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On Zero-Preserving Linear Transformations

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For an arbitrary subset I of $\mathbb{R}$ and for a function f defined on I, the number of zeros of f on I will be denoted by

$$Z_I(f).$$

In this paper we attempt to characterize all linear transformations T taking a linear subspace W of $C(I)$ into functions defined on J ($I, J \subseteq \mathbb{R}$) such that

$$Z_I(f) = Z_J(Tf)$$

for all $f \in W$.

1. INTRODUCTION

Let I and J be arbitrary subsets of $\mathbb{R}$. Let $F(J)$ denote the space of (real-valued) functions defined on J, and let W be a linear subspace of $C(I)$. In this paper we consider linear transformations T from W into $F(J)$ for which

$$Z_I(f) = Z_J(Tf) \tag{1.1}$$

for all $f \in W$, and we attempt to completely characterize such transformations. When we can say something positive, it transpires that T always has

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the specific form

$$(Tf)(y) = q(y)f(r(y)), \tag{1.2}$$

for all $y \in J$, where $q \in F(J)$ does not vanish on J and r is a 1–1 map from J onto I. That these conditions define a linear transformation T satisfying (1.1) is readily checked. It is the converse claim which we will work to verify.

We consider four main cases. If $W = C(I)$ we prove (1.2) if I is compact or if I is an interval (we conjecture that for $W = C(I)$ Eq. (1.2) should hold independent of I). When $W = \Pi$ (the space of all algebraic polynomials, restricted to I), we prove that (1.2) holds for arbitrary $I \subseteq \mathbb{R}$. If $W = \pi_N$ (the space of algebraic polynomials of degree at most N), we show that (1.2) always holds for arbitrary I if N is even. However, for N odd we have only been able to prove this result if I is bounded above or below or if we impose certain additional constraints. (Again we conjecture that this result should always be valid.) We also consider certain subspaces W which are somewhat different (containing complete Chebyshev systems of dimension 3) and look for a mechanism which enables us to verify (1.2) thereon.

Before proving these results, we point out that it is not true that for every $I, J \subseteq \mathbb{R}$, subspace W of $C(I)$, and T satisfying (1.1) we must have T of the form (1.2) for some q and r. As a simple example consider $I = J = (0, 1)$ and the space

$$W = \operatorname{span}\{1, x, (1/x)\sin(1/x)\}.$$

Note that

$$Z_{(0,1)}((1/x)\sin(1/x) + ax + b) = \infty,$$

for every choice of $a, b \in \mathbb{R}$. Let $h \in C(0, 1)$ have this same property; i.e.,

$$Z_{(0,1)}(h(y) + ay + b) = \infty,$$

for every choice of $a, b \in \mathbb{R}$. Then the linear operator T given by $T1 = 1$, $Tx = y$, and

$$T((1/x)\sin(1/x))(y) = h(y)$$

satisfies (1.1), but is not of the desired form. Nevertheless, if $T1 = 1$, $Tx = y$, and

$$Z_{(0,1)}(f) = Z_{(0,1)}(Tf),$$

for all $f \in C(0, 1)$, then necessarily $Tf = f$.

The analytic problem we consider in this paper is related to the following geometric problem: Consider two curves A and B in the plane, and assume that the number of intersections of each straight line with A and B is the same. Does this imply that $A = B$? If A and B are the graphs of two continuous function f and g, with a common domain of definition I, we can reformulate this problem as follows: Assume $f, g \in C(I)$ and

$$Z_I(f(x) - (ax + b)) = Z_I(g(x) - (ax + b)),$$

for all $a, b \in \mathbb{R}$. Does this imply that $f = g$? Defining the linear transformation T by $T1 = 1$, $Tx = x$, and $Tf = g$ brings us back to our setting. (If (1.2) holds then $f = g$.) For this geometric problem, the values $Z_I(f(x) - (ax + b))$ define what is known as the Crofton transform of f. For more on this subject, see Fast [2], Richardson [7], Horwitz [3], and the references therein.

We, however, were led to a consideration of this problem from a totally different perspective. We were interested in the question of characterizing linear transformations T: $\pi_n \to \pi_n$, all n, for which

$$Z_I(p) \le Z_J(Tp)$$

for all $p \in \Pi$. This problem has a long history; see Carnicer *et al.* [1]. The "equality" case seemed simple. But, as so often happens in mathematical research, one thing led to another and the result is this paper.

2. SOME GENERAL RESULTS

In this paper we use three types of convexity. As our domain of definition I need not be connected or compact; we formally define them so as to prevent any misunderstanding.

DEFINITION 2.1. Let $f \in C(I)$, where I is an arbitrary subset of $\mathbb{R}$. We say that f is *convex* on I if for each $\alpha \in I$ there exist constants $a(\alpha), b(\alpha)$ such that

$$f(x) - (a(\alpha)x + b(\alpha)) \ge 0, \tag{2.1}$$

for all $x \in I$, and in addition

$$f(\alpha) - (a(\alpha)\alpha + b(\alpha)) = 0. \tag{2.2}$$

f is said to be *strictly convex* on I if it is convex thereon and the equality in (2.1) only holds for $x = \alpha$. We say that f is *uniformly convex* on I if f is strictly convex and if for each $\alpha \in I$ and $\delta > 0$ there exists a constant $C(\delta, \alpha) > 0$ such that

$$f(x) - (a(\alpha)x + b(\alpha)) \ge C(\delta, \alpha) \tag{2.3}$$

for all $x \in I$ for which $|x - \alpha| \ge \delta$.

For example, x^2 is a strictly and uniformly convex function on every $I \subseteq \mathbb{R}$.

