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Variational problems arising from balancing several error criteria

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Dedicated to the memory of Prof. Aldo Ghizzetti

Riassunto: *Scopo di questo lavoro è la descrizione di vari risultati riguardanti una classe di problemi variazionali, nei quali uno o più criteri d'errore devono esser controllati simultaneamente, in modo ottimo. Questo tipo di problema ha numerose applicazioni nella teoria del controllo ottimo, in teoria dell'approssimazione, nelle stime statistiche. Non pretendiamo che la nostra trattazione sia esaustiva, né per ciò che riguarda le applicazioni presentate, né per l'analisi che sviluppiamo; piuttosto, vogliamo mettere in evidenza certe proprietà generali di quei problemi che risultano di pratica utilità, dare una panoramica di alcuni esempi interessanti che si presentano nelle applicazioni, e fornire un'analisi di certe classi di problemi variazionali, nei quali la totale positività gioca un ruolo centrale.*

Abstract: *The purpose of this paper is to describe various results which pertain to a class of variational problems in which two or more error criteria are to be controlled simultaneously in some optimal fashion. This type of problem has wide application in optimal control, statistical estimation and approximation theory. We make no pretense that our treatment is exhaustive either in the applications we cover or in the analysis we use. Rather, we wish to point out some general properties of these problems that are useful, survey some interesting examples which arise in applications, and give an analysis of certain classes of these variational problems where total positivity plays a central role.*

Key Words and Phrases: *Variational problems – Total positivity*

1 – Introduction

The purpose of this paper is to describe various results which pertain to a class of variational problems in which two or more error criteria are to be controlled simultaneously in some optimal fashion. This type of problem has wide application in optimal control, statistical estimation and approximation theory. We make no pretense that our treatment is exhaustive either in the applications we cover or in the analysis we use. Rather, we wish to point out some general properties of these problems that are useful, survey some interesting examples which arise in applications, and give an analysis of certain classes of these variational problems where total positivity plays a central role.

The problems that we refer to have a general presentation. We suppose that two functionals $G_1(x)$ and $G_2(x)$ are given which both measure the "desirability" of x. Thus we are led to control both simultaneously.

Perhaps the simplest example of what we have in mind is *smoothing splines*. Here we are given data $y_1, \ldots, y_m$ which is assumed to be inaccurate and wish to find a smooth function that passes near these values at $t_1 < \ldots < t_m$ in some interval $[a,b]$. Thus we desire to balance the fidelity of our functions values $f(t_1), \ldots, f(t_m)$ to the noisy data $y_1, \ldots, y_m$ with the smoothness of f. Here a standard choice of functionals is

$$G_1(f) = \sum_{j=1}^{m} \left| f(t_j) - y_j \right|^2 \tag{1.1}$$

and

$$G_2(f) = \int_a^b \left| f^{(n)}(t) \right|^2 dt\,. \tag{1.2}$$

A smoothing spline is the unique function which minimizes a positive combination of both. That is, which solves

$$\min \left\{ G_1(f) + \sigma G_2(f) : f \in W_2^n[a,b] \right\} \tag{1.3}$$

where $W_2^n[a,b]$ is the Sobolev space

$$W_2^n[a,b] = \left\{ f : f, \ldots, f^{(n-1)} \quad \text{abs. cont.}, \quad f^{(n)} \in L_2[a,b] \right\},$$

A.M.S. Classification: 49K10 – 49K20 – 41A29 and $\sigma > 0$ is some prescribed smoothing parameter. Much is known about this problem which we will not dwell upon here. The only point we wish to raise here concerns the choice of σ. There are several strategies. A useful one is based on the technique of cross-validation. That is, a good σ is determined by how well the smoothing spline to a subset of the data fits the remaining data, WAHBA [27]. Another possibility is to choose σ interactively. Thus, if you like the graph of f corresponding to a given σ, fine, if not adjust σ and proceed if necessary. This hit and miss approach although not to be seriously recommended in general may have merit in specific problems. A more reasonable approach can be designed if one had a preferred error tolerance for the data error, say some $\varepsilon > 0$. In this case one might adjust σ so that the solution to (1.3) satisfies $G_1(f) \le \varepsilon$. Even better, we can abandon (1.3) altogether and consider

$$\text{(1.4)} \qquad \begin{aligned} &\text{minimize} \qquad G_2(f) \\ &\text{subject to:}\, G_1(f) \le \varepsilon\,, \qquad f \in W_2^n[a,b]\,. \end{aligned}$$

The fact that these extremal problems (1.3), (1.4) are the same is of no surprise and we will highlight their precise relationship in a general context later in Section 2. As a final remark about smoothing splines we mention that some general facts about smoothing noisy data in a Hilbert space when f is constrained to lie in a convex set appear in MICCHELLI and UTRERAS [19].

A problem of a different type is studied in FORSYTHE [9], SPJOTVOLL [25], WALSH [28], whose motivation comes from both *nonlinear parameter estimation and optimal control*. Here A is some given $n \times n$ matrix and b some fixed vector in $\mathbb{C}^n$. It is then desired to minimize $\|Ax-b\|$ subject to $\|x\| \le 1$ (and also $\|x\| = 1$) where $\|x\|^2 = \sum_{i=1}^{n} |x_i|^2$ is the usual Euclidean norm of $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$. Thus, in this case $G_2(x) = \|Ax - b\|$ and $G_1(x) = \|x\|$. A characterization of the solution and various other facts about this problem are given in the above references.

Another example comes from *optimal design*. Here the problem is to find an optimal impulse response $u(t)$ of a filter so that the response to a given input signal s of finite support lies within specified pointwise bounds and at the same time minimizes the effect of input noise. When the noise is assumed to be zero - mean white noise the output noise power is proportional to the squared L^2-norm of u. Thus the problem takes the form

$$\text{(1.5)} \qquad \text{minimize} \int_{-\infty}^{\infty} |u(t)|^2 dt$$

subject to

$$\varepsilon^-(t) \le \psi(t) := (s * u)(t) = \int_{-\infty}^{\infty} u(\tau)s(t-\tau)d\tau \le \varepsilon^+(t)$$

where ε^- and ε^+ are given functions. If we let: $d := \frac{1}{2}(\varepsilon^+ + \varepsilon^-)$ and $\varepsilon := \frac{1}{2}(\varepsilon^+ - \varepsilon^-)$ then we get the equivalent problem

$$\text{(1.6)} \qquad \min\left\{\|u\|_2^2 : \|Ku - d\|_\infty \le 1\right\}$$

where $Ku := s*u$, $\|u\|_2^2 := \int_{-\infty}^{\infty} |u(t)|^2 dt$ and $\|f\|_\infty := \sup_{-\infty<t<\infty} |f(t)|/\varepsilon(t)$. Thus, here we have $G_2(u) = \|u\|_2^2$ and $G_1(u) = \|Ku - d\|_\infty$. Further details and computational algorithms for solutions of this problem appear in Evans, Fortmann, Cantoni [7] and Evans, Cantoni, Fortmann [8]. In contrast, in the series of papers [3-5], a detailed analysis of a class of optimal filter design problems were given. These problems are surveyed in Berkovitz [3]. We wish to mention only one such problem of this type considered in Berkovitz and Pollard [4]. Specifically, they consider the minimum of

$$\text{(1.7)} \qquad \left(\int_0^{\infty} |y(t)|dt\right)^2 + \int_0^{\infty} |y''(t)|^2 dt$$

where $y \in L_1(0,\infty)$, y' is absolutely continuous, $y'' \in L_2(0,\infty)$ and $y(0) = a, y'(0) = b$ are specified. Equivalently, if we let $f(t) := a + bt$, and $(Kh)(t) := \int_0^x (x-t)h(t)dt$, then (1.7) becomes

$$\text{(1.8)} \qquad \|f - Kh\|_{L_1}^2 + \|h\|_{L_2}^2$$

where

$$\|h\|_{L_1} = \int_0^\infty |h(t)dt, \qquad \|h\|_{L_2}^2 = \int_0^\infty |h(t)|^2 dt$$

and we minimize over all h such that $f - Kh \in L_1(0,\infty)$ and $h \in L_2(0,\infty)$. Thus our functionals are in this case $G_1(h) = \|f - Kh\|_{L_1}^2$, and $G_2(h) = \|h\|_{L_2}^2$.

Next, we describe a problem in *control theory*, GLASHOFF [11] and GLASHOFF and WECK [12]. The problem here is to choose a temperature $u(t,\xi)$, $t \in [0,T]$, $\xi \in \partial\Omega \subseteq \mathbb{R}^n$, of a medium surrounding a body Ω to satisfy prescribed pointwise constraints and so that the temperature distribution $y(T,x)$ at time T of the body is as close as possible to some desired temperature $z(x)$, $x \in \Omega$. Under some regularity hypothesis on Ω we have

$$y(u;t,x) = \int_{\Gamma_t} g(t,x,\tau,\xi)u(\tau,\xi)d\tau\, d\xi$$

for some Green's function g and $\Gamma_t := (0,t) \times \partial\Omega$. Thus the problem reduces to minimizing

$$\|Su - z\|_\infty(\overline{\Omega}) := \operatorname{ess\,sup}_{x \in \overline{\Omega}} |(Su)(x) - z(x)| \tag{1.9}$$

where

$$(Su)(x) := y(u;T,x)$$

subject to $u \in L_\infty(\Gamma)$ where $\Gamma := (0,T) \times \partial\Omega$ and

$$\|u\|_\infty(\Gamma) \le 1 \tag{1.10}$$

which gives us $G_2(u) = \|Su - z\|_\infty$ and $G_1(u) = \|u\|_\infty$.

In the totally different context of functional analysis, the *Peetre K-functional* leads us to similar problems, BUTZER and BERENS [6], BERGH and LÖFSTRÖM [2]. The K-functional is an essential tool in various problems in functional analysis and approximation theory, especially in problems concerned with the Bernstein inverse theorem on order of approximation. The K-functional is defined relative to two Banach spaces X_1, X_2 which are assumed to be continuously imbedded in some linear Hausdorff space X. Then their algebraic sum $X_1 + X_2 = \{x : x = x_1 + x_2, x_i \in$

$X_i, i = 1, 2\}$ is a Banach space relative to the norm (K-functional)

$$K(t, x) = \|x\|_{X_1+X_2} := \inf_{x=x_1+x_2} \left(\|x_1\|_1 + t\|x_2\|_2\right).$$

In this case $G_1(x_2) = \|x - x_2\|_1$ and $G_2(x_2) = \|x_2\|_2$. It is only in rare case that the K-functional of a pair of spaces can be found explicitly and so one does not generally study the variational problem directly. Instead, concrete bounds from above and below for K are sought in terms of some useful measure of x. This may take the form of the modulus of continuity of higher order differences for Sobolev spaces.

Approximation theory provides several further instances of the type of variational problem which interests us here. We mention the problem of *best operator approximation* with a fixed bound on its norm, to a given operator, STECHKIN [26]. The general description of the problem is as follows: Let X, Y be Banach spaces and U an unbounded linear operator of X into Y with domain $D_U \subseteq X$. Let V be another linear operator with domain $D_V \subseteq D_U$ and range in a normed linear space Z. The problem is to study the variational problem

$$\inf_{\|S\| \le N} \sup_{\|Vx\|_Z \le 1} \|Ux - Sx\|_Y$$

where the infimum is taken over all bounded linear operators S from X into Y, and N is a prescribed positive constant. Here our functionals are $G_1(S) = \|S\|$ and $G_2(S) = \sup\{\|Ux - Sx\| : \|Vx\|_Z \le 1\}$.

Vectorial approximation is also very much in the spirit of the problems in which we are interested cf. BACOPOULUS [1], GEARHART [10]. This problem is similar to the rest and has the manifestation of simultaneously best approximating a function f and its derivative f' by polynomials. In its general form, it concerns the following notion. We suppose X is some set on which G_1 and G_2 are defined. Given a subset C of X, we say that $u_0 \in C$ is a best vectorial approximation to f from C if there does not exist a $u \in C$ such that both

$$G_1(f - u) < G_1(f - u_0) \quad \text{and} \quad G_2(f - u) < G_2(f - u_0).$$

We will discuss this problem a bit more in Section 2.

Finally, as our last example we describe the point of view of *optimal recovery* which gives a model of computation that provides a general framework for the study of estimation problems under limited information. We suppose U is a bounded linear operator from a linear space X into a normed linear space Z, K a convex subset of X, and I another bounded linear operator from X into another normed linear space Y. We wish to estimate Ux given $x \in K$ and any observation $y = Ix + w$ where $\|w\|_Y \leq \varepsilon$, $\varepsilon > 0$, fixed. The intrinsic error (in the worst case) is given by

$$E = \inf_A \sup_{\substack{x \in K \\ \|w\|_Y \leq \varepsilon}} \big\|Ux - A(Ix + w)\big\|_Z \tag{1.11}$$

where we minimize over *all mappings* A from $\{Ix + w, x \in K, \|w\| \leq \varepsilon\}$ into Z, Micchelli [14], Micchelli and Rivlin [18]. We can bound the intrinsic error from above by

$$\inf_B \Big\{ \sup_{x \in K} \|Ux - B\,Ix\|_Z + \varepsilon \|B\| \Big\} \tag{1.12}$$

where here we minimize over all *bounded linear* operators B from Y to Z. This bound is sometimes sharp, Micchelli [14], Micchelli and Rivlin [18]. Here again as in the operator approximation problem, we have

$$G_2(B) = \sup_{x \in K} \|Ux - B\,Ix\|_Z$$

and

$$G_1(B) = \|B\|\,.$$

2 – Basic Facts

The first result we wish to present is very general in nature. It concerns simultaneously controlling a family of real-valued functions $G_t(x) := G(t, x)$ where $t \in T$ and $x \in K$, a given convex subset of a linear space X. We make the following hypotheses:

(i) $$\sup_{t \in T} \big|G(t, x)\big| < \infty\,, \qquad x \in K\,,$$

(ii) $$x \to G_t(x) = G(t, x) \quad \text{is convex on} \quad K\,,$$

(iii) given any $x, y \in K$ such that $G(t,x) < G(t,y)$ for all $t \in T$, there exists a positive constant $c > 0$ such that

$$0 < c \le G(t,y) - G(t,x)\,, \qquad t \in T\,.$$

Clearly, (i) and (iii) are satisfied if T is a compact Hausdorff space and $t \to G(t,x)$ is continuous on T for each $x \in K$.

