\bar{B}_0(x) = -\frac{1}{2}$, then $\bar{B}_r'(x) = \bar{B}_{r-1}(x)$, $\bar{B}_r(x) = (-1)^r \bar{B}_r(2\pi - x)$, $r = 1, 2, \cdots$, and $\bar{B}_r(0) = 0$, $r = 3, 5, 7, \cdots$. Thus $\bar{B}_r(x)$ on $[0, 2\pi]$ is what is generally referred to as the Bernouilli polynomial $B_r(x)$, suitably normalized. $\bar{B}_r(x)$ is the 2π-periodic extension of $B_r(x)$ and is known as the Bernouilli monospline. $\bar{B}_r(x)$ is often denoted by $D_r(x)$.

The Bernouilli monospline $\bar{B}_r(x)$ is not CVD_{2n} for any n. In addition the class of function $\tilde{W}_p^{(r)}$ is not simply the image of the rth derivative under convolution by $\bar{B}_r(x)$. However, the methods and ideas of the previous sections may, to a great extent, be applied largely due to the following result.

Proposition 7.1. *Let* $0 \leqq t_1 < \cdots < t_k < 2\pi$, k *odd. Then* $\{1, \bar{B}_r(x-t_2) - \bar{B}_r(x-t_1), \bar{B}_r(x-t_3) - \bar{B}_r(x-t_2), \cdots, \bar{B}_r(x \dot{-} t_k) - \bar{B}_r(x-t_{k-1})\}$ *constitutes a weak Tchebycheff (WT) system of order k on* $[0, 2\pi)$.

Remark 7.1. The above proposition is equivalent to the fact that $A_0 + \sum_{i=1}^{k} C_i\bar{B}_r(x - t_i)$ exhibits at most $k-1$ sign changes on $[0, 2\pi]$ if $\sum_{i=1}^{k} C_i = 0$ and k is odd.

Proof. Let $f(x) = A_0 + \sum_{i=1}^{k} C_i\bar{B}_r(x - t_i)$, k odd, $\sum_{i=1}^{k} C_i = 0$. Assume that $f(x)$ has at least k sign changes on $[0, 2\pi]$. Since $f(x)$ is periodic of period 2π, $\bar{B}'_r(x - t_i) = \bar{B}_{r-1}(x - t_i)$, and $f \in C^{r-1}$, it follows that $\sum_{i=1}^{k} C_i\bar{B}_1(x - t_i)$ (or $A_0 + \sum_{i=1}^{k} C_i\bar{B}_1(x - t_i)$ in the case $r = 1$) has at least k sign changes on $[0, 2\pi]$. Now, $B_1(x) = (\pi - x)/2$ for $x \in [0, 2\pi]$. Thus, $\bar{B}_1(x - t_i) = (-\pi + t_i - x)/2 + \pi(x - t_i)_+^0$ on $[0, 2\pi]$, where $x_+^0 = 1$ if $x \geqq 0$, and 0 otherwise. Since $\sum_{i=1}^{k} C_i = 0$, we obtain for $r \geqq 2$

$$
\begin{aligned}
f^{(r-1)}(x) &= \sum_{i=1}^{k} C_i\bar{B}_1(x - t_i) \\
&= \sum_{i=1}^{k} C_i\left[\frac{-\pi + t_i - x}{2}\right] + \pi\sum_{i=1}^{k} C_i(x - t_i)_+^0 \\
&= \frac{1}{2}\sum_{i=1}^{k} C_i t_i + \pi\sum_{i=1}^{k} C_i(x - t_i)_+^0.
\end{aligned}
$$

Since $f^{(r-1)}(x) = \frac{1}{2}\sum_{i=1}^{k} C_i t_i$ for $x \in [0, t_1)$ and $(t_k, 2\pi]$ (i.e., a sign change does not occur at $0 \equiv 2\pi$) it is necessary, in order that $f^{(r-1)}(x)$ possess k sign changes, that $f^{(r-1)}(x)$ change sign at each t_i, $i = 1, \cdots, k$. However k is odd and $f^{(r-1)}(x)$ has the same sign on $[0, t_1)$ and $(t_k, 2\pi]$. Thus $f^{(r-1)}$ cannot have k sign changes on $[0, 2\pi]$. This contradiction proves the proposition. If $r = 1$ then the same analysis holds.