Our first result concerns the form of T when restricted to convex functions if W contains the functions 1 and x.

PROPOSITION 2.1. *Let $I, J \subseteq \mathbb{R}$, and assume that W contains the functions 1, x and a function g strictly convex on I. Assume T is a linear transformation from W to $F(J)$ satisfying*

$$Z_I(f) = Z_J(Tf)$$

for all $f \in W$. Then there exists a $q \in F(J)$ which does not vanish on J and an $r \in F(J)$ which is a bijection (1–1 *onto map*) *from J to I, such that*

$$(T1)(y) = q(y), \qquad (Tx)(y) = q(y)r(y),$$

and

$$(Tf)(y) = q(y)f(r(y)),$$

for every convex $f \in W$.

Proof. Set $q(y) = (T1)(y)$. Since $Z_I(1) = 0$ we have $Z_J(q) = 0$. Thus $q \in F(J)$ and q does not vanish on J.

Set $(Tx)(y) = q(y)r(y)$; i.e., let

$$r(y) = \frac{(Tx)(y)}{q(y)}.$$

Since q does not vanish, the function r is well-defined. By assumption,

$$Z_J(r(y) - c) = Z_J(q(y)r(y) - cq(y)) = Z_I(x - c) = \begin{cases} 1, & c \in I, \\ 0, & c \notin I. \end{cases}$$

Thus r is also 1–1 and the range of r is all of I.

It remains to prove that

$$(Tf)(y) = q(y)f(r(y)),$$

for every convex $f \in W$. Since T is linear, it suffices to prove this fact for strictly convex f. (We first prove this result for the strictly convex g and then for $f + g$, which is also strictly convex for any convex f.)

For each $\alpha \in I$, set

$$h(x; \alpha) = f(x) - (a(\alpha)x + b(\alpha)),$$

where $a(\alpha), b(\alpha)$ are as in Definition 2.1. Let

$$p(y; \alpha) = [Th(x; \alpha)](y).$$

Thus

$$Z_J(p(y;\alpha)) = Z_I(h(x;\alpha)) = 1, \tag{2.4}$$

for all $\alpha \in I$. Furthermore, from (2.1) we have that for all $c > 0$,

$$0 = Z_I(h(x;\alpha)+c) = Z_J(p(y;\alpha)+cq(y)) = Z_J\left(\frac{p(y;\alpha)}{q(y)}+c\right).$$

Thus $p(y;\alpha)/q(y)+c$ has no zero in J for all $c > 0$, which implies that

$$\frac{p(y;\alpha)}{q(y)} \geq 0$$

for all $y \in J$.

Set

$$v(y) = (Tf)(y).$$

Now

$$\begin{aligned} p(y;\alpha) &= [Th(x;\alpha)](y) \\ &= T(f(x)-(a(\alpha)x+b(\alpha)))(y) \\ &= v(y)-(a(\alpha)q(y)r(y)+b(\alpha)q(y)) \\ &= v(y)-q(y)f(r(y))+q(y)[f(r(y))-(a(\alpha)r(y)+b(\alpha))]. \end{aligned}$$

Thus

$$0 \leq \frac{p(y;\alpha)}{q(y)} = \left[\frac{v(y)}{q(y)}-f(r(y))\right]+h(r(y);\alpha). \tag{2.5}$$

For $y \in J$, let $\alpha = r(y)$. Substituting in the above it follows from (2.2) that

$$\frac{v(y)}{q(y)}-f(r(y)) \geq 0, \tag{2.6}$$

for all $y \in J$.

For each $\alpha \in I$ there exists a $\tau(\alpha) \in J$ such that

$$p(\tau(\alpha);\alpha) = 0.$$

This follows from (2.4). Substituting in (2.5) we have from (2.6) and (2.1) that

$$\frac{p(\tau(\alpha))}{q(\tau(\alpha))}-f(r(\tau(\alpha))) = 0$$

and

$$h(r(\tau(\alpha));\alpha) = 0. \tag{2.7}$$

Now $r(\tau(\alpha)) \in I$ and we therefore know that (2.7) implies that $r(\tau(\alpha)) = \alpha$ since f is strictly convex (see (2.2) and the definition of strict convexity). As r is a 1–1 map from J onto I, it thus follows that as α ranges over all of I, $\tau(\alpha)$ ranges over all of J. Thus for each $y \in J$,

$$\frac{v(y)}{q(y)} - f(r(y)) = 0;$$

i.e.,

$$(Tf)(y) = v(y) = q(y)f(r(y)).$$

∎

We can use the linearity of T and the fact that it is of the desired form on convex functions to prove the following result.

THEOREM 2.2. *Let $I, J \subseteq \mathbb{R}$, and let W be a linear subspace of $C(I)$ containing $1, x$, and a uniformly convex g. Assume T is a linear transformation from W to $F(J)$ satisfying*

$$Z_I(f) = Z_J(Tf)$$

for all $f \in W$. There exists a $q \in F(J)$ which does not vanish on J and an $r \in F(J)$ which is a bijection from J to I such that for any $f \in W$ which is also bounded on I we have

$$(Tf)(y) = q(y)f(r(y)).$$

Proof. The q and r are, of course, as given in Proposition 2.1; i.e., $T1 = q$ and $Tx = qr$. We use the method of proof of the well-known Bohman–Korovkin Theorem, which may be found in the book by Korovkin [5].

Fix $\alpha \in I$. Since f is continuous at α, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$|x - \alpha| < \delta,$$

then

$$|f(x) - f(\alpha)| < \varepsilon.$$

We assume f is bounded on W and let $M \geq |f(x)|$ for all $x \in I$.