To state the result we have in mind we let $B(T)$ be the space of real-valued bounded functions on T and let $\le$ be the natural pointwise ordering on $B(T)$, i.e. $f \le g, f, g \in B(T)$ means $f(t) \le g(t)$, for all $t \in T$. A linear function L in the algebraic dual of $B(T)$ is said to be nonnegative if $f \ge 0$ implies $Lf \ge 0$.

We say that $x_0 \in K$ is a best G-approximation from K if there does not exist an $x \in K$ such that $G(t,x) < G(t,x_0)$ for all $t \in T$.

THEOREM 2.1. *An element $x_0 \in K$ is a best G-approximation if and only if there exists a nonnegative nontrivial linear functional L on $B(T)$ such that*

$$L\big(G(\cdot, x_0)\big) = \min_{x \in K} L\big(G(\cdot, x)\big)\,. \tag{2.1}$$

REMARK 2.1. The functional $x \to L\big(G(\cdot,x)\big)$ is convex and so this theorem reduces the problem of finding a G-approximation to that of minimizing some convex function. In particular, for a finite set of convex functionals $\{G_1, \dots, G_m\}$ this says that a G-approximation comes from minimizing

$$\sigma_1 G_1(x) + \ldots + \sigma_m G_m(x)\,, \qquad x \in K \tag{2.2}$$

for some choice of $\sigma_1 \ge 0, \dots, \sigma_m \ge 0$, $\sum\limits_{i=1}^{m} \sigma_i > 0$.

PROOF. Suppose L and $x_0 \in K$ satisfy (2.1) and x_0 is not a best G-approximation. Then there is an $\bar{x} \in K$ such that $G(t,\bar{x}) < G(t,x_0)$ for all $t \in T$. Let e be the function in $B(T)$ which is identically one. Since L is a nonnegative nontrivial linear functional on $B(T)$ it follows that $Le > 0$. From property (iii) there is a constant c such that

$$G(t,\bar{x}) \le G(t,x_0) - c\,, \qquad t \in T\,. \tag{2.3}$$

Hence by (2.1), (2.3) and the nonnegativity of L we get

$$L(G(\cdot,x_0)) \le L(G(\cdot,\bar{x})) \le L(G(\cdot,x_0)) - cLe\,,$$

which is a contradiction.

Conversely, let x_0 be a best G-approximation. We introduce two subsets of $B(T)$, namely,

$$C_1 = \{f : f \in B(T), \quad f(t) < G(t,x_0)\,, \quad t \in T\}$$

and

$$C_2 = \text{ convexhull } \quad \{G(\cdot,x) : x \in K\}\,.$$

Clearly C_1 and C_2 are convex subsets of $B(T)$. They are disjoint, since otherwise for some $\lambda_1 \ge 0, \dots, \lambda_n \ge 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $x_i \in K$, we would have by property (ii)

$$G\left(t, \sum_{i=1}^{n} \lambda_i x_i\right) \le \sum_{i=1}^{n} \lambda_i G(t,x_i) < G(t,x_0)\,,$$

contradicting the fact that x_0 is a best G-approximation. From property (i) there is a $d \in \mathbb{R}$ such that $d \le G(t,x_0)$ for all $t \in T$. It then follows that $f_0 = (d-\varepsilon)e$ is an internal point of C_1, for any $\varepsilon > 0$.

Thus by the basic separation theorem for convex sets, see Royden [24, p.176], there is a nontrivial linear functional L and an $\alpha \in \mathbb{R}$ such that

$$Lf \le \alpha\,, \qquad f \in C_1 \tag{2.4}$$

$$Lf \ge \alpha\,, \qquad f \in C_2\,. \tag{2.5}$$

For any $f \in B(T)$, $f(t) \le 0$, $t \in T$ and any $d^1 < d$ we have $d^1 + \lambda f \in C_1$ for all $\lambda \ge 0$. Hence by (2.4), $L(\lambda f + d^1) \le \alpha$ and so we get $Lf \le 0$. That is, L is a nonnegative linear functional. Similarly, because $G(\cdot,x_0) + \lambda \in C_1$, for $\lambda < 0$ we get by (2.4), $L(G(\cdot,x_0)) \le \alpha$ while (2.5) gives $L(G(\cdot,x)) \ge \alpha$ for all $x \in K$. This completes the proof. □

This result shows us that in the special case of two convex functionals $\{G_1, G_2\}$, G-approximation is equivalent to the minimum problem

$$\inf_{x\in K} \sigma_1 G_1(x) + \sigma_2 G_2(x)\,, \quad \sigma_1, \sigma_2 \geq 0\,, \quad \sigma_1 + \sigma_2 = 1\,. \tag{2.6}$$

We now turn out attention to properties of (2.6) as a function of σ_1 and σ_2. To this end, it will be more convenient (and suffices) for us to discuss the function

$$(G_1 + G_2)(\sigma) := \inf_{x\in K} \{G_1(x) + \sigma G_2(x)\}\,, \qquad \sigma > 0\,. \tag{2.7}$$

Our main concern is the relationship of (2.7) to the two problems

$$(G_1/G_2)(t) := \inf\{G_1(x) : \ x \in K\,, \quad G_2(x) \leq t\}\,, \tag{2.8}$$

where $t > \mu(G_2) := \inf_{x\in K} G_2(x)$ and

$$(G_2/G_1)(t) := \inf\{G_2(x) : G_1(x) \leq t\}\,, \tag{2.9}$$

where $t > \mu(G_1)$. Note that G_1/G_2 is nonincreasing and so we can define $(G_1/G_2)(t)$ when $t = \mu(G_2)$ as

$$\begin{aligned}&\lim_{\delta\to 0^+} (G_1/G_2)\big(\mu(G_2) + \delta\big) = \sup\Big\{\inf\{G_2(x) : G_1(x) \leq t\}\Big\}\\ &= \sup\Big\{\inf\{G_2(x) : G_1(x) \leq t\} : t > \mu(G_2)\Big\}\,.\end{aligned}$$

Similarly, for $G_2\,/\,G_1$. Hence the domain of definition of $G_1\,/\,G_2$ is $\big[\mu(G_2), \infty\big)$ and for G_2/G_1 it is $\big[\mu(G_1), \infty\big)$. By definition both G_1/G_2 and G_2/G_1 are continuous (from the right) at the left hand endpoint of their domain of definition.

In what follows, we also assume that G_1 and G_2 are bounded below on K. Thus, without loss of generality we will henceforth require that G_1 and G_2 are nonnegative. We start with some general properties of $(G_1/G_2)(t)$ and $(G_2/G_1)(t)$.

Let

$$\Gamma = \{y = (y_1, y_2) : G_i(x) \leq y_i, \quad i = 1, 2, \quad \text{for some} \quad x \in K\}\,.$$

Γ is a generalization of Gagliardo diagrams studied in connection with the K-functional (see e.g. Bergh, Löfström [2, Chap. 3 and 7]). Since each G_i is convex and nonnegative, it easily follows that Γ is a convex unbounded subset in the first quadrant of $\mathbb{R}^2$. To understand quite explicitly the usefulness of Γ, set

$$\mu[G_1; G_2] = \lim_{t \to \mu(G_2)^+} (G_1/G_2)(t) = \sup_{t > \mu(G_2)} (G_1/G_2)(t)$$

and similarly, we define $\mu[G_2; G_1]$. The graph of Γ is given by

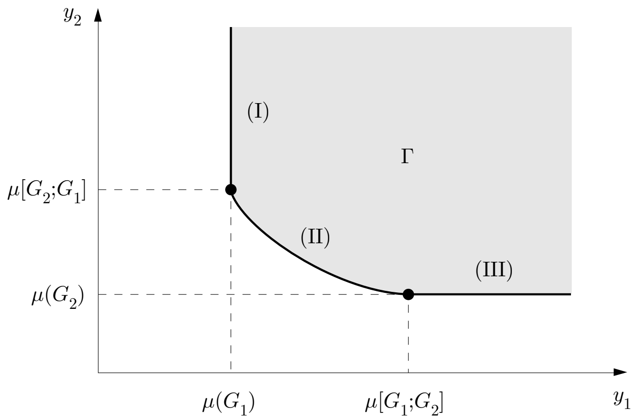

Fig. 1

We remark that each of these segments (I), (II) and (III) of $\partial\Gamma$ may be empty. The boundary sections (II) and (III) (without the left-endpoint) are the "essential" graph of $(G_2/G_1)(t)$ (where $t = y_1$). At the left-endpoint of (II), i.e., at $y_1 = t = \mu(G_1)$, it may be that $(G_2/G_1)(\mu(G_1))$ is not defined. This will occur if $\inf\{G_1(x) : x \in K\}$ is not attained. Moreover, even if $\min\{G_1(x) : x \in K\}$ does exist, then $(G_2/G_1)(\mu(G_1)) \geq \mu[G_2; G_1]$ and equality need not hold. The graph of $(G_1/G_2)(t)$ is obtained by interchanging the axes. To see that this is indeed the case, we list and prove the following facts.

Theorem 2.2. *Let G_1, G_2 be convex nonnegative functions on a convex subset K of a given linear space. Then*

(1) $(G_1/G_2)(t)$ *is nonincreasing and convex on its domain of definition, and continuous on the interior thereof.*

(2) $(G_1/G_2)(t) = \mu(G_1)$ *for* $t > \mu[G_2; G_1]$.

(3) $(G_1/G_2)(t)$ *is strictly decreasing on* $\big(\mu(G_2), \mu[G_2; G_1]\big)$.

(4) $(G_2/G_1)\big((G_1/G_2)(t)\big) = t$ *for* $t \in \big(\mu(G_2), \mu[G_2; G_2]\big)$.

Proof. (1). It is obvious that $(G_1/G_2)(t)$ is nonincreasing. The convexity is a simple consequence of the convexity of Γ. The continuity follows from general considerations regarding convex nonnegative functions.

(2). Since $G_1(x) \geq \mu(G_1)$ for all x, we have $(G_1/G_2)(t) \geq \mu(G_1)$ for all t. $(G_2/G_1)(t)$ is nonincreasing. Thus from the definition of $\mu[G_2; G_1]$, when $t > \mu[G_2; G_1]$, there exists a sequence $\{x_n\}$ in K such that $G_2(x_n) \leq t$ while $G_1(x_n) \leq \mu(G_1) + \dfrac{1}{n}$. But then $(G_1/G_2)(t) \leq \mu(G_1) + \dfrac{1}{n}$, implying that $(G_1/G_2)(t) \leq \mu(G_1)$.

(3). Suppose that there are two values $t_1, t_2 \in \big(\mu(G_2), \mu[G_2; G_1]\big)$ such that $(G_1/G_2)(t_1) = (G_1/G_2)(t_2)$, $t_1 < t_2$. Since (G_1/G_2) is convex and nonincreasing it easily follows that $(G_1/G_2)(t) =$ constant for $t \geq t_1$. Thus (2) implies that $(G_1/G_2)(t) = \mu(G_1)$, for all $t \geq t_1$, for some $t_1 \in \big(\mu(G_2), \mu[G_2; G_1]\big)$. Consequently, given any $\delta > 0$ there is an $x_\delta \in K$ with

$$G_1(x_\delta) \leq \mu(G_1) + \delta\,, \qquad G_2(x_\delta) \leq t\,.$$

Hence $(G_2/G_1)\big(\mu(G_1) + \delta\big) \leq t$ for all $\delta > 0$ and therefore by the definition of $\mu[G_2; G_1]$ we get

$$\mu[G_2; G_1] \leq t\,, \quad \text{for any} \quad t \geq t_1\,.$$

But $t_1 < \mu[G_2, G_1]$, which is a contradiction.

(4). We first note that for $t \in \big[\mu(G_2), \infty\big)$, $(G_1/G_2)(t)$ is in the domain of (G_2/G_1) because by definition $(G_1/G_2)(t) \geq \mu(G_2)$. For $t \in \big(\mu_2(G_2), \mu(G_2; G_1)\big)$ we have by (2) that

$$\mu(G_1) < (G_1/G_2)(t) < \mu[G_1; G_2]\,.$$

In this case, we let $\bar{t} := (G_2/G_1)\big((G_1/G_2)(t)\big)$ and choose any $\delta > 0$. From the definition of $(G_1/G_2)(t)$ there exists $x_\delta \in K$ with

$$\begin{aligned} &G_1(x_\delta) \leq (G_1/G_2)(t) + \delta \\ &G_2(x_\delta) \leq t\,. \end{aligned}$$

Hence

$$(G_2/G_1)\big((G_1/G_2)(t)+\delta\big) \le t$$

which by the continuity of G_2/G_1 in the open interval $\big(\mu(G_1), \mu[G_1;G_2]\big)$ gives, as $\delta \to 0^+$ that $\bar{t} \le t$. For the reverse inequality, let $\delta > 0$. Then there is an $x_\delta \in K$ such that

$$G_2(x_\delta) \le \bar{t} + \delta\,, \qquad G_1(x_\delta) \le (G_1/G_2)(t)\,.$$

Consequently, $(G_1/G_2)(\bar{t}+\delta) \le (G_1/G_2)(t)$ and so $\bar{t}+\delta \ge t$, for otherwise we would contradict the strict decrease of (G_1/G_2) established in (2). Letting $\delta \to 0^+$ proves the result. ∎

Before studying the function $(G_1+G_2)(\sigma)$, $\sigma > 0$, we consider the relationship between it and the functions $(G_1/G_2)(t)$ and $(G_2/G_1)(t)$. This relationship is geometrically given as follows. For $\sigma > 0$, consider the tangent line to Γ with slope $-1/\sigma$. The y_1-intercept of this tangent line is the value $(G_1+G_2)(\sigma)$. To obtain $(G_2/G_1)(t)$ from $(G_2+G_1)(\sigma)$ for $t > \mu(G_1)$, we do the reverse. That is, consider the line with slope $-1/\sigma$ and y_1-intercept $(G_1+G_2)(\sigma)$. The supremum of the y_2-values of this line at t is $(G_2/G_1)(t)$. This is all equivalent to the following analytic statements.