Let $f_n^*(x) = (1/\pi)\int_0^{2\pi} \bar{B}_r(x - t)h_n^*(t)dt$, where $h_n^*(t)$ is as defined in Section 3, i.e., $h_n^*(t) = (-1)^i$, for $t \in [i\pi/n, (i+1)\pi/n)$, $i = 0, 1, \cdots, 2n - 1$. Note that $\int_0^{2\pi} h_n^*(t)dt = 0$. Let $\tilde{B}_p^{(r)}$ denote the unit ball in $\tilde{W}_p^{(r)}$, i.e., $\tilde{B}_p^{(r)} = \{f : f \in \tilde{W}_p^{(r)}, \|f^{(r)}\|_p \leqq 1\}$. It is well-known that

$$
\|f_n^*\|_\infty = K_r = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu(r+1)}}{(2\nu + 1)^{r+1}}.
$$

We are interested in proving the following theorems.

Theorem 7.1. *With the above assumptions and notation*

$$
d_{2n-1}(\tilde{B}_\infty^{(r)}; L^\infty) = d_{2n}(\tilde{B}_\infty^{(r)}; L^\infty) = \delta_{2n-1}(\tilde{B}_\infty^{(r)}; L^\infty) = \delta_{2n}(B_\infty^{(r)}; L^\infty) = \|f_n^*\|_\infty.
$$

Furthermore,

(I) *T_{n-1} is an optimal subspace for $d_{2n-1}(\tilde{B}_\infty^{(r)}; L^\infty)$, and*

$$P_{2n-1}f(x)=\frac{1}{2\pi}\int_0^{2\pi} f(t)dt+\frac{1}{\pi}\int_0^{2\pi} t_{n-1}^*(x-t)f^{(r)}(t)dt$$

is an optimal linear map for $\delta_{2n-1}(\tilde{B}_\infty^{(r)}; L^\infty)$, *where* $t_{n-1}^*(x)$ *is the best* L^1-*approximation to* $B_r(x)$ *on* $[0,2\pi]$.

(II) $S_{2n}^*=\{A+\sum_{i=1}^{2n} C_i\bar{B}_r(x-\pi i/n)\colon \sum_{i=1}^{2n} C_i=0\}$ *is a* $2n$-*dimensional optimal subspace for* $d_{2n}(\tilde{B}_\infty^{(r)}; L^\infty)$, *and interpolation to* $f\in\tilde{B}_\infty^{(r)}$ *at the zeros of* f_n^* *from* S_{2n}^* *is an optimal linear map.*

Theorem 7.2. *With the above assumptions and notation*

$$d_{2n}(\tilde{B}_\infty^{(r)}; L^p)=\delta_{2n}(\tilde{B}_\infty^{(r)}; L^p)=\|f_n^*\|_p$$

and part (II) *of Theorem* 7.1 *applies,* $1\leqq p<\infty$.

Theorem 7.3. *With the above assumptions and notation,*

$$d_{2n-1}(\tilde{B}_1^{(r)}; L^1)=d_{2n}(\tilde{B}_1^{(r)}; L^1)=\delta_{2n-1}(\tilde{B}_1^{(r)}; L^1)=\delta_{2n}(\tilde{B}_1^{(r)}; L^1)=\|f_n^*\|_\infty$$

and parts (I) *and* (II) *of Theorem* 7.1 *hold with the obvious changes.*

Theorem 7.4. *With the above assumptions and notation*

$$d_{2n-1}(\tilde{B}_p^{(r)}; L^1)=d_{2n}(\tilde{B}_p^{(r)}; L^1)=\delta_{2n}(\tilde{B}_p^{(r)}; L^1)=\|f_n^*\|_q$$

for $1\leqq p\leqq\infty$, *where* $1/p+1/q=1$. *Furthermore*

(I) T_{n-1} *is an optimal subspace for* $d_{2n-1}(\tilde{B}_p^{(r)}; L^1)$.