Let $g \in W$ be uniformly convex and $C(\delta, \alpha)$ be as defined in (2.3) with respect to g. Set

$$p_u(x) = f(\alpha) + \varepsilon + \frac{2M(g(x) - (a(\alpha)x + b(\alpha)))}{C(\delta, \alpha)}$$

and

$$p_\ell(x) = f(\alpha) - \varepsilon - \frac{2M(g(x) - (a(\alpha)x + b(\alpha)))}{C(\delta, \alpha)}.$$

We claim that $p_u(x) > f(x) > p_\ell(x)$ for all $x \in I$. We prove only the first inequality. The proof of the second inequality is totally analogous. (In fact, we construct $-p_\ell$ with respect to $-f$ in the same way as we construct p_u with respect to f.)

If $x = \alpha$, then

$$p_u(\alpha) - f(\alpha) = \varepsilon > 0.$$

If $0 < |x - \alpha| < \delta$ then $|f(x) - f(\alpha)| < \varepsilon$, and

$$p_u(x) - f(x) > \frac{2M(g(x) - (a(\alpha)x + b(\alpha)))}{C(\delta, \alpha)} > 0.$$

If $|x - \alpha| \geq \delta$, then $|f(x) - f(\alpha)| \leq 2M$ and $(g(x) - (a(\alpha)x + b(\alpha))) \geq C(\delta, \alpha)$. Thus

$$p_u(x) - f(x) \geq \varepsilon - 2M + 2M > 0.$$

Since $p_u(x) > f(x)$ for all $x \in I$ we have

$$Z_I(p_u - f + c) = 0,$$

for all $c \geq 0$. Thus

$$Z_J(Tp_u - Tf + cq) = 0,$$

for all $c \geq 0$. As q does not vanish on J, this implies that

$$\frac{(Tp_u)(y) - (Tf)(y)}{q(y)} > 0,$$

for every $y \in J$. Similarly,

$$\frac{(Tf)(y) - (Tp_\ell)(y)}{q(y)} > 0,$$

for every $y \in J$. Now from Proposition 2.1,

$$(Tp_u)(y) = q(y)p_u(r(y)),$$
$$(Tp_\ell)(y) = q(y)p_\ell(r(y)).$$

Thus

$$p_u(r(y)) > \frac{(Tf)(y)}{q(y)} > p_\ell(r(y))$$

for all $y \in J$. As r is a 1–1 map from J onto I, there exists a $z \in J$ such that $r(z) = \alpha$. Set $y = z$ in the above to obtain

$$f(r(z)) + \varepsilon = p_u(r(z)) > \frac{(Tf)(z)}{q(z)} > p_\ell(r(z)) = f(r(z)) - \varepsilon.$$

Thus

$$\left| \frac{(Tf)(z)}{q(z)} - f(r(z)) \right| < \varepsilon.$$

As the choice of $\varepsilon > 0$ was arbitrary,

$$(Tf)(z) = q(z) f(r(z)).$$

Since r is a 1–1 map from J onto I, this equality is valid for every $z \in J$. ∎

Remark. Once we identify q and r, we can define

$$(Sf)(x) = \frac{(Tf)(r^{-1}(x))}{q(r^{-1}(x))}.$$

Note that for $f \in C(I)$ we have $(Tf)(y) = q(y)f(r(y))$ if and only if $(Sf)(x) = f(x)$. In Theorem 2.2, S is the identity on 1, x, and a uniformly convex function. In addition, a property which we very essentially used in the proof of Theorem 2.2 translates into the fact that S is a positive linear operator. This is why the method of proof of the Bohman–Korovkin Theorem applies. Nonetheless, it actually makes little difference if we work with T or S. We find it more convenient to work with T.

If I is compact, then every continuous function on I is bounded. In addition, on a compact set every strictly convex function is uniformly convex. We have, as a consequence of Theorem 2.2,

COROLLARY 2.3. *Let $I, J \subseteq \mathbb{R}$ and assume I is compact. Let W be any linear subspace of $C(I)$ containing* 1, x, *and a strictly convex function. Assume T is a linear transformation from W to $F(J)$ satisfying*

$$Z_I(f) = Z_J(Tf),$$

for all $f \in W$. There exists a $q \in F(J)$ which does not vanish on J and an $r \in F(J)$ which is a bijection from J to I such that for every $f \in W$,

$$(Tf)(y) = q(y)f(r(y)).$$

3. THE CASE $W = C(I)$

A special case of Corollary 2.3 is when $W = C(I)$. We formally state this.

COROLLARY 3.1. *Let $I, J \subseteq \mathbb{R}$ and assume I is compact. Assume T is a linear transformation from $C(I)$ to $F(J)$ satisfying*

$$Z_I(f) = Z_J(Tf)$$

for all $f \in C(I)$. There exists a $q \in F(J)$ which does not vanish on J and an $r \in F(J)$ which is a bijection from J to I such that for every $f \in C(I)$,

$$(Tf)(y) = q(y)f(r(y)).$$

What if I is not compact? Since T is linear, we have, by Proposition 2.1, the following result. (For n even, x^n is convex. For n odd, x^n can be written as the difference of two easily defined convex functions in $C(\mathbb{R})$ and thus in $C(I)$.)

COROLLARY 3.2. *Let $I, J \subseteq \mathbb{R}$. Assume T is a linear transformation from $C(I)$ to $F(J)$ satisfying*

$$Z_I(f) = Z_J(Tf),$$

for all $f \in C(I)$. There exist a $q \in F(J)$ which does not vanish on J and an $r \in F(J)$ which is a bijection from J to I such that

$$(Tx^n)(y) = q(y)r^n(y),$$

for all $n = 0, 1, \dots$.