Proposition 2.3. *Assume G_1 and G_2 are convex nonnegative functions on a convex subset K of a given linear space. Then*

(1) $(G_1+G_2)(\sigma) = \inf_{t>\mu(G_1)} \big(t+\sigma(G_2/G_1)(t)\big) = \inf_{t>\mu(G_2)} \big((G_1/G_2)(t)+\sigma t\big)\,.$

(2) *For $t > \mu(G_1)$*

$$(G_2/G_1)(t) = \sup_{\sigma>0}\left(\frac{(G_1+G_2)(\sigma)-t}{\sigma}\right).$$

Proof. We prove only the first part of (1). Pick any $\delta > 0$. Then there is an $x_\delta \in K$ with $G_2(x_\delta) \le \delta + (G_2/G_1)(t)$ and $G_1(x_\delta) \le t$. Thus

$$(G_1 + G_2)(\sigma) \le G_1(x_\delta) + \sigma G_2(x_\delta) \le t + \sigma\big(\delta + (G_2/G_1)(t)\big)$$

which gives $(G_1 + G_2)(\sigma) \le \inf_t \big(t + \sigma(G_2/G_1)(t)\big)$ by letting $\delta \to 0^+$. For the other inequality choose $y^\delta \in K$ with

$$G_1(y^\delta) + \sigma G_2(y^\delta) \le (G_1 + G_2)(\sigma) + \delta\,.$$

Set $t_0 := G_1(y^\delta)$ so that

$$\begin{aligned}\inf_{t>\mu(G_1)} \big(t + \sigma(G_2/G_1)(t)\big) &\le t_0 + \sigma(G_2/G_1)(t_0)\\ &\le G_1(y^\delta) + \sigma G_2(y^\delta) \le (G_1 + G_2)(\sigma) + \delta\,.\end{aligned}$$

The other statements of the proposition follow similarly. ∎

We now consider some general properties of the function $(G_1+G_2)(\sigma)$ for $\sigma \ge 0$. For this purpose we introduce for $\sigma \ge 0$ the set

$$X_\sigma = \{x : (G_1 + G_2)(\sigma) = G_1(x) + \sigma G_2(x)\}$$

which may be empty for any particular σ. If $X_\sigma \ne \phi$, set

$$G_i^+(\sigma) = \sup_{x\in X_\sigma} G_i(x)\,, \qquad i = 1, 2$$

and

$$G_i^-(\sigma) = \inf_{x\in X_\sigma} G_i(x)\,, \qquad i = 1, 2\,.$$

Proposition 2.4. *Let G_1 and G_2 be nonnegative convex functions on K. Then*

(a) *$(G_1+G_2)(\sigma)$ is a nondecreasing continuous concave function of σ on $(0,\infty)$, and $(G_1 + G_2)(\sigma)/\sigma$ is nonincreasing in σ. Furthermore, if there exists an $\hat{x} \in K$ such that $G_2(\hat{x}) = 0$, then $(G_1 + G_2)(\sigma)$ is bounded.*

(b) *$\lim_{\sigma\to 0^+} (G_1 + G_2)(\sigma) = \mu(G_1)$ and $\lim_{\sigma\to\infty} (1/\sigma)(G_1 + G_2)(\sigma) = \mu(G_2)$.*

(c) *If there exists an* $x^* \in K$ *for which* $G_1(x^*) = \mu(G_1)$*, then for* $0 < \sigma_1 < \sigma_2 < \infty$*,*

$$\frac{(G_1 + G_2)(\sigma_2) - (G_1 + G_2)(\sigma_1)}{\sigma_2 - \sigma_1} \leq G_2(x^*)\,.$$

(d) *If* $0 < \sigma_1 < \sigma_2 < \infty$*, and* X_{σ_1}*,* X_{σ_2} *are both non-empty, then*

(i) $$G_2^+(\sigma_2) \leq G_2^-(\sigma_1)$$

(ii) $$G_1^+(\sigma_1) \leq G_1^-(\sigma_2)\,.$$

(e) *Assume* K *is closed in* X*, and* G_1, G_2 *are lower semi-continuous on* K*. If there exist* $x_\sigma \in X_\sigma$ *and* x_∞ *such that*

$$\lim_{\sigma\to\infty} x_\sigma = x_\infty\,,$$

then

(i) $$G_1(x_\infty) = (G_1/G_2)\big(\mu(G_2)\big)$$

(ii) $$G_2(x_\infty) = \mu(G_2)\,.$$

If there exist $x_\sigma \in X_\sigma$ *and* x_0 *such that*

$$\lim_{\sigma\to 0^+} x_\sigma = x_0\,,$$

then

(iii) $$G_2(x_0) = (G_2/G_1)\big(\mu(G_1)\big)$$

(iv) $$G_1(x_0) = \mu(G_1)\,.$$

Proof. (a). By definition, $(G_1+G_2)(\sigma)$ is a nondecreasing function of σ. Consequently, since $(G_1+G_2)(\sigma)/\sigma=(G_2+G_1)(1/\sigma)$ this function in a nonincreasing function of σ. If there exists an $\hat{x}\in K$ for which $G_2(\hat{x})=0$, then

$$0\le (G_1+G_2)(\sigma)\le G_1(\hat{x})+\sigma G_2(\hat{x})=G_1(\hat{x})$$

for all σ. Thus $(G_1+G_2)(\sigma)$ is bounded. To prove the concavity, let $\sigma_1,\sigma_2\in(0,\infty)$, $\lambda\in[0,1]$, and $\sigma=\lambda\sigma_1+(1-\lambda)\sigma_2$. Then

$$\begin{aligned}
(G_1+G_2)(\sigma) &= \inf_{x\in K}\big(G_1(x)+\sigma G_2(x)\big)\\
&= \inf_{x\in K}\Big[\lambda\big(G_1(x)+\sigma_1 G_2(x)\big)+(1-\lambda)\big(G_1(x)+\sigma_2 G_2(x)\big)\Big]\\
&\ge \inf_{x\in K}\lambda\big(G_1(x)+\sigma_1 G_2(x)\big)+\inf_{x\in K}(1-\lambda)\big(G_1(x)+\sigma_2 G_2(x)\big)\\
&= \lambda(G_1+G_2)(\sigma_1)+(1-\lambda)(G_1+G_2)(\sigma_2)\,.
\end{aligned}$$

Thus $(G_1+G_2)(\sigma)$ is concave and nondecreasing on $(0,\infty)$. Every such function is necessarily continuous on $(0,\infty)$.

(b). Since $(G_1+G_2)(\sigma)$ is nondecreasing and bounded below (by zero) on $(0,\infty)$, the limit

$$\lim_{\sigma\to 0^+}(G_1+G_2)(\sigma)$$

necessarily exists. Furthermore,

$$\begin{aligned}
\lim_{\sigma\to 0^+}(G_1+G_2)(\sigma) &= \inf_{\sigma>0}(G_1+G_2)(\sigma)\\
&= \inf_{x\in K}\inf_{\sigma>0}\big(G_1(x)+\sigma G_2(x)\big)=\inf_{x\in K}G_1(x)=\mu(G_1)\,.
\end{aligned}$$

Since $\dfrac{1}{\sigma}(G_1+G_2)(\sigma)=(G_2+G_1)\Big(\dfrac{1}{\sigma}\Big)$, the second result of (b) is verified.

(c). Let $G_1(x^*)=\mu(G_1)$. Since $(G_1+G_2)(\sigma)$ is concave on $(0,\infty)$, we have for any $0<\sigma_0<\sigma_1<\sigma_2<\infty$,

$$\frac{(G_1+G_2)(\sigma_2)-(G_1+G_2)(\sigma_1)}{\sigma_2-\sigma_1}\le\frac{(G_1+G_2)(\sigma_1)-(G_1+G_2)(\sigma_0)}{\sigma_1-\sigma_0}\,.$$

Now,

$$(G_1 + G_2)(\sigma_1) \le G_1(x^*) + \sigma_1 G_2(x^*) = \mu(G_1) + \sigma_1 G_2(x^*)\,.$$

From (b), $\lim_{\sigma_0 \to 0+} (G_1 + G_2)(\sigma_0) = \mu(G_1)$. Substituting these two facts on the right hand side of the initial inequality, we obtain

$$\frac{(G_1 + G_2)(\sigma_2) - (G_1 + G_2)(\sigma_1)}{\sigma_2 - \sigma_1} \le \frac{\mu(G_1) + \sigma_1 G_2(x^*) - \mu(G_1)}{\sigma_1} = G_2(x^*)\,.$$

(d). Let $x_i \in X_{\sigma_i}$, $i = 1, 2$. Then by definition,

$$\begin{aligned} &G_1(x_1) + \sigma_1 G_2(x_1) \le G_1(x_2) + \sigma_1 G_2(x_2) \\ &= G_1(x_2) + \sigma_2 G_2(x_2) - (\sigma_2 - \sigma_1) G_2(x) \\ &\le G_1(x_1) + \sigma_2 G_2(x_1) - (\sigma_2 - \sigma_1) G_2(x_2)\,. \end{aligned}$$

Since $\sigma_2 - \sigma_1 > 0$, this implies that $G_2(x_2) \le G_2(x_1)$ which proves (i). Because

$$G_1(x_1) + \sigma_1 G_2(x_1) \le G_1(x_2) + \sigma_1 G_2(x_2)$$

and $G_2(x_2) \le G_2(x_1)$, we have $G_1(x_1) \le G_1(x_2)$, which proves (ii).

(e). We prove (i) and (ii). The proofs of (iii) and (iv) are totally analogous. To prove (ii), note that

$$\frac{1}{\sigma} G_1(x_\sigma) + G_2(x_\sigma) \le \frac{1}{\sigma} G_1(x) + G_2(x)$$

for any $x \in K$. Let $\sigma \to \infty$. Then in the limit,

$$G_2(x_\infty) \le G_2(x)$$

for all $x \in K$. Thus $G_2(x_\infty) = \mu(G_2)$. To prove (i), let $\hat{x}$ be any element of K such that $G_2(\hat{x}) = \mu(G_2)$. Then

$$G_1(x_\sigma) + \sigma G_2(x_\sigma) \le G_1(\hat{x}) + \sigma G_2(\hat{x}) \le G_1(\hat{x}) + \sigma G_2(x_\sigma)\,.$$

Thus $G_1(x_\sigma) \le G_1(\hat{x})$ for all $\sigma > 0$, implying that $G_1(x_\infty) \le G_1(\hat{x})$ for all $\hat{x} \in K$ satisfying $G_2(\hat{x}) = \mu(G_2)$. Thus

$$G_1(x_\infty) \le (G_1/G_2)(\mu(G_2))\,.$$

Now

$$(G_1/G_2)(\mu(G_2)) = \inf_{G_2(x)=\mu(G_2)} G_1(x) \le G_1(x_\infty)$$

since $G_2(x_\infty) = \mu(G_2)$. Thus $G_1(x_\infty) = (G_1/G_2)(\mu(G_2))$. ∎

3 – Operators on Banach Spaces

We shall discuss in some detail the function $(G_1 + G_2)(\sigma)$ for a class of G_1 and G_2. To this end, we introduce the following notation. Let X and Y be normed linear spaces, and T a bounded linear operator from X to Y. Associated with T is its adjoint T^*, a bounded linear operator from Y^* to X^*, where $X^*(Y^*)$ is the continuous dual of $X(Y)$. For a fixed $f \in X^*$, we set

$$G_1(g) = \|f - T^*g\|_{X^*} \tag{3.1}$$

and

$$G_2(g) = \|g\|_{Y^*} \tag{3.2}$$

for any $g \in Y^*$. Thus for $\sigma \ge 0$,

$$(G_1 + G_2)(\sigma) = \inf_{g \in Y^*} \|f - T^*g\|_{X^*} + \sigma\|g\|_{Y^*}\,. \tag{3.3}$$

We work with the dual spaces X^* and Y^*, and the adjoint operator T^* because in the next result we prove that the above infimum is attained (for $\sigma > 0$), and we also identify the dual problem.

Theorem 3.1. *Let X^*, Y^* and T^* be as above. For $f \in X^*$, $f \ne 0$, and $\sigma > 0$,*

$$\sup_{\substack{\|h\|_X \le 1 \\ \|Th\|_Y \le \sigma}} (f, h) = \min_{g \in Y^*} \|f - T^*g\|_{X^*} + \sigma\|g\|_{Y^*}\,. \tag{3.4}$$

Remark 3.1. By (f,h) we mean $f(h)$ since $f\in X^*$ and $h\in X$.

Proof. To prove the theorem we will use a result in Micchelli [14]. Namely, for every $\sigma>0$

$$\sup_{\substack{\|h\|_X\le 1\\ \|Th\|_Y\le\sigma}}(f,h)=\min_{g\in Y^*}\left\{\sup_{\|h\|_X\le 1}\big|(f,h)-(g,Th)\big|+\sigma\|g\|_{Y^*}\right\},$$

which actually holds in a more general setting, and is used in the theory of optimal recovery. Now,

$$\sup_{\|h\|_X\le 1}\big|(f,h)-(g,Th)\big|=\sup_{\|h\|_X\le 1}\big|(f-T^*g,h)\big|=\|f-T^*g\|_{X^*}$$

since $f-T^*g\in X^*$. Thus (3.4) holds. ∎

Generally, the supremum on the left side of (3.4) is not attained. As an example, let $X=l_1=c_0^*$, $Y=\mathbb{R}$, $\mathbf{f}=(1,1/2,1/3,1/4,\dots)\in c_0$, and $T\mathbf{h}=\sum_{n=1}^{\infty}h_n$. Then it is not difficult to verify that

$$\sup_{\substack{\|\mathbf{h}\|_{l_1}\le 1\\ |T\mathbf{h}|\le 1/2}}(\mathbf{f},\mathbf{h})=3/4\,, \tag{3.5}$$

but the supremum is not attained. To see that the supremum is at least 3/4 observe that

$$\mathbf{h}^n=(3/4,\dots,0,-1/4,0,\dots,0)\,, \tag{3.6}$$

satisfies $(\mathbf{f},\mathbf{h}^n)=3/4-1/4n\to 3/4$, $\|\mathbf{h}^n\|_{l_1}$, $T\mathbf{h}^n=1/2$. Next, we show that the supremum cannot exceed 3/4 and that is not attained. To this end, we observe that $(\mathbf{f},\mathbf{h}^n)\to(\mathbf{f},\mathbf{h}^0)=3/4$ where $\mathbf{h}^0=(3/4,0,0,\dots)$, $\|\mathbf{h}^0\|_{l_1}\le 1$ but $T\mathbf{h}^0=3/4$. Moreover, if $(\mathbf{f},\mathbf{h})\ge 3/4$, then $3/4\le\sum_1^{\infty}(f_i-1/2)h_i+1/2T\mathbf{h}\le\|\mathbf{f}-1/2\|_{l_\infty}\|\mathbf{h}\|_{l_1}+1/2|T\mathbf{h}|\le 1/2+1/4=3/4$. The vector $\mathbf{f}-1/2$ takes its maximum 1/2 only on its first component. Thus if $\mathbf{h}$ achieves the supremum in (3.5) then $\mathbf{h}=(\rho,0,\dots)$ for some constant ρ. Moreover, since we must have $\|\mathbf{h}\|_{l_1}=1$ we get $|\rho|=1$ while $|T\mathbf{h}|=1/2$ requires $|\rho|=1/2$, which is a contradiction. (Note that T is also compact.)