(II) *Part* (II) *of Theorem* 7.1 *obtains with the obvious changes.*

Remark 7.2. The following results were known. Tichomirov [14] had proved that

$$d_{2n-1}(\tilde{B}_\infty^{(r)}; \tilde{C})=d_{2n}(\tilde{B}_\infty^{(r)}; \tilde{C})=\delta_{2n-1}(\tilde{B}_\infty^{(r)}; \tilde{C})=\delta_{2n}(\tilde{B}_\infty^{(r)}; \tilde{C})=\|f_n^*\|_\infty,$$

where $\tilde{C}$ represents the space of 2π-periodic continuous functions on $[0,2\pi)$. This condition, as was noted earlier, is unnecessary. The result $d_{2n-1}(\tilde{B}_1^{(r)}; L^1)=\|f_n^*\|_\infty$ had been proven independently by Makovoz [4] and Subbotin [12]. The result $d_{2n-1}(\tilde{B}_\infty^{(r)}; L^1)=\|f_n^*\|_1$ may be found in Makovoz [5].

Proof of Theorems 7.1–7.4. All the results of Theorems 7.1–7.4 except part (I) of Theorem 7.4 are analogues of results of Sections 3, 5, and 6. Part (I) of

Theorem 7.4 depends on the fact, due to Taikov [13], that $\|f_n^*\|_q$ represents an upper bound on the degree of approximation of $\tilde{B}_p^{(r)}$ by trigonometric polynomials of degree $\leq n-1$, i.e., of $t_{n-1} \in T_{n-1}$, in the L^1 norm.

A proof of the remaining upper bounds follows much the same pattern as the proof found in Tichomirov [14]. The method of proof given in Section 3 may also be modified in order to obtain the upper bounds.

To prove the optimality of the subspace T_{n-1} in Theorems 7.1 and 7.3 one should note that the orthogonality condition $\int_0^{2\pi} f^{(r)}(t)dt = 0$ may be dropped. The proof of $S_c^-(\bar{B}_r - t_{n-1}) \leq 2n$ for any $t_{n-1} \in T_{n-1}$ is a standard Rolle's theorem argument.

The major innovation here is the lower bound argument.

Let $\tilde{\Xi}_{2n} = \{\xi\colon \xi = (\xi_1, \cdots, \xi_{2n}),\quad \xi_0 = 0 \leq \xi_1 \leq \cdots \leq \xi_{2n} \leq \xi_{2n+1} = 2\pi,$ $\sum_{i=0}^{2n}(-1)^i(\xi_{i+1} - \xi_i) = 0\}$, and let $h_\xi(x)$ be defined, for each $\xi \in \tilde{\Xi}_{2n}$, as in Section 3, i.e., $h_\xi(x) = (-1)^i$, $\xi_i \leq x < \xi_{i+1}$, $i = 0, 1, \cdots, 2n$. The condition $\sum_{i=0}^{2n}(-1)^i(\xi_{i+1} - \xi_i) = 0$ is equivalent to $\int_0^{2\pi} h_\xi(x)dx = 0$. The lower bound argument depends upon the following analogue of Lemma 3.3 and Theorem 5.2.

Proposition 7.2. *For each $\xi \in \tilde{\Xi}_{2n}$, set*

$$f_\xi(x) = \frac{1}{\pi}\int_0^{2\pi} \bar{B}_r(x-t)h_\xi(t)dt.$$

Then for each $p \in [1, \infty]$,

$$\min_{a \in \mathbf{R},\, \xi \in \tilde{\Xi}_{2n}} \|a + f_\xi\|_p = \|f_n^*\|_p.$$

The proof follows the same lines as the proof of Lemma 3.3 and Theorem 5.2. The details are somewhat more tedious.

Let X_{2n} be any $2n$-dimensional subspace of $L^p[0, 2\pi]$, $1 < p < \infty$. In order that

$$\sup_{f \in \tilde{B}_\infty^{(r)}} \inf_{g \in X_{2n}} \|f - g\|_p < \infty,$$

it is necessary that X_{2n} contains the constant function. Thus we may define a basis for X_{2n}, $X_{2n} = \operatorname{span}\{g_0, g_1, \cdots, g_{2n-1}\}$, where $g_0(x) = 1$.