The idea used in the proof of Theorem 2.2 can be applied to the case where $W = C(I)$ and I is any interval.

THEOREM 3.3. *Let I be an interval. Assume T is a linear transformation from $C(I)$ to $F(J)$ satisfying*

$$Z_I(f) = Z_J(Tf)$$

for all $f \in C(I)$. Then there exists a $q \in F(J)$ which does not vanish on J and an $r \in F(J)$ which is a bijection from J to I such that for every $f \in C(I)$ we have

$$(Tf)(y) = q(y)f(r(y)).$$

Proof. Our proof is based on the method of proof of Theorem 2.2. Given any $f \in C(I)$, $\alpha \in I$, and $\varepsilon > 0$, we will construct a convex $p_u \in C(I)$ and a concave $p_\ell \in C(I)$ such that

$$p_u(x) > f(x) > p_\ell(x),$$

for all $x \in I$, while

$$p_u(\alpha) = f(\alpha) + \varepsilon, \qquad p_\ell(\alpha) = f(\alpha) - \varepsilon.$$

This is exactly what is needed in the proof of Theorem 2.2. Previously, we had to take into consideration the demand that p_u and p_ℓ be elements of W. Here $W = C(I)$, so we have more freedom in our construction of p_u and p_ℓ.

We will explain how to construct p_u. The construction of p_ℓ is analogous. Let $a < b$ be the endpoints of I and choose sequences $\{a_n\}_1^\infty$, $\{b_n\}_1^\infty$ such that a_n strictly decreases to a while b_n strictly increases to b. Assume $a_1 < a_0 = \alpha = b_0 < b_1$. (If α is an endpoint we alter things very slightly.) We construct p_u as follows: p_u will be continuous; strictly decreasing for $x < \alpha$; strictly increasing for $x > \alpha$; linear on each (b_{m-1}, b_m) and (a_m, a_{m-1}); $m = 1, 2 \ldots$; convex; and will satisfy $p_u(x) > f(x)$ for all $x \in I$ and $p_u(\alpha) = f(\alpha) + \varepsilon$. This is in fact easily accomplished. Assume, for example, that p_u has already been defined on $[a_m, b_m]$ satisfying the above properties. Thus p_u is linear on $[b_{m-1}, b_m]$ with slope $c_m > 0$ and $p(b_m) > f(b_m)$. On $[b_m, b_{m+1}]$ the function f is continuous and thus bounded. We simply let p_u on $[b_m, b_{m+1}]$ be any linear function with slope $c_{m+1} > c_m$ (this gives the convexity) which is continuous at b_m and satisfies $p_u(x) > f(x)$ for all $x \in [b_m, b_{m+1}]$. A similar construction is done on $[a_{m+1}, a_m]$. The remaining details are left to the reader. ∎

Remark. If I is the union of two disjoint open intervals, then it is not true that every $f \in C(I)$ can be bounded above by a function which is convex in the above sense.

Remark. If I, J are arbitrary sets in $\mathbb{R}$, and if T is a linear transformation from $C(I)$ to $F(J)$ satisfying

$$Z_I(f) = Z_J(Tf),$$

for all $f \in C(I)$, then we conjecture that we always have

$$(Tf)(y) = q(y)f(r(y)),$$

for q and r, as previously defined.

4. THE POLYNOMIAL CASE

If $1, x \in W$, and $x^{2n} \in W$, for $n \in \mathbb{N}$, then since x^{2n} is strictly convex, it follows from Proposition 2.1 that

$$(Tx^{2n})(y) = q(y)r^{2n}(y).$$

For $x^{2n-1} \in W$, we would have this same result if, for example, we could express x^{2n-1} as the difference of two strictly convex functions in W (see Corollary 3.2). (There are many ways in which x^{2n-1} can be expressed as the difference of two convex functions. However, it is not necessarily true that these two functions are in W.) We now consider the special case where W is not all of $C(I)$, but rather $W = \Pi$, the space of all algebraic polynomials, or $W = \pi_N$, the space of all algebraic polynomials of degree at most N.

PROPOSITION 4.1. *Let $I, J \subseteq \mathbb{R}$, and let Π be the linear subspace of algebraic polynomials. Assume T is a linear transformation from Π to $F(J)$ satisfying*

$$Z_I(p) = Z_J(Tp),$$

for all $p \in \Pi$. There exists a $q \in F(J)$ which does not vanish on J and an $r \in F(J)$ which is a bijection from J to I such that for every $p \in \Pi$ we have

$$(Tp)(y) = q(y)p(r(y)).$$

Proof. Our proof is simple. The q and r are of course given by $T1 = q$ and $Tx = qr$. We also have from Proposition 2.1 that

$$(Tx^{2n})(y) = q(y)r^{2n}(y), \tag{4.1}$$

for every $n \in \mathbb{N}$.

Let $n \geq 2$. Given any $a \neq 0$, we claim that there exists a $b > 0$ such that

$$x^{2n} + ax^{2n-1} + bx^2$$

is strictly convex on all of $\mathbb{R}$, and thus on any $I \subseteq \mathbb{R}$. To prove this, simply consider the second derivative

$$2n(2n-1)x^{2n-2} + a(2n-1)(2n-2)x^{2n-3} + 2b.$$

For given a there exists an N such that for all $|x| > N$,

$$2n(2n-1)x^{2n-2} + a(2n-1)(2n-2)x^{2n-3} > 0.$$

Thus for $b > 0$ sufficiently large

$$2n(2n-1)x^{2n-2} + a(2n-1)(2n-2)x^{2n-3} + 2b > 0,$$

for all $x \in \mathbb{R}$.