Throughout this section, we always demand that

Assumption I. *T is a bounded linear operator from X to Y where X has a pre-dual. The element f is in the pre-dual of X and*

$$K_\sigma = \{h : \|h\|_X \le 1, \|Th\|_Y \le \sigma\}$$

is weak**-closed in X for all $\sigma > 0$.*

Assumption I implies that the supremum in (3.4) is attained. It holds when Y has a pre-dual (recall that X is assumed to have a pre-dual) and there exists a bounded linear operator S from the pre-dual of Y to the pre-dual of X such that T it its adjoint. In this case T is continuous in the weak*-topologies. Thus, if h_n converges weak* to h_0 in S, (the unit ball in X), and $\|Th_n\|_Y \le \sigma$ for all n, then $\|Th_0\|_Y \le \sigma$. If, for example, X and Y are Hilbert spaces then this is necessarily satisfied by every bounded linear operator T from X to Y.

For case of exposition, rather than using the notation of the previous section, we set

$$E(\sigma) = \inf_{g \in Y^*} \|f - T^* g\|_{X^*} + \sigma \|g\|_{Y^*} \tag{3.7}$$

for $\sigma \in [0, \infty)$. For $\sigma > 0$ this infimum is attained (Theorem 3.1) and $g_\sigma \in Y^*$ will denote any function attaining this infimum. Note that the G_1 and G_2 of (3.1) and (3.2) satisfy the assumptions of the previous section. Thus, as a consequence of Proposition 2.4, we have

Proposition 3.2. (i) *$E(\sigma)$ is a bounded nondecreasing, concave, continuous function of σ on $(0, \infty)$ and $E(\sigma)/\sigma$ is nonincreasing.*
(ii) $\lim_{\sigma \to 0^+} E(\sigma) = \inf_{g \in Y^*} \|f - T^* g\|_{X^*}$. *Thus $E(\cdot)$ is continuous on $[0, \infty)$.*
(iii) *If there exists a $g_0 \in Y^*$ such that*

$$\|f - T^* g_0\|_{X^*} = \inf_{g \in Y^*} \|f - T^* g\|_{X^*}\,,$$

then for all $0 < \sigma_1 < \sigma_2 < \infty$,

$$\frac{E(\sigma_2) - E(\sigma_1)}{\sigma_2 - \sigma_1} \le \|g_0\|_{Y^*}\,.$$

(iv) *For* $0 < \sigma_1 < \sigma_2 < \infty$,

(a)
$$\|g_{\sigma_1}\|_{Y^*} \geq \|g_{\sigma_2}\|_{Y^*}$$
(b)
$$\|f - T^* g_{\sigma_1}\|_{X^*} \leq \|f - T^* g_{\sigma_2}\|_{X^*}\,.$$

(v) *For* $\sigma > 0$,

$$\|g_\sigma\|_{Y^*} = \min\left\{\|g_\sigma + g\|_{Y^*} : g \in Y^*,\ T^* g = 0\right\}.$$

REMARK 3.2. Statement (v) does not follow from Proposition 2.4, but is an immediate consequence of the definition of g_σ.

According to (3.4) we see that for $\sigma > \|T\|$ we have $E(\sigma) = \|f\|_{X^*}$. This suggest that we introduce the smallest value $\tilde{\sigma}$ in $[0,\infty)$ for which $E(\sigma)$ is a constant on $[\tilde{\sigma},\infty)$. $\tilde{\sigma}$ always exists and $E(\sigma) = \|f\|_{X^*}$ for $\sigma \geq \tilde{\sigma}$. In fact, we have the following proposition.

PROPOSITION 3.3. *Suppose Assumption I holds. Then* $E(\sigma) = \|f\|_{X^*}$ *if and only if* $\sigma \geq \tilde{\sigma}$, *where*

$$\tilde{\sigma} = \inf\left\{\frac{\|Th\|_Y}{\|h\|_X} : h \in X,\ h \neq 0,\ (f,h) = \|f\|_{X^*}\|h\|_X\right\}. \tag{3.8}$$

Furthermore, if $\tilde{\sigma} > 0$ *then the above infimum is attained.*

REMARK 3.3. Since, by Assumption I, f is in the pre-dual of X, there exists an $h \in X$, $h \neq 0$, satisfying $(f,h) = \|f\|_{X^*}\|h\|_X$. Thus the infimum in (3.8) is taken over a nonempty set.

PROOF. By Assumption I and (3.4), $E(\sigma) = \|f\|_{X^*}$ for $\sigma > 0$ if and only if

$$\|f\|_{X^*} = \max_{\substack{\|h\|_X \leq 1 \\ \|Th\|_Y \leq \sigma}} (f,h)\,. \tag{3.9}$$

($\Longrightarrow$). Assume $E(\sigma) = \|f\|_{X^*}$, $\sigma > 0$. Then there exists an $h \in X$, $h \neq 0$, which is admissible in (3.9), i.e., $\|h\|_X \leq 1$, $\|Th\|_Y \leq \sigma$, and $(f,h) = \|f\|_{X^*}$. Since

$$\|f\|_{X^*} = (f,h) \leq \|f\|_{X^*}\|h\|_X \leq \|f\|_{X^*}\,,$$

we have $\|h\|_X = 1$. Thus
$$\frac{\|Th\|_Y}{\|h\|_X} \le \sigma$$
which implies that $\tilde{\sigma} \le \sigma$.

($\Longleftarrow$). Assume $\sigma > \tilde{\sigma}$. Thus there exists an $\tilde{h} \in X$, $\tilde{h} \neq 0$, such that $(f, \tilde{h}) = \|f\|_{X^*}\|\tilde{h}\|_X$, and
$$\frac{\|T\tilde{h}\|_Y}{\|\tilde{h}\|_X} < \sigma \, .$$
Normalize $\tilde{h}$ so that $\|\tilde{h}\|_X = 1$. Thus $\|T\tilde{h}\|_Y < \sigma$ and so we also have
$$\|f\|_{X^*} = (f, \tilde{h}) \le \max_{\substack{\|h\|_X \le 1 \\ \|Th\|_Y \le \sigma}} (f, h) = E(\sigma) \, .$$
But $E(\sigma) \le \|f\|_{X^*}$ for every $\sigma \ge 0$. Thus
$$E(\sigma) = \|f\|_{X^*} \, .$$

If $\tilde{\sigma} > 0$, then $E(\tilde{\sigma}) = \|f\|_{X^*}$ and from the first part of the proof of this proposition, there exists an $h \in X$, $h \neq 0$, satisfying $\|h\|_X = 1$, $\|Th\|_Y \le \tilde{\sigma}$ and $(f, h) = \|f\|_{X^*}\|h\|_X$. If $\|Th\|_Y < \tilde{\sigma}$, we contradict our definition of $\tilde{\sigma}$. The infimum in (3.8) is attained. ∎

Remark 3.4. It may happen that $\tilde{\sigma} = 0$, i.e., $E(\sigma) = \|f\|_{X^*}$ for all $\sigma \ge 0$. This will occur if and only if
$$\inf_{g \in Y^*} \|f - T^*g\|_{X^*} = \|f\|_{X^*} \, .$$

If $f = T^*g^*$ for some $g^* \in Y^*$ ($g^* \neq 0$), then $E(0) = 0$. It is then also obvious that for all $\sigma > 0$,

$$E(\sigma) \le \sigma\|g^*\|_{Y^*} \, . \tag{3.10}$$

If equality holds in (3.10) for some $\sigma = \sigma^*$, then necessarily
$$E(\sigma) = \sigma\|g^*\|_{Y^*}$$
for all $0 \le \sigma \le \sigma^*$. This is an immediate consequence of either (iii) or (iv) (b) of Proposition 3.2. It is therefore of interest to characterize the

largest possible value of σ^* for which (3.10) holds. This largest possible value we will denote by $\hat{\sigma}$. If $f \neq T^*g$ for any $g \in Y^*$, then $\hat{\sigma}$ does not exist. If $f = T^*g^*$ for some $g^* \in Y^*$, then $\hat{\sigma}$ does exist, but may equal 0. Prior to characterizing $\hat{\sigma}$, we present a lemma which will be used in subsequent results.

LEMMA 3.4. *Assume* $\sigma > 0$.
i) *If* $g_0 \neq 0$, *then there exists an* $h_\sigma \in X$ *satisfying* $\|h_\sigma\|_X \leq 1$, $\|Th_\sigma\|_Y = \sigma$, *and*

$$(g_\sigma, Th_\sigma) = \sigma \|g_\sigma\|_{Y^*} \,. \tag{3.11}$$

ii) *If* $f \neq T^*g_\sigma$, *then there exists an* $h_\sigma \in X$ *satisfying* $\|h_\sigma\|_X = 1$, $\|Th_\sigma\|_Y \leq \sigma$, *and*

$$(f - T^*g_\sigma, h_\sigma) = \|f - T^*g_\sigma\|_{X^*} \,. \tag{3.12}$$

iii) *If* $g_\sigma \neq 0$ *and* $f \neq T^*g_\sigma$, *then there exists an* $h_\sigma \in X$ *satisfying* $\|h_\sigma\|_X = 1$, $\|Th_\sigma\|_Y = \sigma$, (3.11) *and* (3.12). *Thus*

$$\sigma = \frac{\|Th_\sigma\|_Y}{\|h_\sigma\|_X} \,.$$

PROOF. Statement (iii) is a simple consequence of (i) and (ii), and is stated for convenience. Since $\sigma > 0$, from Assumption I and Theorem 3.1,

$$\max_{\substack{\|h\|_X \leq 1 \\ \|Th\|_Y \leq \sigma}} (f, h) = \|f - T^*g_\sigma\|_{X^*} + \sigma \|g_\sigma\|_{Y^*} \,.$$

Let h_σ attain the maximum on the left hand side. Then $\|h_\sigma\|_X \leq 1$, $\|Th_\sigma\|_Y \leq \sigma$, and

$$\begin{aligned} &\|f - T^*g_\sigma\|_{X^*} + \sigma\|g_\sigma\|_{Y^*} = (f, h_\sigma) = (f - T^*g_\sigma, h_\sigma) + (T^*g_\sigma, h_\sigma) \\ &= (f - T^*g_\sigma, h_\sigma) + (g_\sigma, Th_\sigma) \leq \|f - T^*g_\sigma\|_{X^*}\|h_\sigma\|_X + \|g_\sigma\|_{Y^*}\|Th_\sigma\|_Y \\ &\leq \|f - T^*g_\sigma\|_{X^*} + \sigma\|g_\sigma\|_{Y^*} \,. \end{aligned}$$

If $\|g_\sigma\|_{Y^*} \neq 0$, we must have

$$\sigma \|g_\sigma\|_{Y^*} = (g_\sigma, Th_\sigma) = \|g_\sigma\|_{Y^*} \|Th_\sigma\|_Y \,.$$

Thus $\|Th_\sigma\|_Y = \sigma$ and (3.11) holds.
If $f \neq T^* g_\sigma$, i.e., $\|f - T^* g_\sigma\|_{X^*} \neq 0$ we must have

$$\|f - T^* g_\sigma\|_{X^*} = (f - T^* g_\sigma, h_\sigma) = \|f - T^* g_\sigma\|_{X^*}\|h_\sigma\|_X \,.$$

Thus $\|h_\sigma\|_X = 1$ and (3.12) holds. ∎

We now present a characterization of $\hat{\sigma}$.

Proposition 3.5. *Suppose $f = T^* g^*$ and $f \neq 0$. Then $E(\sigma) = \sigma\|g^*\|_{Y^*}$ for some $\sigma > 0$ if and only if there is an h with $Th \neq 0$ such that $(g^*, Th) = \|g^*\|_{Y^*}\|Th\|_Y$. Moreover, in this case*

$$\|g^*\|_{Y^*} = \min\left\{\|g^* + g\|_{Y^*} : g \in Y^*,\ T^* g = 0\right\} \tag{3.13}$$

and

$$\hat{\sigma} = \max\left\{\frac{\|Th\|_Y}{\|h\|_X} : (g^*, Th) = \|g^*\|_{Y^*}\|Th\|_Y,\ Th \neq 0\right\} \tag{3.14}$$

is the largest value for which $E(\sigma) = \sigma\|g^*\|_{Y^*}$ for $\sigma \in [0, \hat{\sigma}]$.

Proof. Assume $E(\sigma) = \sigma\|g^*\|_{Y^*}$ for some $\sigma > 0$ where $f = T^* g^*$. That is, $g_\sigma = g^*$. From Proposition 3.2 (v), we obtain (3.13). Since $g_\sigma \neq 0$, we have from Lemma 3.4 (i) the existence of $h_\sigma \in X$ satisfying $\|h_\sigma\|_X \leq 1$, $\|Th_\sigma\|_Y = \sigma$, and

$$(g^*, Th_\sigma) = \|g^*\|_{Y^*}\|Th_\sigma\|_Y \,.$$

Thus

$$\sigma \leq \frac{\|Th_\sigma\|_Y}{\|h_\sigma\|_X} \leq \hat{\sigma} \,.$$

Conversely, assume that there is an h such that $Th \neq 0$ and $(g^*, Th) = \|g^*\|_{Y^*}\|Th\|_Y$. Choose an $\hat{h} \in X$ such that $(g^*, T\hat{h}) = \|g^*\|_{Y^*}\|T\hat{h}\|_Y$, and

$$\hat{\sigma} = \frac{\|T\hat{h}\|_Y}{\|\hat{h}\|_X}$$

where $\hat{\sigma}$ is defined by (3.14). Normalize $\hat{h}$ so that $\|\hat{h}\|_X = 1$. Thus $\|T\hat{h}\|_Y = \hat{\sigma}$. Therefore

$$\begin{aligned}\hat{\sigma}\|g^*\|_{Y^*} \geq E(\hat{\sigma}) &= \max_{\substack{\|h\|_X \leq 1 \\ \|Th\|_Y \leq \hat{\sigma}}} (f, h) = \max_{\substack{\|h\|_X \leq 1 \\ \|Th\|_Y \leq \hat{\sigma}}} (T^* g^*, h) \\ &\geq (T^* g^*, \hat{h}) = (g^*, T\hat{h}) = \hat{\sigma}\|g^*\|_{Y^*}\,.\end{aligned}$$

Thus $E(\hat{\sigma}) = \hat{\sigma}\|g^*\|_{Y^*}$. From Proposition 3.2, (iii) or (iv) (b), we get $E(\sigma) = \sigma\|g^*\|_{Y^*}$ for all $\sigma \in [0, \hat{\sigma}]$. Moreover, for any $g \in Y^*$ with $T^* g = 0$ we have

$$\|g^*\|_{Y^*}\|T\hat{h}\|_Y = \big|(g^*, T\hat{h})\big| = \big|(g^* + g, T\hat{h})\big| \leq \|T\hat{h}\|_Y \|g^* + g\|_{Y^*}\,.$$

This proves (3.13). □

We introduce the number σ' which we set equal to 0 if $\hat{\sigma}$ does not exist and otherwise set $\sigma' = \hat{\sigma}$. Thus we have identified $E(\sigma)$ for $\sigma \geq \tilde{\sigma}$ and for $\sigma \leq \sigma'$. There is, in general, no formula which explicitly gives $E(\sigma)$ for $\sigma \in (\sigma', \tilde{\sigma})$. However, we do have the following result.