Let $S_{2n} = \{t\colon t = (t_1, \cdots, t_{2n+1})\ \sum_{i=1}^{2n+1} t_i^2 = 2\pi\}$, and for each $t \in S_{2n}$, define $\xi(t) \in \Xi_{2n}$ by $\xi_0(t) = 0$, $\xi_i(t) = \sum_{j=1}^{i} t_j^2$, $j = 1, \cdots, 2n+1$. Thus, $0 = \xi_0(t) \leq \xi_1(t) \leq \cdots \leq \xi_{2n}(t) \leq \xi_{2n+1}(t) = 2\pi$. Define $h_{\xi(t)}(x) = \operatorname{sgn} t_i$, $\xi_{i-1}(t) \leq x < \xi_i(t)$, $i = 1, 2, \cdots, 2n+1$. Now, let $\tilde{S}_{2n}$ be the subset of S_{2n} determined by $t \in S_{2n}$ for which $\int_0^{2\pi} h_{\xi(t)}(x)dx = 0$, i.e.,

$$\tilde{S}_{2n} = \left\{t\colon t \in S_{2n},\ \int_0^{2\pi} h_{\xi(t)}(x)dx = 0\right\}.$$

$\tilde{S}_{2n}$ may be represented as the boundary of a bounded, open, symmetric neighborhood of 0 in $\mathbf{R}^{2n}$. Let A be the mapping of $\tilde{S}_{2n}$ to $(a_1^p(t), \cdots, a_{2n-1}^p(t))$, where $a_i^p(t)$ is the coefficient of $g_i(x)$, $i = 1, \cdots, 2n-1$, in the best L^p-approximation to $(1/\pi)\int_0^{2\pi} \bar{B}_r(x-t)h_{\xi(t)}(t)dt$ from X_{2n}. Since $1 < p < \infty$, this mapping is well-defined, continuous, and odd. Thus by the Borsuk Antipodensatz there exists a $t^* \in \tilde{S}_{2n}$ for which $a_i(t^*) = 0$, $i = 1, \cdots, 2n-1$, i.e., the best approximation to $-a_0^p(t^*) + (1/\pi)\int_0^{2\pi} \bar{B}_r(x-t)h_{\xi(t^*)}(t)dt$ from X_{2n} in L^p is the zero function. For $p = 1$ and $p = \infty$, we take limits as in Section 3. Thus the lower bound is proven for Theorems 7.1 and 7.2.

To obtain the lower bounds in Theorems 7.3 and 7.4, we utilize the duality between L^p and L^q, $1/p + 1/q = 1$. Let X_{2n} be any $2n$-dimensional subspace of $L^1[0, 2\pi]$. In order that

$$\sup_{f \in \tilde{B}_p^{(r)}} \inf_{g \in X_{2n}} \|f - g\|_1 < \infty,$$

it is necessary that the constant function be in X_{2n}. Thus we may again assume that $X_{2n} = \text{span}\,\{1, g_1, g_2, \cdots, g_{2n-1}\}$. Now

$$\sup_{\substack{\|h\|_p \leq 1 \\ \int_0^{2\pi} h(t)dt = 0}} \inf_{g \in X_{2n}} \left\| \frac{1}{\pi} \int_0^{2\pi} \bar{B}_r(x-t)h(t)dt - g(x) \right\|_1$$

$$= \sup_{\substack{\|h\|_p \leq 1 \\ \int_0^{2\pi} h(t)dt = 0}} \sup_{\substack{\|f\|_\infty \leq 1 \\ f \perp X_{2n}}} \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \bar{B}_r(x-t)h(t)f(x)dtdx$$

$$= \sup_{\substack{\|f\|_\infty \leq 1 \\ f \perp X_{2n}}} \inf_{a \in \mathbf{R}} \left\| \frac{1}{\pi} \int_0^{2\pi} \bar{B}_r(x-t)f(x)dx - a \right\|_q.$$

By the Hobby–Rice theorem, there exists a $\xi \in \Xi_{2n}$ such that $h_\xi \perp X_{2n}$. This implies that $\int_0^{2\pi} h_\xi(t)dt = 0$, i.e, $\xi \in \tilde{\Xi}_{2n}$. Therefore

$$\sup_{\substack{\|h\|_p \leq 1 \\ \int_0^{2\pi} h(t)dt = 0}} \inf_{g \in X_{2n}} \left\| \frac{1}{\pi} \int_0^{2\pi} \bar{B}_r(x-t)h(t)dt - g(x) \right\|_1$$

$$\geq \inf_{\xi \in \tilde{\Xi}_{2n}} \inf_{a \in \mathbf{R}} \left\| \frac{1}{\pi} \int_0^{2\pi} \bar{B}_r(x-t)h_\xi(t)dt - a \right\|_q$$

$$= \|f_n^*\|_q.$$

Thus $d_{2n}(\tilde{B}_p^{(r)}; L^1) \geq \|f_n^*\|_q$. The theorem is proven.

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(Received September 10, 1978)