We now use the fact that T is linear, Proposition 2.1, and Eq. (4.1), which prove that

$$(Tx^{2n-1})(y) = q(y)r^{2n-1}(y),$$

for every $n \in \mathbb{N}$. ∎

Remark. When considering algebraic polynomials it is often natural to consider Z^* rather than Z, where Z^* counts zeros, including multiplicities. Is there any difference in our results if, for example, we are asked to characterize all linear transformations T mapping Π, defined on I, to $C^\infty(J)$ and satisfying

$$Z_I^*(p) = Z_J^*(Tp),$$

for all $p \in \Pi$? The q and r must now be in $C^\infty(J)$ and must satisfy the previous properties. There is only one further difference, and that is that r' must not vanish on J.

Let us now restrict ourselves to T restricted to polynomials p and such that Tp is also a polynomial. If we have

$$Z_I(p) = Z_J(Tp),$$

for all $p \in \Pi$ and $Tp \in \Pi$, then from Proposition 4.1 we must have $q \in \Pi$ and also $qr^n \in \Pi$ for all n. This latter condition implies that $r \in \Pi$. Furthermore, as r is a 1–1 map from J onto I this restricts the pair of permissible I and J. For example, if J is an interval, then I must be an interval of the same type. However, the converse need not hold. What if, in addition, $T\colon \pi_n \to \pi_n$, for every n, where π_n denotes the set of polynomials of degree at most n? Then it necessarily follows that q is a nonzero constant and r is a linear function (which is not a constant). This implies that the I and J are related by translation and dilation; i.e., $I = aJ + b$, $a \neq 0$.

We formally state this as follows:

COROLLARY 4.2. *Let $I, J \subseteq \mathbb{R}$, and let π_n be the linear subspace of algebraic polynomials of degree at most n. Assume T is a linear transformation from π_n to π_n, for all n, satisfying*

$$Z_I(p) = Z_J(Tp),$$

for all $p \in \Pi$. Then

$$(Tp)(y) = cp(ay + b),$$

for some constants $a, b, c \in \mathbb{R}$; $a, c \neq 0$. Furthermore, $I = aJ + b$.

If $T: \pi_N \to \pi_N$ for some fixed N, then from the method of proof of Proposition 4.1, it follows that if N is even we have

$$(Tp)(y) = q(y)p(r(y)),$$

for all $p \in \pi_N$, without any a priori restriction on I or J. If N is odd, then we necessarily have

$$(Tp)(y) = q(y)p(r(y)),$$

for all $p \in \pi_{N-1}$ (again from the method of proof of Proposition 4.1). If, in addition, I is bounded, either above or below, then $A(x-\alpha)^N$ is strictly convex on I for some A and α, and we therefore also have

$$(Tx^N)(y) = q(y)r^N(y).$$

Another case in which the desired result holds is the following. T will be a map from π_N to $C^\infty(J)$ (with restrictions on J), and we will demand that

$$Z_I^*(p) = Z_J^*(Tp)$$

for all $p \in \pi_N$, where $Z_I^*(f)$ counts the number of zeros, including multiplicities, of f on I. As $f \in C^\infty(I)$, we have no problem with such a count.

PROPOSITION 4.3. *Let $I, J \subseteq \mathbb{R}$, and assume that J is a countable union of connected intervals, none of which is a singleton. Assume $T: \pi_N \to C^\infty(J)$ for N odd, and*

$$Z_I^*(p) = Z_J^*(Tp),$$

for all $p \in \pi_N$. There exists a $q \in C^\infty(J)$ which does not vanish on J, and an $r \in C^\infty(J)$ which is a bijective map from J to I for which also r' does not vanish on J, such that

$$(Tp)(y) = q(y)p(r(y)),$$

for all $p \in \pi_N$.

Proof. We assume that $N = 2m+1 \geq 3$ is odd. Set $q(y) = (T1)(y)$, and $q(y)r(y) = (Tx)(y)$. This defines q and r with the desired properties. As stated above, we have from Corollary 3.2 and the proof of Proposition 4.1 that

$$(Tx^n)(y) = q(y)r^n(y), \tag{4.2}$$

for $n = 0, 1, \dots, 2m$. It remains to prove that

$$(Tx^{2m+1})(y) = q(y)r^{2m+1}(y).$$

For $\alpha \in I$, set

$$p(y;\alpha) = (T(x-\alpha)^{2m+1})(y).$$

Thus

$$Z_J^*(p(y;\alpha)) = Z_I^*((x-\alpha)^{2m+1}) = 2m+1.$$

We claim that $p(y;\alpha)$ has a zero in J of multiplicity at least 2. This follows from the fact that for each $\alpha \in I$,

$$Z_I^*((x-\alpha)^{2m+1} + c) \le 1,$$

for all $c \in \mathbb{R}$, $c \ne 0$. Thus

$$Z_J^*(p(y;\alpha) + cq(y)) \le 1,$$

for all $c \in \mathbb{R}$, $c \ne 0$. The function $p(y;\alpha)$ has $2m+1$ zeros in J, counting multiplicity. Each zero lies in a component of J which is connected, but not a singleton. For each such zero the function $p(y;\alpha) + cq(y)$ contains at least one nearby zero for all c sufficiently small and of one fixed sign (since q does not vanish). But then $p(y;\alpha)$ has at most two distinct zeros. Since $2m+1 \ge 3$, one of these zeros must be of multiplicity at least 2.