Proposition 3.6. *Assume $\sigma \in (\sigma', \tilde{\sigma})$. Then $g_\sigma \in Y^*$ is a solution to $E(\sigma)$ if and only if $g_\sigma \neq 0$, $f \neq T^* g_\sigma$, and there exists an $h_\sigma \in X$ with $Th_\sigma \neq 0$ satisfying*

$$(f - T^* g_\sigma, h_\sigma) = \|f - T^* g_\sigma\|_{X^*}\|h_\sigma\|_X \tag{3.15}$$

$$(g_\sigma, Th_\sigma) = \|g_\sigma\|_{Y^*}\|Th_\sigma\|_Y\,. \tag{3.16}$$

Furthermore,

$$\sigma = \frac{\|Th_\sigma\|_Y}{\|h_\sigma\|_X}\,. \tag{3.17}$$

Proof. ($\Longrightarrow$). Since $\sigma < \tilde{\sigma}$, it follows that $g_\sigma \neq 0$ while $\sigma > \sigma'$ insures that $f \neq T^* g_\sigma$. Thus (3.15) - (3.17) follow from (iii) of Lemma 3.4.

($\Longleftarrow$). Assume h_σ and g_σ are as above satisfying (3.15) - (3.17) and normalize h_σ so that $\|h_\sigma\|_X = 1$. Thus by (3.17) $\|Th_\sigma\|_Y = \sigma$ and so

$$\begin{aligned} E(\sigma) = \max_{\substack{\|h\|_X \le 1 \\ \|Th\|_Y \le \sigma}} (f,h) \ge (f,h_\sigma) &= (f - T^* g_\sigma, h_\sigma) + (g_\sigma, Th_\sigma) \\ &= \|f - T^* g_\sigma\|_{X^*} + \sigma \|g_\sigma\|_{Y^*} \end{aligned}$$

from which we conclude that g_σ is a solution to $E(\sigma)$. ∎

REMARK 3.5. The converse direction does not imply that $\hat{\sigma} < \sigma < \tilde{\sigma}$. However, from Proposition 3.2, (iv), it follows that if (3.15) - (3.17) hold for $g_\sigma \ne 0$ and $f \ne T^* g_\sigma$, then $\hat{\sigma} \le \sigma \le \tilde{\sigma}$.

Proposition 3.4 delineates the infinite interval on which $E(\sigma)$ is a constant. Proposition 3.5 (if $\hat{\sigma} > 0$ exists) gives us an interval on which $E(\sigma)$ is linear. Is it true that $E(\sigma)$ is strictly concave on the complement of these two intervals? The next result gives a condition implying the strict concavity of $E(\sigma)$ for $\sigma \in (\sigma', \tilde{\sigma})$.

PROPOSITION 3.7. *Let $I = (\sigma', \tilde{\sigma})$. If X is strictly convex, then E is strictly concave on I.*

PROOF. If $E(\sigma)$ is not strictly concave on I, then $E(\sigma)$ is linear on some proper subinterval $[\sigma_1, \sigma_2]$ of I. For $\lambda \in [0,1]$, set $\sigma_\lambda = \lambda\sigma_1 + (1 - \lambda)\sigma_2$. Then

$$E(\sigma_\lambda) = \lambda E(\sigma_1) + (1-\lambda) E(\sigma_2)$$

for $\lambda \in [0,1]$. From the definition of $E(\sigma)$, it follows that if $g_{\sigma_\lambda} \in Y^*$ is a solution to $E(\sigma_\lambda)$ for some $\lambda \in (0,1)$, then it is also a solution to $E(\sigma_1)$ and $E(\sigma_2)$ and thus to $E(\sigma)$ for all $\sigma \in [\sigma_1, \sigma_2]$. Set $g_{\sigma_\lambda} = \hat{g}$ so that

$$E(\sigma) = \|f - T^* \hat{g}\|_{X^*} + \sigma \|\hat{g}\|_{Y^*}$$

for all $\sigma \in [\sigma_1, \sigma_2]$.

Now by Proposition 3.6, $\hat{g} \ne 0$ and $f \ne T^* \hat{g}$ and thus there exists for each $\sigma \in [\sigma_1, \sigma_2]$ an $h_\sigma \in X$, $Th_\sigma \ne 0$, satisfying (3.15), (3.16) and (3.17). Normalize h_σ so that $\|h_\sigma\|_X = 1$. Since we also have $\|Th_\sigma\|_Y = \sigma$, we conclude that $h_{\sigma_1} \ne \alpha h_{\sigma_2}$ for any $\alpha \in \mathbb{R}$. Let $\lambda \in (0,1)$, and

$$\hat{h}_{\sigma_\lambda} = \lambda h_{\sigma_1} + (1-\lambda) h_{\sigma_2}\,.$$

Then $\|T\hat{h}_{\sigma_\lambda}\|_Y \le \sigma_\lambda$, and since X is strictly convex $\|\hat{h}_{\sigma_\lambda}\|_X < 1$. Thus

$$\begin{aligned} E(\sigma_\lambda) = \max_{\substack{\|h\|_X \le 1 \\ \|Th\|_Y \le \sigma}} (f,h) \ge (f,\hat{h}_{\sigma_\lambda}) = \lambda(f,h_{\sigma_1}) + (1-\lambda)(f,h_{\sigma_2}) \\ = \lambda E(\sigma_1) + (1-\lambda)E(\sigma_2) = E(\sigma_\lambda) \end{aligned}$$

and since equality holds throughout we get

$$E(\sigma_\lambda) = (f,\hat{h}_{\sigma_\lambda}) .$$

But

$$\begin{aligned} (f,\hat{h}_{\sigma_\lambda}) = (f - T^*\hat{g},\hat{h}_{\sigma_\lambda}) + (\hat{g}, T\hat{h}_{\sigma_\lambda}) \le \|f - T^*\hat{g}\|_{X^*}\|\hat{h}_{\sigma_\lambda}\|_X \\ + \|\hat{g}\|_{Y^*}\|T\hat{h}_{\sigma_\lambda}\|_Y < \|f - T^*\hat{g}\|_{X^*} + \sigma_\lambda\|\hat{g}\|_{Y^*} = E(\sigma_\lambda) , \end{aligned}$$

a contradiction which proves the proposition. □

REMARK 3.6. If T is one-to-one and Y is strictly convex, then this same result also holds.

Let us now return to the case where $f = T^*g^*$ for some $g^* \in Y^*$ satisfying (3.13). In this case we have the following inequalities.

LEMMA 3.8. *If* $f = T^*g^*$, $g^* \in Y^*$ *satisfying* (3.13), $f \ne 0$, *then*

$$\sigma' \le \frac{\|T^*g^*\|_{X^*}}{\|g^*\|_{Y^*}} \le \tilde{\sigma} .$$

PROOF. By (3.8),

$$\tilde{\sigma} = \inf\left\{ \frac{\|Th\|_Y}{\|h\|_X} : h \in X,\ h \ne 0,\ (f,h) = \|f\|_{X^*}\|h\|_X \right\}.$$

Let $h \in X$, $h \ne 0$, satisfy

$$(f,h) = \|f\|_{X^*}\|h\|_X .$$

Such h exist since f is in the pre-dual of X. By definition, $f = T^*g^*$. Thus

$$(T^*g^*, h) = \|T^*g^*\|_{X^*}\|h\|_X$$

and furthermore,

$$(T^*g^*, h) = (g^*, Th) \le \|g^*\|_{Y^*}\|Th\|_Y\,.$$

Therefore

$$\frac{\|T^*g^*\|_{X^*}}{\|g^*\|_{Y^*}} \le \frac{\|Th\|_Y}{\|h\|_X}$$

implying that

$$\frac{\|T^*g^*\|_{X^*}}{\|g^*\|_{Y^*}} \le \tilde{\sigma}\,.$$

There is nothing to prove unless $\hat{\sigma} > 0$. In this case (3.14) implies

$$\hat{\sigma} = \max\left\{\frac{\|Th\|_Y}{\|h\|_X} : (g^*, Th) = \|g^*\|_{Y^*}\|Th\|_Y,\ h \ne 0\right\}.$$

Moreover, if $h \in X$, $h \ne 0$, and

$$(g^*, Th) = \|g^*\|_{Y^*}\|Th\|_Y\,,$$

then since

$$(g^*, Th) = (T^*g^*, h) \le \|T^*g^*\|_{X^*}\|h\|_X\,,$$

we have

$$\frac{\|Th\|_Y}{\|h\|_X} \le \frac{\|T^*g^*\|_{X^*}}{\|g^*\|_{Y^*}}$$

and consequently

$$\hat{\sigma} \le \frac{\|T^*g^*\|_{X^*}}{\|g^*\|_{Y^*}}\,.$$

□

As a simple consequence of Lemma 3.8, and its proof, we can identify when $\tilde{\sigma} = \sigma'$. Note that from Lemma 3.8, when $f = T^*g^*$, $f \ne 0$, we must have $\tilde{\sigma} > 0$.

PROPOSITION 3.9. *Let $f = T^*g^*$, $g^* \in Y^*$ satisfying* (3.13)*, $f \ne 0$. Then $\tilde{\sigma} = \hat{\sigma}$ (> 0) if and only if there exists an $h^* \ne 0$, satisfying*

$$\|g^*\|_{Y^*}\|Th^*\|_Y = (g^*, Th^*) = (T^*g^*, h^*) = \|T^*g^*\|_{X^*}\|h^*\|_X\,. \tag{3.18}$$

In this case

$$\tilde{\sigma} = \hat{\sigma} = \frac{\|T^*g^*\|_{X^*}}{\|g^*\|_{Y^*}}\,. \tag{3.19}$$

Proof. ($\Longrightarrow$). If $\tilde{\sigma} = \hat{\sigma}$, then from Lemma 3.8, (3.19) holds. Let $h^* \in X$, $h^* \neq 0$, satisfy (3.8) for $\tilde{\sigma}$ (> 0). Then

$$\frac{\|Th^*\|_Y}{\|h^*\|_X} = \tilde{\sigma} = \frac{\|T^*g^*\|_{X^*}}{\|g^*\|_{Y^*}} \tag{3.20}$$

and $(T^*g^*, h^*) = \|T^*g^*\|_{X^*}\|h^*\|_X$. Since $(T^*g^*, h^*) = (g^*, Th^*)$ and $\|T^*g^*\|_{X^*}\|h^*\|_X = \|Th^*\|_Y\|g^*\|_{Y^*}$ by (3.20), we obtain (3.18).

($\Longleftarrow$). Assume $h^* \in X$, $h^* \neq 0$, satisfies (3.18). Then from Propositions 3.3 and 3.5 (with the h^* of (3.18)),

$$\tilde{\sigma} \le \frac{\|Th^*\|_Y}{\|h^*\|_X} \le \hat{\sigma}\,.$$

However $\hat{\sigma} \le \tilde{\sigma}$ by definition. Thus $\tilde{\sigma} = \hat{\sigma}$, and (3.19) is a consequence of Lemma 3.8. ∎

Remark 3.7. An application of Proposition 3.9 is given in the next section.

We now turn our attention to the question of the uniqueness of the solution g_σ to $E(\sigma)$.

Proposition 3.10. *If X^* and Y^* are strictly convex, then for $\sigma \neq \tilde{\sigma}$, the solution g_σ for $E(\sigma)$ is unique.*

Proof. *Case 1.* $\sigma > \tilde{\sigma}$.

From Proposition 3.2, (iv) (a), if $0 < \sigma_1 < \sigma_2 < \infty$ and g_{σ_i} is any solution for $E(\sigma_i)$, $i = 1, 2$, then $\|g_{\sigma_1}\|_{Y^*} \ge \|g_{\sigma_2}\|_{Y^*}$. Since $g_{\tilde{\sigma}} = 0$ is a solution for $E(\tilde{\sigma})$ it follows that $g_\sigma = 0$ is the only solution for $E(\sigma)$ for $\sigma > \tilde{\sigma}$.

Case 2. $0 < \sigma < \tilde{\sigma}$.

Assume $0 < \sigma < \tilde{\sigma}$ and g_1, g_2 are two distinct solutions for $E(\sigma)$ with $g_1 \neq g_2$. Since $\sigma < \tilde{\sigma}$, we have $g_1 \neq 0$ and $g_2 \neq 0$. The solution set of $E(\sigma)$ is convex, and in fact, if $\lambda \in [0, 1]$, and $g_\lambda = \lambda g_1 + (1 - \lambda) g_2$, then the triangle inequality gives

$$\|f - T^*g_\lambda\|_{X^*} = \lambda\|f - T^*g_1\|_{X^*} + (1 - \lambda)\|f - T^*g_2\|_{X^*} \tag{3.21}$$

and

$$\|g_\lambda\|_{Y^*} = \lambda\|g_1\|_{Y^*} + (1-\lambda)\|g_2\|_{Y^*}\,. \tag{3.22}$$

Since Y^* is strictly convex, we have from (3.22) that $g_1 = ag_2$, $a > 0$. If $a = 1$ we are finished, otherwise we have

$$g_1 = ag_2, \qquad a > 0, \qquad a \neq 1\,. \tag{3.23}$$

If $T^*g_1 = T^*g_2$, then a contradiction ensues from (3.23) (since $g_1 \neq 0$, $g_2 \neq 0$) unless $T^*g_1 = T^*g_2 = 0$. But in this latter case we must have $E(\sigma) = \|f\|_{X^*}$ from which it would follow that $\sigma \geq \tilde{\sigma}$, a contradiction. Since $T^*g_1 \neq T^*g_2$, and X^* is strictly convex, we have from (3.22) that $f - T^*g_1$ and $f - T^*g_2$ are linearly dependent. By interchanging the roles of g_1 and g_2, if necessary, we conclude that there is a positive constant b such that

$$f - T^*g_1 = b(f - T^*g_2)\,. \tag{3.24}$$

Since $T^*g_1 \neq T^*g_2$ obviously $b \neq 1$.

Now, from (3.23) and (3.24)

$$\begin{aligned}\|f - T^*g_2\|_{X^*} + \sigma\|g_2\|_{Y^*} &= \|f - T^*g_1\|_{X^*} + \sigma\|g_1\|_{Y^*}\\ &= b\|f - T^*g_2\|_{X^*} + \sigma a\|g_2\|_{Y^*}\,.\end{aligned}$$

Thus

$$(1-b)\|f - T^*g_2\|_{X^*} = \sigma(a-1)\|g_2\|_{Y^*}$$

with $a, b > 0$, $a \neq 1$, $b \neq 1$. Furthermore, from (3.24)

$$(1-b)f = T^*g_1 - bT^*g_2 = (a-b)T^*g_2$$

and consequently

$$\begin{aligned}\sigma\frac{(a-1)}{(1-b)}\|g_2\|_{Y^*} = \|f - T^*g_2\|_{X^*} &= \left\|\left(\frac{a-b}{1-b}\right)T^*g_2 - T^*g_2\right\|_{X^*}\\ &= \left\|\left(\frac{a-1}{1-b}\right)T^*g_2\right\|_{X^*}\,,\end{aligned}$$

which simplifies to

$$\sigma\|g_2\|_{Y^*} = \|T^*g_2\|_{X^*}\,.$$

Since $\sigma < \tilde{\sigma}$, we have $E(\sigma) < \|f\|_{X^*}$ and so

$$\begin{aligned}\|f\|_{X^*} > E(\sigma) &= \|f - T^*g_2\|_{X^*} + \sigma\|g_2\|_{Y^*}\\ &\geq \|f\|_{X^*} - \|T^*g_2\|_{X^*} + \sigma\|g_1\|_{Y^*} = \|f\|_{X^*}\,.\end{aligned}$$

This contradiction proves the proposition. ∎

Remark 3.8. For $\sigma > \tilde{\sigma}$, no assumption on X^* or Y^* are needed to prove that $g_\sigma = 0$ is unique solution for $E(\sigma)$. If $f = T^*g^*$, $g^* \in Y^*$ satisfies (3.13), $f \neq 0$, and $\hat{\sigma} > 0$, then using Proposition 3.2, (iv) (b), it follows that $g_\sigma = g^*$ is the unique solution for $E(\sigma)$ for $\sigma < \hat{\sigma}$ if T^* is $1-1$ or Y^* is strictly convex.

4 – An Example

Let X and Y be Hilbert spaces, and T a compact operator from X to Y. In this case T satisfies Assumption I and from Proposition 3.3, it immediately follows that $\tilde{\sigma} = \|Tf\|_Y/\|f\|_X$. The identification of $\hat{\sigma}$ is not as simple. However, it is possible to identify exactly when $\tilde{\sigma} = \hat{\sigma}$. To this end, we recall that the operator TT^* from Y into itself is compact, self-adjoint, and nonnegative. It has eigenvectors and eigenvalues (which are nonnegative and whose square roots are called the singular values or s-numbers of T).

Proposition 4.1. *Let $f \in X$, $f \neq 0$. Then under the above assumptions, $\tilde{\sigma} = \hat{\sigma}$ (> 0) if and only if $f = T^*g^*$ where*

$$TT^*g^* = \lambda g^*$$

for some $\lambda > 0$. In this case $\tilde{\sigma} = \hat{\sigma} = \lambda^{1/2}$.

REMARK 4.1. Note that in this case both the functions 0 and g^* are solutions to $E(\sigma)$ for $\sigma = \tilde{\sigma} = \hat{\sigma}$. The solution set is convex and thus αg^* is a solution for all $\alpha \in [0,1]$ (see Proposition 3.10).

PROOF. ($\Longrightarrow$). Assume $\tilde{\sigma} = \hat{\sigma}$ and $f = T^*g^*$, $g^* \in Y$ satisfying (3.13). From Proposition 3.9 there exists an $h^* \in X$, $h^* \neq 0$, satisfying (3.18). The equalities in (3.18) imply that $Th^* = \alpha g^*$ and $T^*g^* = \beta h^*$ for some $\alpha, \beta > 0$. Thus

$$TT^*g^* = \lambda g^*$$

where $\lambda = \alpha\beta > 0$. From (3.19) it easily follows that $\tilde{\sigma} = \hat{\sigma} = \lambda^{1/2}$.

($\Longleftarrow$). Assume $f = T^*g^*$, and $TT^*g^* = \lambda g^*$ for some $\lambda > 0$. Set $h^* = T^*g^*$. Then it follows that

$$(h^*, h^*) = (T^*g^*, h^*) = (g^*, Th^*) = (g^*, TT^*g^*) = \lambda(g^*, g^*)\,.$$

Furthermore,

$$\|g^*\|_Y \|Th^*\|_Y = \|g^*\|_Y \|TT^*g^*\|_Y = \lambda \|g^*\|_Y^2 = \lambda(g^*, g^*)\,,$$

while

$$\|T^*g^*\|_X \|h^*\|_X = \|h^*\|_X \|h^*\|_X = \|h^*\|_X^2 = (h^*, h^*)\,.$$

Thus (3.18) holds, and from (3.19) $\tilde{\sigma} = \hat{\sigma} = \lambda^{1/2}$. □

Let us now assume that $\sigma \in (\sigma', \tilde{\sigma})$ and proceed to characterize g_σ. From Proposition 3.6, there exists an $h_\sigma \in X$, $Th_\sigma \neq 0$, satisfying (3.15) - (3.17). In particular (3.15) gives

$$\alpha h_\sigma = f - T^*g_\sigma$$

for some $\alpha > 0$ (since $f \neq T^*g_\sigma$), and from (3.16)

$$\beta g_\sigma = Th_\sigma$$

for some $\beta > 0$ (since $g_\sigma \neq 0$). Thus it follows that for some positive constant $\lambda(\sigma)$

$$T(f - T^*g_\sigma) = \lambda(\sigma) g_\sigma\,. \tag{4.1}$$

Since TT^* is compact, self-adjoint and nonnegative, the operator $TT^*+\lambda I$ is invertible for all $\lambda > 0$ and so

$$g_\sigma = \big(TT^* + \lambda(\sigma)\big)^{-1} Tf\,. \tag{4.2}$$

From (4.2) and (3.17) it may be shown that

$$\lambda(\sigma) = \sigma \frac{\|f - T^* g_\sigma\|_X}{\|g_\sigma\|_Y}\,. \tag{4.3}$$

If (4.1) and (4.3) hold, then so do (3.15), (3.16) and (3.17) for $h_\sigma = f - T^* g_\sigma$. Thus these equations determine g_σ. It remains to explain how the function $\lambda(\sigma)$ is obtained. To this end, note that from Proposition 3.2, (iv) (a), $\|g_\sigma\|_Y$ is a nonincreasing function of σ, and from Proposition 3.2, (iv) (b), $\|f - T^* g_\sigma\|_X$ is a nondecreasing function of σ. Thus for $\sigma \in (\sigma', \tilde\sigma)$, the function $\lambda(\sigma)$ is a strictly increasing function of σ. Furthermore, it is not difficult to ascertain that $\lambda(\sigma)$ is also a continuous function of σ in $(\sigma', \tilde\sigma)$, and

$$\lim_{\sigma \uparrow \tilde\sigma} \lambda(\sigma) = \infty$$

(since $\lim_{\sigma \uparrow \tilde\sigma} \|g_\sigma\|_Y = 0$), while

$$\lim_{\sigma \downarrow \sigma'} \lambda(\sigma) = 0\,.$$

For $\lambda > 0$, set

$$\hat g_\lambda = (TT^* + \lambda I)^{-1} Tf$$

and define

$$\zeta(\lambda) = \frac{\lambda \|\hat g_\lambda\|_Y}{\|f - T^* \hat g_\lambda\|_X}\,.$$

Then according to (4.2) $\zeta\big(\lambda(\sigma)\big) = \sigma$ for $\sigma \in (\sigma', \tilde\sigma)$. That is, ζ is the inverse of λ and hence in this sense determines λ. More can be said about g_σ when $f = T^* g^*$, for some $g^* \in Y \backslash \{0\}$ such that $TT^* g^* \neq \lambda g^*$ for all $\lambda > 0$. In this case, $\sigma' < \tilde\sigma$. Let $\{\lambda_i\}_{i=1}$ denote the nonzero eigenvalues of TT^* listed to their algebraic multiplicity. Let $\{g_i\}_{i=1}$ denote a corresponding set of orthonormal eigenvectors, that is,

$$TT^* g_i = \lambda_i g_i\,.$$

Since the $\{g_i\}_{i=1}$ are complete in the range of TT^*, and since we may assume that g^* satisfies (3.13), it then follows that

$$g^* = \sum_i \alpha_i g_i$$

for some $\{\alpha_i\}_{i=1}$ in l_2.

For $\sigma' < \sigma < \tilde{\sigma}$, it follows from (4.2), or using the results of the previous section, that

$$g_\sigma = \sum_i \left(\frac{\lambda_i \alpha_i}{\lambda_i + \lambda}\right) g_i\,,$$

where

$$\sigma = \frac{\left(\sum_i \dfrac{\lambda_i^2 \alpha_i^2}{(\lambda_i+\lambda)^2}\right)^{1/2}}{\left(\sum_i \dfrac{\lambda_i \alpha_i^2}{(\lambda_i+\lambda)^2}\right)^{1/2}}\,.$$

Note that as $\lambda \uparrow \infty$, the above quantity tends to

$$\tilde{\sigma} = \frac{\left(\sum_i \lambda_i^2 \alpha_i^2\right)^{1/2}}{\left(\sum_i \lambda_i \alpha_i^2\right)^{1/2}}\,,$$

while as $\lambda \downarrow 0$, it tends to

$$\hat{\sigma} = \frac{\left(\sum_i \alpha_i^2\right)^{1/2}}{\left(\sum_i \alpha_i^2/\lambda_i\right)^{1/2}}\,,$$

which is understood to be zero if the series in the denominator diverges.

There are two additional examples of the results from Section 3 which we have examined in detail. The first is the matrix extremal problem;

$$E_{pq}(\sigma) = \min_{\mathbf{g}} \|\mathbf{f} - D\mathbf{g}\|_p + \sigma \|\mathbf{g}\|_q \tag{4.4}$$

where $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$, $p, q \in [1, \infty]$, $\sigma \geq 0$, and D is an $n \times n$ diagonal matrix given by $D = \mathrm{diag}\{d_1, \ldots, d_n\}$. By $\|\cdot\|_p$ and $\|\cdot\|_q$ we mean the usual l_p^n and l_q^n norms, respectively. A continuous analogue of this is the following

$$\min_{g \in L^q} \|f - dg\|_p + \sigma \|g\|_q$$

where d is a fixed function.

Let $\nu_1, \ldots, \nu_n \in C(\Omega)$, where Ω is a compact subset of $\mathbb{R}^m$. For $1 \leq p \leq \infty$, $1 < q \leq \infty$, $\mathbf{f} = (f_1, \ldots, f_n) \in \mathbb{R}^n$, and $\sigma > 0$, define

$$E_{pq}(\sigma) = \min_{g \in L^q} \left(\sum_{i=1}^{n} \left| f_i - \int_{\Omega} v_i(y) g(y) dy \right|^p \right)^{1/p} + \sigma \|g\|_q \tag{4.5}$$

where $L^q = L^q(\Omega)$ is the usual L^q-space with associated Lebesgue measure on Ω.

For $q = 1$ instead of (4.5) the appropriate form is

$$E_{p1}(\sigma) = \min_{\mu \in C^*(\Omega)} \left(\sum_{i=1}^{n} \left| f_i - \int_{\Omega} v_i(y) d\mu(y) \right|^p \right)^{1/p} + \sigma \|\mu\|_{T.V.} \tag{4.6}$$

where $C^*(\Omega)$ denotes the dual space of $C(\Omega)$. We identify this as the space of all Borel measures μ on Ω, with norm denoted $\|\mu\|_{T.V.}$, the total variation of μ over Ω. Both problem (4.4) and (4.5) - (4.6) have been studied in detail in Micchelli and Pinkus [17].

5 – Totally Positive Kernels

We consider in this section the particular case of our general problem where the operator T^* is given by an integral operator whose kernel is strictly totally positive (STP). Before dealing with the specifics of this problem, we first present the general framework.

We let $[a, b]$ or $[c, d]$ denote closed nontrivial intervals of $\mathbb{R}$. For $1 \leq p \leq \infty$, L^q will be the usual space with associated Lebesgue measure on either $[a, b]$ or $[c, d]$. As previously, for any interval $[a, b]$, $C^*[a, b]$ will denote the dual space of $C[a, b]$ which we identify as all Borel measures μ on $[a, b]$ normed with total variation $\|\mu\|_{T.V.}$.