Set

$$v(y) = (Tx^{2m+1})(y).$$

A simple calculation based on (4.2) shows that

$$p(y;\alpha) = v(y) - q(y)r^{2m+1}(y) + q(y)(r(y)-\alpha)^{2m+1}. \tag{4.3}$$

We first prove that

$$\frac{1}{r'(y)}\left(\frac{p(y;\alpha)}{q(y)}\right)' \ge 0,$$

for all $y \in J$.

We know that r' never vanishes on J. Assume that there exists a point $y_0 \in J$ at which

$$\frac{1}{r'(y_0)}\left(\frac{p(y_0;\alpha)}{q(y_0)}\right)' < 0.$$

Then for some constants $a, b \in \mathbb{R}$, $a > 0$,

$$\frac{p(y;\alpha)}{q(y)} + a(r(y)+b)$$

has a double zero at y_0. Thus

$$\begin{aligned} 2 &\le Z_J^*\left(\frac{p(y;\alpha)}{q(y)} + a(r(y)+b)\right) \\ &= Z_J^*(p(y;\alpha) + aq(y)(r(y)+b)) \\ &= Z_I^*((x-\alpha)^{2m+1} + ax + ab)) \\ &\le Z_{\mathbb{R}}^*((x-\alpha)^{2m+1} + a(x-\alpha) + a(b+\alpha)) \\ &= Z_{\mathbb{R}}^*(z^{2m+1} + az + c) \end{aligned}$$

Since $a > 0$, it easily follows that

$$Z_{\mathbb{R}}^*(z^{2m+1} + az + c) = 1,$$

for any $c \in \mathbb{R}$. This contradiction implies the above claim.

From (4.3),

$$\frac{p(y;\alpha)}{q(y)} = \frac{v(y)}{q(y)} - r^{2m+1}(y) + (r(y)-\alpha)^{2m+1}.$$

Differentiate each side and then divide by $r'(y)$ (which is nonzero) to obtain

$$\begin{aligned} \frac{1}{r'(y)}\left(\frac{p(y;\alpha)}{q(y)}\right)' = \frac{1}{r'(y)}\left(\frac{v(y)}{q(y)}\right)' &- (2m+1)r^{2m}(y) \\ &+ (2m+1)(r(y)-\alpha)^{2m}, \end{aligned}$$

for all $y \in J$, $\alpha \in I$. We now follow the proof of Proposition 2.1 almost word-for-word. We have shown that the left-hand side is nonnegative on all of J. This easily implies that

$$\frac{1}{r'(y)}\left(\frac{v(y)}{q(y)}\right)' - (2m+1)r^{2m}(y) \ge 0,$$

for all $y \in J$, while obviously

$$(2m+1)(r(y)-\alpha)^{2m} \ge 0,$$

for all $y \in J$. By a previous claim, $p(y;\alpha)$ has a double zero in J, and at such a point $(p(y;\alpha)/q(y))' = 0$. It therefore follows, using the previous reasoning, that at this zero $\tau(\alpha)$,

$$r(\tau(\alpha)) = \alpha,$$

and thus we obtain for all $y \in J$,

$$\frac{1}{r'(y)}\left(\frac{v(y)}{q(y)}\right)' = (2m+1)r^{2m}(y).$$

That is,

$$\left(\frac{v(y)}{q(y)}\right)' = (2m+1)r'(y)r^{2m}(y),$$

for all $y \in J$, which implies that

$$v(y) = q(y)r^{2m+1}(y) + cq(y),$$

for some constants $c \in \mathbb{R}$, which may differ from component to component of J. Substituting in (4.3) we get

$$p(y;\alpha) = cq(y) + q(y)(r(y) - \alpha)^{2m+1}.$$

Setting $y = \tau(\alpha)$ (and since q does not vanish) we see that $c = 0$ on the component containing $\tau(\alpha)$. But as α varies over all I, $\tau(\alpha)$ varies over all J. Thus $c = 0$ on every component of J, which proves the result. ∎

We do believe that the result of Proposition 4.3 should hold without these unnecessary topological conditions.

Remark. The arguments of this paper are analytic. We have used very little geometry. To use the geometry in some meaningful way, it seems necessary to restrict the sets I and J and the range of T. For example, let $I = J = [a, b]$, and let T satisfy $T1 = 1$, $Tx = x$, and $Tp = q$, where p and q are some fixed polynomials. Assume

$$Z_I^*(ap + bx + c) = Z_I^*(aq + bx + c),$$

for all possible constants a, b, c. Thus T is defined only on the span $\{1, x, p\}$. It follows from Horwitz [3, Theorem 3] that necessarily $p = q$. It seems that we can in fact replace Z_I^* with Z_I.

The next proposition tells us about the possible forms of T if it is a map from π_N to π_N which satisfies (1.2).

PROPOSITION 4.4. *Assume T: $\pi_N \to \pi_N$ for some fixed N and*

$$(Tp)(y) = q(y)p(r(y)),$$

for every $p \in \pi_N$, where q does not identically vanish and r is not a constant function. Then r has the form

$$r(y) = \frac{ay+b}{cy+d},$$

with $ad - bc \neq 0$, and $q(y) = A(cy+d)^N$, for some constants a, b, c, d, and A.

Proof. Since q and qr are in π_N, it follows that q is a polynomial of degree at most N and r is a rational function with numerator and denominator in π_N. Assume $r = g/h$, where $g, h \in \pi_N$ have no common factors.

If h is a constant function then since r is not a constant function, g must be of degree at least 1. Since $qr^N \in \pi_N$, this implies that g is of degree exactly 1, and as a consequence q is a constant function. (This is the case $c = 0$.)