Let $K(x,y)$ be a jointly continuous kernel for $(x,y) \in [a,b] \times [c,d]$, i.e., $K \in C([a,b] \times [c,d])$. For functions h and g defined on $[a,b]$ and $[c,d]$, respectively, set

$$(Th)(y) = (K^T h)(y) = \int_a^b K(x,y)h(x)dx$$

and

$$(T^* g)(x) = (Kg)(x) = \int_c^d K(x,y)g(y)dy\,.$$

(K^T here denotes the transpose of K). For $\nu \in C^*[a,b]$ and $\mu \in C^*[c,d]$, we set

$$(T\nu)(y) = (K^T \nu)(y) = \int_a^b K(x,y)d\nu(x)$$

and

$$(T^*\mu)(x) = (K\mu)(x) = \int_c^d K(x,y)d\mu(y)\,.$$

We also set for $\sigma > 0$ and $p, q \in [1,\infty]$

$$E_{pq}(\sigma) = \begin{cases} \min\limits_{g \in L^q} \|f - Kg\|_p + \sigma\|g\|_q\,, & 1 \le p \le \infty, \quad 1 < q \le \infty\,, \\ \min\limits_{\mu \in C^*[a,b]} \|f - K\mu\|_p + \sigma\|\mu\|_{T.V.}\,, & 1 \le p \le \infty, \quad q = 1 \end{cases}$$

and

$$e_{pq}(\sigma) = \begin{cases} \max\limits_{\substack{\|h\|_{p'} \le 1 \\ \|K^T h\|_{q'} \le \sigma}} (f,h)\,, & 1 \le p < \infty, \quad 1 \le q \le \infty \\ \max\limits_{\substack{\|\nu\|_{T.V.} \le 1 \\ \|K^T \nu\|_{q'} \le \sigma}} (f,\nu)\,, & p = \infty, \quad 1 \le q \le \infty\,. \end{cases}$$

Proposition 5.1. *For any $f \in C[a,b]$, $\sigma > 0$ and $p, q \in [1,\infty]$ we have*

$$E_{pq}(\sigma) = e_{pq}(\sigma)\,.$$

Proof. This follows from an application of Theorem 3.1 where some care needs to be taken in the choice of the spaces X and Y. We divide the possibilities into six cases

$$(X,Y) = \begin{cases} (L^{p'}, L^{q'}), \quad 1 < p < \infty, \quad 1 < q \le \infty \\ (L^{p'}, C[c,d]), \quad 1 < p < \infty, \quad q = 1 \\ (L^{\infty}, L^{q'}), \quad p = 1, \quad 1 < q \le \infty \\ (L^{\infty}, C[c,d]), \quad p = 1, \quad q = 1 \\ (C^*[a,b], L^{q'}), \quad p = \infty \quad 1 < q \le \infty \\ (C^*[a,b], C[c,d]), \quad p = \infty \quad q = 1 \end{cases}$$

and note that in all possibilities Assumption I holds. Also, it need be noted for the third case that whenever $f \in C[a,b]$

$$\|f\|_{(L^\infty)^*} = \|f\|_1$$

and in the fifth case that

$$\|f\|_{C^{**}[a,b]} = \|f\|_\infty\,.$$

We also observe when $1 < p < \infty$ that Proposition 3.3 implies that

$$\tilde{\sigma} = \frac{\|K^T \tilde{h}\|_{q'}}{\|\tilde{h}\|_{p'}}$$

where $\tilde{h}(x) = |f(x)|^{p-1} \operatorname{sgn}(f(x))$. Similarly if $1 < p < \infty$, $1 < q < \infty$ and $\sigma \in (\sigma', \tilde{\sigma})$, then the solution $g_\sigma \in L^q$ to $E_{pq}(\sigma)$ is unique (Proposition 3.10), and is uniquely given by a solution to

$$K^T\big(|f - Kg_\sigma|^{p-1}\operatorname{sgn}(f - Kg_\sigma)\big)(y) = \lambda |g_\sigma(y)|^{q-1} \operatorname{sgn}(g_\sigma(y)) \tag{5.1}$$

a.e. on $[c,d]$ for some choice of $\lambda > 0$ (see Proposition 3.6) depending on σ.

To proceed further we assume that the kernel K is STP. That is, for every $a \le x_1 < \ldots < x_n \le b$, $c \le y_1 < \ldots < y_n \le d$ and $n = 1, 2, \ldots$

$$\det\big(K(x_i, y_j)\big)_{i,j=1}^n > 0\,.$$

The two main facts connected with STP kernels which we shall use are the following, cf. KARLIN [13].

a) For any $h \in L^1[c,d]$,
$$\widetilde{Z}(Kh) \le S(h)$$
where $\widetilde{Z}(f)$ is the number of zeros of $f\,(f \in C[a,b])$ on $[a,b]$, with the convention that a zero of f in (a,b) is counted twice if f does not change sign at that zero. $S(h)$ is the number of essential sign changes of the function h. This is called the *variation diminishing property* of the kernel K.

b) If
$$\widetilde{Z}(Kh) = S(h) < \infty\,,$$
and h is positive near d, then $(Kh)(b) > 0$, or $(Kh)(b-\varepsilon) < 0$ for all $\varepsilon > 0$ sufficiently small. We then say that Kh and h have the same sign *orientation*.

Thus, for example, for the g_σ as defined above in (5.1), we get from (a) that $\widetilde{Z}(Kg_\sigma) \le S(g_\sigma) \le \widetilde{Z}(g_\sigma) \le S(f - Kg_\sigma)$ for each $\sigma \in (\sigma', \tilde{\sigma})$.

Under this condition on K we turn our attention to the problem $E_{\infty\infty}(\sigma)$. According to Proposition 5.1 we have

$$E_{\infty\infty}(\sigma) = \min_{g \in L^\infty} \|f - Kg\|_\infty + \sigma\|g\|_\infty = \min_{\substack{\|\mu\|_{T.V.} \le 1 \\ \|K^T\mu\|_1 \le \sigma}} (f,\mu)\,. \tag{5.2}$$

We recall the following result proved in PINKUS [21].

THEOREM 5.2. *Let K be STP, $f \in C[a,b]$, and $\tau > 0$. Assume $f \notin A_\tau := \{Kg : \|g\|_\infty \le \tau\}$. Then there exists a unique solution g^τ to*

$$\min\{\|f - Kg\|_\infty : \|g\|_\infty \le \tau\} \tag{5.3}$$

characterized as follows: there exists a nonnegative integer $m(\tau)$, points

$$c = \xi_0 < \xi_1 < \ldots < \xi_{m(\tau)} < \xi_{m(\tau)+1} = d\,,$$
$$a \le \eta_1 < \ldots < n_{m(\tau)+1} \le b$$

(all dependent on τ), and some $\varepsilon \in \{-1,1\}$ such that

$$\begin{aligned} &\text{a) } \varepsilon(-1)^i g^\tau(y) = \|g^\tau\|_\infty = \tau \ \textit{a.e. for } y \in (\xi_{i-1},\xi_i),\ i = 1,\ldots,m(\tau)+1 \\ &\text{b) } \varepsilon(-1)^i (f - Kg^\tau)(\eta_i) = \|f - Kg^\tau\|_\infty, \quad i = 1,\ldots,m(\tau)+1\,. \end{aligned} \tag{5.4}$$

We remark here that a discrete analogue of Theorem 5.2 may be found in Pinkus and Strauss [23, Theorem 4.2]. As noted in Section 2, the two problems (5.2) and (5.3) are related. The solution g_σ to $E_{\infty\infty}(\sigma)$ is equal to g^τ of Theorem 5.2 for $\tau = \|g_\sigma\|_\infty$. Furthermore g^τ also solves $E_{\infty\infty}(\sigma)$ for some σ. To be precise, let g^τ be as in Theorem 5.2 with associated $\{\xi_i\}_{i=1}^{m(\tau)}$ and $\{\eta_i\}_{i=1}^{m(\tau)+1}$. Since K is STP, there exists a unique set of coefficients $\{a_i^*\}_{i=1}^{m(\tau)+1}$ satisfying

$$(5.5)\qquad \begin{aligned} &\text{a)} \quad \left[\sum_{i=1}^{m(\tau)+1} a_i^* K(\eta_i, y)\right] g^\tau(y) \geq 0 \quad \text{a.e.} \quad y \in [c,d] \\ &\text{b)} \quad \sum_{i=1}^{m(\tau)+1} |a_i^*| = 1 \\ &\text{c)} \quad \varepsilon(-1)^i a_i^* > 0, \qquad i = 1, \dots, m(\tau)+1\,. \end{aligned}$$

Define $\mu^* \in C^*[a,b]$ by

$$(5.6)\qquad d\mu^* = \sum_{i=1}^{m(\tau)+1} a_i^* \delta_{\eta_i}\,.$$

Then $\|\mu^*\|_{T.V.} = \sum\limits_{i=1}^{m(\tau)+1} |a_i^*| = 1$, and from (5.4) and (5.5),

$$(f - Kg^\tau, \mu^*) = \|f - Kg^\tau\|_\infty \|\mu^*\|_{T.V.}$$

and

$$(g^\tau, K^T\mu^*) = \|g^\tau\|_\infty \|K^T\mu^*\|_1\,.$$

Thus from Proposition 3.6, we get that g^τ solves (5.2) for $\sigma = \|K^T\mu^*\|_1$. In addition, it follows that $\sigma' \leq \sigma \leq \tilde{\sigma}$.

Now, it need not be that the $\{\eta_i\}_{i=1}^{m(\tau)+1}$ are uniquely defined for a given τ. That is, $f - Kg^\tau$ may attain its norm at more than $m(\tau)+1$ points. In this case the μ^* of (5.6) is not uniquely defined. Consequently, there may exist a range of values for $\|K^T\mu^*\|_1$ where μ^* is associated with a fixed g^τ, as above. This range of values is necessarily an interval. On this interval the same g^τ is a solution to $E_{\infty\infty}(\sigma)$, and thus $E_{\infty\infty}(\sigma)$ is linear. This corresponds to a lack of differentiability at a point of the associated Gagliardo diagram. On the other hand, it may be that distinct τ give rise to the same μ^*. In this case we will have nonuniqueness of the solution to $E_{\infty\infty}(\sigma)$ for the associated σ. We discuss this point later in some detail.

Let us now consider the relationship between σ, τ and $m(\tau)$ for $\sigma \in (\sigma', \tilde{\sigma})$. From Proposition 3.2 (iv), if $0 < \sigma_1 < \sigma_2 < \infty$, then

$$\text{(5.7)} \qquad \begin{aligned} &\text{a)} \quad \|g_{\sigma_1}\|_\infty \geq \|g_{\sigma_2}\|_\infty \\ &\text{b)} \quad \|f - Kg_{\sigma_1}\|_\infty \leq \|f - Kg_{\sigma_2}\|_\infty \,. \end{aligned}$$

Let $\sigma' < \sigma_1 < \sigma_2 < \tilde{\sigma}$. Denote by g_{σ_i} a solution to $E_{\infty\infty}(\sigma)$ for $\sigma = \sigma_i$, $i = 1, 2$. By our previous remarks $g_{\sigma_i} = g^{\tau_i}$, where $\tau_i = \|g_{\sigma_i}\|_\infty$, $i = 1, 2$. Thus from (5.7) (a) $\tau_1 \geq \tau_2$. Associated with each $g_{\sigma_i} = g^{\tau_i}$ is the number $m(\tau_i)$ as given in (5.4).

Proposition 5.3. *For $\sigma' < \sigma_1 < \sigma_2 < \tilde{\sigma}$, it follows that $m(\tau_1) \geq m(\tau_2)$.*

Proof. Assume $g_{\sigma_1} \neq g_{\sigma_2}$. Since $\|g_{\sigma_1}\|_\infty \geq \|g_{\sigma_2}\|_\infty$ and $|g_{\sigma_1}(y)| = \|g_{\sigma_1}\|_\infty$ for all y, with $S(g_{\sigma_1}) = m(\tau_1)$, we have that

$$S(g_{\sigma_1} - g_{\sigma_2}) \leq m(\tau_1)\,.$$

Now $\|f - Kg_{\sigma_2}\|_\infty \geq \|f - Kg_{\sigma_1}\|_\infty$, and from (5.4) (b), $f - Kg_{\sigma_2}$ attains its norm, alternately, on at least $m(\tau_2) + 1$ points. Thus

$$\widetilde{Z}\big((f - Kg_{\sigma_2}) - (f - Kg_{\sigma_1})\big) \geq m(\tau_2)\,.$$

Thus

$$\text{(5.8)} \qquad \begin{aligned} m(\tau_2) &\leq \widetilde{Z}\big((f - Kg_{\sigma_2}) - (f - Kg_{\sigma_1})\big) = \widetilde{Z}\big(K(g_{\sigma_1} - g_{\sigma_2})\big) \\ &\leq S(g_{\sigma_1} - g_{\sigma_2}) \leq m(\tau_1)\,. \end{aligned}$$

This proves the proposition. ∎

A more detailed analysis provides us with this next result.

Proposition 5.4. *Assume* $m(\tau_1) = m(\tau_2)$, $\sigma' < \sigma_1 < \sigma_2 < \tilde{\sigma}$, *and* $g_{\sigma_1} \neq g_{\sigma_2}$. *Then*

1) $\tau_1 = \|g_{\sigma_1}\|_\infty > \|g_{\sigma_2}\|_\infty = \tau_2$,
2) $\|f - Kg_{\sigma_1}\|_\infty < \|f - Kg_{\sigma_2}\|_\infty$,
3) g_{σ_1} *and* g_{σ_2} *have the same orientation,*
4) $f - Kg_{\sigma_2}$ *attains its norm, alternately, at exactly* $m(\tau_2) + 1$ *points.*

Proof. Both (1) and (2) are immediate consequences of the uniqueness of the solution in (5.3). We know that $\tau_1 \geq \tau_2$. If $\tau_1 = \tau_2$, then g_{σ_1} and g_{σ_2} are distinct solutions to (5.3) for the same τ. Thus (1) holds. From (5.7) (b), $\|f - Kg_{\sigma_1}\|_\infty \geq \|f - Kg_{\sigma_2}\|_\infty$. If equality holds, then g_{σ_1} and g_{σ_2} would again both be solutions to (5.3) for $\tau = \tau_1$. This proves (2).

To prove (4), note that equality holds in (5.8), and thus $f - Kg_{\sigma_2}$ cannot attain its norm, alternately, at more than $m(\tau_2) + 1$ points.

Equality in (5.8) also implies that $K(g_{\sigma_1} - g_{\sigma_2})$ and $g_{\sigma_1} - g_{\sigma_2}$ have the same sign orientation. The sign orientation of $K(g_{\sigma_1} - g_{\sigma_2}) = (f - Kg_{\sigma_2}) - (f - Kg_{\sigma_1})$ is determined by the sign orientation of $f - Kg_{\sigma_2}$, and this, by (5.4), is in turn determined by the sign pattern of g_{σ_2}. The sign orientation of $g_{\sigma_1} - g_{\sigma_2}$ is determined by the sign pattern of g_{σ_1}. As such a little bookkeeping shows that g_{σ_1} and g_{σ_2} have the same orientation. This proves (3). ∎

For fixed $\sigma \in (\sigma', \tilde{\sigma})$, the above analysis does *not* imply that there is a unique solution to $E_{\infty\infty}(\sigma)$. Nonuniqueness may occur but only when the following is satisfied.