Assume h is not the constant function. As $qr^N \in \pi_N$ and as g and h have no common factors, we must have $q/h^N \in \pi_N$. As $q \in \pi_N$ this implies that h must be a linear polynomial of the form $cy + d$ and $q(y) = A(cy + d)^N$. Thus $qr^N = Ag^N \in \pi_N$ and hence g is a nonzero polynomial of degree at most 1. ∎

Remark. Aside from the conditions of Proposition 4.4, we also demand that r map J to I in a 1–1 manner. This poses additional restrictions. For example, if $I = J = \mathbb{R}$, then we must have $c = 0$. If $I = J = [0, 1]$, then there exist constants $A \neq 0$ and $a, b > 0$ such that either

$$r(y) = \frac{ay}{ay + b(1-y)} \qquad \text{and} \qquad q(y) = A(ay + b(1-y))^N,$$

or

$$r(y) = \frac{b(1-y)}{ay + b(1-y)} \qquad \text{and} \qquad q(y) = A(ay + b(1-y))^N.$$

(If $a = b$, then r is linear.)

5. W DOES NOT CONTAIN 1 OR x

Let us consider what we can say if W does not contain the function 1 or x.

We will assume that I is an arbitrary subset of $[a, b]$ and that we are given $\{e_0, e_1, e_2\}$ which is a CT-system (complete Chebyshev system) on $[a, b]$. This means that each of $\{e_0\}$, $\{e_0, e_1\}$, and $\{e_0, e_1, e_2\}$ is a T-system (Chebyshev system) on $[a, b]$. (A finite-dimensional subspace U is a T-system on $[a, b]$ if each $u \in U \backslash \{0\}$ has at most $\dim U - 1$ distinct zeros in $[a, b]$.) What we will need, at least in our proof, is, for each $\alpha \in I$, the existence of a function

$$h(x; \alpha) = e_2(x) + \gamma e_1(x) + \delta e_0(x),$$

which is strictly positive on $[a, b]/\{\alpha\}$ and vanishes at α. (We may have to multiply e_2 by -1, but this is a minor technicality.) As $\{e_0, e_1, e_2\}$ is a CT-system on $[a, b]$, it follows from a theorem of Krein [6], see also Karlin and Studden [4, p. 28], that this can be done for all $\alpha \in (a, b)$. However, the endpoints may be problematic. As we also need the endpoint result, we will impose additional minor conditions. We could, for example, demand that $\{e_0, e_1, e_2\}$ be a CT-system on (a', b'), where $a' < a < b < b'$, or we might demand that $\{e_0, e_1, e_2\}$ be an ECT-system (extended complete Chebyshev system) on $[a, b]$. (An ECT-system differs from a CT-system simply in that zeros are counted according to their multiplicity.)

Let $\widetilde{W}$ be a linear subspace of $C[a, b]$ and W be the restriction of $\widetilde{W}$ to I. Thus each $f \in W$ is also a bounded function. Assume that $e_0, e_1, e_2 \in \widetilde{W}$. Under the above assumptions we will prove the following result.

THEOREM 5.1. *Let the previous assumptions apply and T be a linear transformation from W to $F(J)$ ($J \subseteq \mathbb{R}$) satisfying*

$$Z_I(f) = Z_J(Tf),$$

for all $f \in W$. There exist a $q \in F(J)$ which does not vanish on J and an $r \in F(J)$ which is a 1–1 map from J onto I such that

$$(Tf)(y) = q(y)f(r(y)),$$

for every $f \in W$.

Proof. In the proof of this theorem we use the methods of proof of Proposition 2.1 and Theorem 2.2. We first construct q and r. Previously, q and r were easily defined. Here there is a bit more work.

Let $Te_i = g_i$, $i = 0, 1, 2$. As $\{e_0\}$ is a T-system on $[a, b]$, it does not vanish thereon. We assume, without loss of generality, that e_0 is positive on $[a, b]$. (If not, simply multiply it by -1.) Thus

$$0 = Z_I(e_0) = Z_J(Te_0) = Z_J(g_0),$$

and g_0 does not vanish on J. By assumption, $\{e_0, e_1\}$ is a T-system on $[a, b]$. Thus

$$Z_I(\beta e_0 + \gamma e_1) \leq 1,$$

for every choice of $\beta, \gamma \in \mathbb{R}$, $(\beta, \gamma) \neq (0, 0)$. Set

$$K = \left\{ \frac{e_1(x)}{e_0(x)} : x \in I \right\}.$$

For each $\alpha \in \mathbb{R}$,

$$Z_J(g_1 - \alpha g_0) = Z_I(e_1 - \alpha e_0) = \begin{cases} 1, & \alpha \in K \\ 0, & \alpha \notin K \end{cases}. \tag{5.1}$$

Then from (5.1) it also follows that

$$K = \left\{ \frac{g_1(y)}{g_0(y)} : y \in J \right\}.$$

In fact, the two maps

$$\frac{e_1}{e_0}: I \to K, \qquad \frac{g_1}{g_0}: J \to K$$

are 1–1 maps onto K. As such,

$$\left(\frac{e_1}{e_0}\right)^{-1}$$

exists. From our assumption that $\{e_0, e_1\}$ is a T-system on all of $[a, b]$ (and not only on I) it actually follows that $(e_1/e_0)^{-1}$ is continuous. Set

$$r(y) = \left(\frac{e_1}{e_0}\right)^{-1}\left(\left(\frac{g_1}{g_0}\right)(y)\right); \tag{5.2}$$

i.e.,

$$\frac{e_1(r(y))}{e_0(r(y))} = \frac{g_1(y)}{g_0(y)}.$$

The function r is a 1–1 map from J onto I. Set

$$q(y) = \frac{g_0(y)}{e_0(r(y))}. \tag{5.3}$$

Thus $q \in F(J)$ and does not vanish thereon. These definitions of q and r imply (from (5.3)) that

$$(Te_0)(y) = g_0(y) = q(y)e_0(r(y))$$

and from (5.2) that

$$(Te_1)(y) = g_1(y) = \frac{g_0(y)e_1(r(y))}{e_0(r(y))} = q(y)e_1(r(y)).$$

We now prove, paralleling the proof of Proposition 2.1, that

$$(Te_2)(y) = q(y)e_2(r(y)).$$

Let

$$h(x;\alpha) = e_2(x) + \gamma e_1(x) + \delta e_0(x),$$

where $h(x;\alpha) > 0$ for all $x \in [a, b]$, and $h(\alpha;\alpha) = 0$. Set

$$p(y;\alpha) = [Th(x;\alpha)](y).$$

Thus

$$Z_J(p(y;\alpha)) = Z_I(h(x;\alpha)) = 1.$$

Since h is nonnegative and e_0 is strictly positive on $[a,b]$, we have for all $c>0$,

$$\begin{aligned}0 &= Z_I(h(x;\alpha)+ce_0(x)) = Z_J(p(y;\alpha)+cq(y)e_0(r(y)))\\ &= Z_J\left(\frac{p(y;\alpha)}{q(y)}+ce_0(r(y))\right).\end{aligned}$$

Thus $p(y;\alpha)/q(y)+ce_0(r(y))$ has no zero in J for all $c>0$, which implies that

$$\frac{p(y;\alpha)}{q(y)} \geq 0,$$

for all $y\in J$.

Now

$$\begin{aligned}p(y;\alpha) &= [Th(x;\alpha)](y)\\ &= T(e_2(x)+\gamma e_1(x)+\delta e_0(x))(y)\\ &= g_2(y)+\gamma q(y)e_1(r(y))+\delta q(y)e_0(r(y))\\ &= g_2(y)-q(y)e_2(r(y))+q(y)[e_2(r(y))+\gamma e_1(r(y))+\delta e_0(r(y))].\end{aligned}$$

Thus

$$0\leq \frac{p(y;\alpha)}{q(y)} = \left[\frac{g_2(y)}{q(y)}-e_2(r(y))\right]+[e_2(r(y))+\gamma e_1(r(y))+\delta e_0(r(y))].$$

For $y\in J$, let $\alpha=r(y)$. Then $\alpha\in I$ and substituting in the above it follows that

$$\frac{g_2(y)}{q(y)}-e_2(r(y))\geq 0.$$

Thus

$$\frac{g_2(y)}{q(y)}-e_2(r(y))\geq 0,$$

for all $y\in J$. We continue the proof, word for word, as in the proof of Proposition 2.1, to obtain

$$(Te_2)(y) = q(y)e_2(r(y)).$$

We now consider any $f\in W$. We will use the method of proof of Theorem 2.2, i.e., that based on the idea of the proof of the Bohman–Korovkin Theorem, to prove the desired result. As the proof is much the same we just mention where the differences arise and how to deal with them.

Fix $\alpha \in I$. Given any $\varepsilon > 0$, it follows from the facts that f is continuous at α and that e_0 is continuous and strictly positive on $[a, b]$ that there exists an $\eta > 0$ such that for

$$|x - \alpha| < \eta,$$

we have

$$|f(x) - f(\alpha)| < \frac{\varepsilon}{2}$$

and

$$\frac{e_0(x)}{e_0(\alpha)} > \frac{1}{2}.$$

As $f \in W$ has an extension to a continuous function on $[a, b]$, it follows that f is bounded on W. Let $M \geq |f(x)|$ for all $x \in I$.

We now set

$$p_u(x) = f(\alpha) + \varepsilon \frac{e_0(x)}{e_0(\alpha)} + A(\eta; \alpha)[e_2(x) + \gamma e_1(x) + \delta e_0(x)]$$

and

$$p_\ell(x) = f(\alpha) - \varepsilon \frac{e_0(x)}{e_0(\alpha)} - B(\eta; \alpha)[e_2(x) + \gamma e_1(x) + \delta e_0(x)].$$

We claim that $p_u(x) > f(x) > p_\ell(x)$ for all $x \in I$ for some judicious choice of positive $A(\eta; \alpha)$ and $B(\eta; \alpha)$. We prove only the first inequality. The proof of the second inequality is totally analogous.

If $x = \alpha$, then

$$p_u(\alpha) - f(\alpha) = \varepsilon > 0.$$

If $0 < |x - \alpha| < \eta$ then $|f(x) - f(\alpha)| < \varepsilon/2$, and

$$p_u(x) - f(x) > -\varepsilon/2 + \varepsilon/2 + A(\eta; \alpha)[e_2(x) + \gamma e_1(x) + \delta e_0(x)] > 0.$$

If $|x - \alpha| \geq \eta$, then $|f(x) - f(\alpha)| \leq 2M$ and

$$e_2(x) + \gamma e_1(x) + \delta e_0(x)$$

is thereon strictly positive and also bounded below away from zero. We now choose $A(\eta; \alpha) > 0$ so that on this set

$$A(\eta; \alpha)[e_2(x) + \gamma e_1(x) + \delta e_0(x)] > 2M.$$

This then implies that

$$p_u(x) - f(x) > 0$$

on $|x - \alpha| \geq \eta$.

The rest of the proof now follows the proof of Theorem 2.2. ∎

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