Proposition 5.5. *Let* $\sigma \in (\sigma', \tilde{\sigma})$. *Assume* g_σ^1 *and* g_σ^2 *are two distinct solutions to* $E_{\infty\infty}(\sigma)$. *Then*

1) $g_\sigma^1(y) = \alpha g_\sigma^2(y)$ *a.e. for some* $\alpha > 0$.
2) *If* $S(g_\sigma^1) = S(g_\sigma^2) = m$, *then there exist* $a \leq \eta_1 < \ldots < \eta_{m+1} \leq b$ *and* $\varepsilon \in \{-1, 1\}$ *such that*

$$\varepsilon(-1)^i (f - Kg_\sigma^j)(\eta_i) = \|f - Kg_\sigma^j\|_\infty$$

for $i = 1, \ldots, m+1$ *and* $j = 1, 2$.

3)
$$\sigma = \frac{\varepsilon(-1)^i (Kg_\sigma^j)(\eta_i)}{\|g_\sigma^j\|_\infty}\,, \quad i = 1, \ldots, m+1, \quad j = 1, 2\,.$$

Proof. Let $\|g_\sigma^j\|_\infty = \tau_j$, $j = 1, 2$. Since (5.3) has a unique solution and $g_\sigma^1 \neq g_\sigma^2$, it follows that $\tau_1 \neq \tau_2$.

Set $g_\lambda = \lambda g_\sigma^1 + (1-\lambda) g_\sigma^2$ for any fixed $\lambda \in (0, 1)$. Then

$$\begin{aligned} E_{\infty\infty}(\sigma) &\leq \|f - Kg_\lambda\|_\infty + \sigma \|g_\lambda\|_\infty \\ &\leq \lambda\big(\|f - Kg_\sigma^1\|_\infty + \sigma\|g_\sigma^1\|_\infty\big) + (1-\lambda)\big(\|f - Kg_\sigma^2\|_\infty + \sigma\|g_\sigma^2\|_\infty\big) \\ &= E_{\infty\infty}(\sigma)\,. \end{aligned}$$

Equality implies that g_λ is also a solution to $E_{\infty\infty}(\sigma)$. Thus

$$\big\|\lambda g_\sigma^1 + (1-\lambda) g_\sigma^2\big\|_\infty = \lambda\|g_\sigma^1\|_\infty + (1-\lambda)\|g_\sigma^2\|_\infty \tag{5.9}$$

and

$$\begin{aligned} &\big\|\lambda(f - Kg_\sigma^1) + (1-\lambda)(f - Kg_\sigma^2)\big\|_\infty = \\ &\qquad = \lambda\|f - K_\sigma^1\|_\infty + (1-\lambda)\|f - Kg_\sigma^2\|_\infty\,. \end{aligned} \tag{5.10}$$

Since g_λ is a solution to $E_{\infty\infty}(\sigma)$ for $\sigma \in (\sigma', \tilde\sigma)$, it is also a solution to (5.3), and thus $|g_\lambda(y)| = \|g_\lambda\|_\infty$ a.e. y. This fact together with (5.9), and the form of g_σ^1 and g_σ^2, implies that

$$g_\sigma^1(y) = \alpha g_\sigma^2(y)\,, \qquad \text{a.e.} \quad y$$

for $\alpha = \|g_\sigma^1\|_\infty / \|g_\sigma^2\|_\infty > 0$.

Let $S(g_\lambda) = m\big(= S(g_\sigma^1) = S(g_\sigma^2)\big)$. There then exist $a \leq \eta_1 < \ldots < \eta_{m+1} \leq b$ and $\varepsilon \in \{-1, 1\}$ (determined as in Theorem 5.2) such that

$$\varepsilon(-1)^i (f - Kg_\lambda)(\eta_i) = \|f - Kg_\lambda\|_\infty$$

for all $i = 1, \ldots, m+1$. From this fact and (5.10) we get

$$\varepsilon(-1)^i (f - Kg_\sigma^j)(\eta_i) = \|f - Kg_\sigma^j\|_\infty$$

for $i = 1, \ldots, m+1$ and $j = 1, 2$.

Finally

$$\begin{aligned}
&\sigma\left[\|g^2_\sigma\|_\infty - \|g^1_\sigma\|_\infty\right] = \|f - Kg^1_\sigma\|_\infty - \|f - Kg^2_\sigma\|_\infty \\
&= \varepsilon(-1)^i\left[(f - Kg^1_\sigma)(\eta_i) - (f - Kg^2_\sigma)(\eta_i)\right] \\
&= \varepsilon(-1)^i K(g^2_\sigma - \alpha g^2_\sigma)(\eta_i) \\
&= \left[\frac{\|g^2_\sigma\|_\infty - \|g^1_\sigma\|_\infty}{\|g^2_\sigma\|_\infty}\right]\varepsilon(-1)^i K g^2_\sigma(\eta_i)
\end{aligned}$$

and thus

$$\sigma = \frac{\varepsilon(-1)^i K g^2_\sigma(\eta_i)}{\|g^2_\sigma\|_\infty}.$$

Substituting from (1), we get the same equality but with g^1_σ replacing g^2_σ. □

We now turn our attention to the problem $E_{11}(\sigma)$. In this case, Proposition 5.1 gives

$$E_{11}(\sigma) = \min_{\mu \in C^*[c,d]} \|f - K\mu\|_1 + \sigma\|\mu\|_{T.V.} = \max_{\substack{\|h\|_\infty \le 1 \\ \|K^T h\|_\infty \le \sigma}} (f, h).$$

For this problem we review some results from Micchelli and Pinkus [16] which require that $f \in C[a,b]$ be in the *convexity cone of* K. By that we mean that for all $m \ge 0$, and all points $a \le x_1 < \ldots < x_{m+1} \le b$ and $c \le y_1 < \ldots < y_m \le d$,

$$\begin{vmatrix} K(x_1, y_1) & \ldots & K(x_1, y_m) & f(x_1) \\ \vdots & & \vdots & \vdots \\ K(x_{m+1}, y_1) & \ldots & K(x_{m+1}, y_m) & f(x_{m+1}) \end{vmatrix} > 0.$$

For $m = 0$, we take this to simply mean that $f(x) > 0$ for all $x \in [a,b]$. We explain how to choose for each $\sigma \in (0, \tilde\sigma)$ (here $\tilde\sigma = \left\| \int_c^d K(\cdot, y)dy \right\|_\infty$) a solution μ^* to $E_{11}(\sigma)$ in a simple way which depends linearly on the particular f in the class.

Given any nonnegative integer m, there exist $a = \xi_0 < \xi_1 < \ldots < \xi_m < \xi_{m+1} = b$ and $c \le \eta_1 < \ldots < \eta_{m+1} \le d$ such that for h_ξ defined by

$$h_\xi(x) = (-1)^{i+m+1}, \qquad x \in (\xi_{i-1}, \xi_i), \qquad i = 1, \ldots, m+1,$$

we have

$$(K^T h_\xi)(\eta_j) = (-1)^{j+m+1}\|K^T h_\xi\|_\infty\,, \qquad j = 1,\dots,m+1\,.$$

In fact, these conditions uniquely define $\xi = (\xi_1,\dots,\xi_m)$. Let

$$\sigma_m = \|K^T h_\xi\|_\infty\,.$$

Then it is easily shown that $\{\sigma_m\}_{m=0}^\infty$ is a strictly decreasing sequence ($\sigma_0 = \tilde\sigma$) and $\lim_{m\to\infty}\sigma_m = 0$. Now for $\sigma \in (\sigma_m, \sigma_{m-1})$, there exist, for $k = 1, 2$, points $a = \xi_0^k < \xi_1^k < \ldots < \xi_{m+1}^k = b$ and $c \le \eta_1^k < \ldots < \eta_m^k \le d$ such that for h_{ξ^k} defined by

$$\text{(5.11)}\quad h_{\xi^k}(x) = (-1)^{i+m+1},\ x \in (\xi_{i-1}^k, \xi_i^k),\ i = 1,\dots,m+1\,;\ k = 1,2\,,$$

we have

$$\text{(5.12)}\quad (K^T h_{\xi^k})(\eta_j^k) = (-1)^{j+m+k}\|K^T h_{\xi^k}\|_\infty, \quad j = 1,\dots,m,\ k = 1,2\,.$$

In fact these conditions uniquely determine the ξ^k, $k = 1, 2$.

For f in the convexity cone of K, an optimal μ^* for $E_{11}(\sigma)$ is given as follows. For $\sigma \in (\sigma_m, \sigma_{m-1})$, let

$$\text{(5.13)}\qquad d\mu^* = \sum_{j=1}^m a_j^* \delta_{\eta_j^2}\,,$$

where the $\{a_j^*\}_{j=1}^m$ are uniquely determined by the conditions

$$\text{(5.14)}\qquad f(\xi_i^2) = \sum_{j=1}^m a_j^* K(\xi_i^2, \eta_j^2), \qquad i = 1,\dots,m\,.$$

Conditions (5.13) and (5.14), together with the fact that f is in the convexity cone of K imply that

$$\text{(5.15)}\qquad a_j^*(-1)^{j+m} > 0, \qquad j = 1,\dots,m\,,$$

and

$$\text{(5.16)}\qquad h_{\xi^2}(x)\big[f(x) - (K\mu^*)(x)\big] \ge 0\,,$$

for all $x \in [a,b]$. Thus from (5.12), (5.13) and (5.15),

$$(K^T h_{\xi^2}, \mu^*) = \|K^T h_{\xi^2}\|_\infty \|\mu^*\|_{T.V.} ,$$

while from (5.11) and (5.16),

$$(h_{\xi^2}, f - K\mu^*) = \|h_{\xi^2}\|_\infty \|f - K\mu^*\|_1 .$$

Applying Proposition 3.6, we see that μ^* is indeed optimal for $E_{11}(\sigma)$. For $\sigma = \sigma_m$, we do much the same using the associated $\{\xi_1, \ldots, \xi_m\}$ and $\{\eta_2, \ldots, \eta_{m+1}\}$ (note here that η_1 is an extra extrema of $K^T h_\xi$).

In case f satisfies

$$\begin{vmatrix} f(x_1) & K(x_1,y_1) & \ldots & K(x_1,y_m) \\ \vdots & \vdots & & \vdots \\ f(x_{m+1}) & K(x_{m+1},y_1) & \ldots & K(x_{m+1},y_m) \end{vmatrix} > 0$$

for all $m \ge 0$, points $a \le x_1 < \ldots < x_{m+1} \le b$ and $c \le y_1 < \ldots < y_m \le d$ we use the values $\{\xi_i^1\}_{i=1}^m$ and $\{\eta_j^1\}_{j=1}^m$ (or $\{\eta_j\}_{j=1}^m$ for $\sigma = \sigma_m$) to solve $E_{11}(\sigma)$.

The last problem we consider is the question of when $\tilde{\sigma} = \hat{\sigma}$. Our first remark does not require that K be STP. Specifically, if $1 < p < \infty$ and $1 < q < \infty$, and in addition $f = Kg^*$ for some $g^* \in L^q$, then from Proposition 3.9, $\tilde{\sigma} = \hat{\sigma}$ if and only if

$$K^T\big(|Kg^*|^{p-1}\operatorname{sgn}(Kg^*)\big)(y) = \lambda |g^*(y)|^{q-1}\operatorname{sgn}\big(g^*(y)\big)$$

a.e. on $[c,d]$, where

$$\lambda = \frac{\|Kg^*\|_p^p}{\|g^*\|_q^q} .$$

In this generality we know very little about these equations. However, for $p = q \in (1,\infty)$ and K strictly totally positive, these were studied in Pinkus [22]. Its solutions (there are a countable number, up to multiplication by constants) are related to certain n-width problems. In fact for K STP, many of the extremal problems considered in the study of n-widths seem to relate to exactly the problem of when $\tilde{\sigma} = \hat{\sigma}$. We describe some of these cases in what follows.

In MICCHELLI and PINKUS [15] it was shown that for any fixed non-negative integer m,

$$g_t(y) = (-1)^{i+m+1}, \qquad t \in (t_{i-1}, t_i), \quad i = 1, \dots, m+1$$

and $p \in [1, \infty]$, the problem

$$\min_t \|Kg_t\|_p , \qquad t = (t_1, \dots, t_m)$$

has a solution $g^* = g_\xi$ (dependent on p) where ξ satisfies $c = \xi_0 < \xi_1 < \dots < \xi_m < \xi_{m+1} = d$ and

$$\int_a^b \left|Kg^*(x)\right|^{p-1} \left[\operatorname{sgn} Kg^*(x)\right] K(x, \xi_i) dx = 0 , \qquad i = 1, \dots, m ,$$

for $1 \le p < \infty$, while for $p = \infty$ there exist $a \le \eta_1 < \dots < \eta_{m+1} \le b$ such that

$$(-1)^{i+m+1} Kg^*(\eta_i) = \|Kg^*\|_\infty, \qquad i = 1, \dots, m+1 .$$

For $1 \le p < \infty$ set

$$h^*(x) = \left|Kg^*(x)\right|^{p-1} \operatorname{sgn}\left(Kg^*(x)\right)$$

so that

$$(Kg^*, h^*) = \|Kg^*\|_p \|h^*\|_{p'} , \tag{5.17}$$

where $1/p + 1/p' = 1$. Moreover, from the variation diminishing property of K (and K^T) we obtain that

$$g^*(y)(K^T h^*)(y) \ge 0$$

for all $y \in [c, d]$ and so

$$(g^*, K^T h^*) = \|g^*\|_\infty \|K^T h^*\|_1 . \tag{5.18}$$

Applying Proposition 3.9, we get the following result.

Proposition 5.6. *If $f = Kg^*$ where g^* is as above for some m and some $p \in [1,\infty)$, then in the problem $E_{p\infty}(\sigma)$,*

$$\tilde{\sigma} = \hat{\sigma} = \frac{\|Kg^*\|_p}{\|g^*\|_\infty} = \frac{\|K^T h^*\|_1}{\|h^*\|_{p'}} .$$

To handle the problem $E_{\infty\infty}(\sigma)$ we define $\mu^* \in C^*[a,b]$ with $\|\mu^*\|_{T.V.} = 1$ by

$$d\mu^* = \sum_{i=1}^{m+1} a_i^* \delta_{\eta_i} ,$$

where the $\{a_i^*\}_{i=1}^{m+1}$ are the unique set of coefficients satisfying

a) $\left[\sum_{i=1}^{m+1} a_i^* K(\eta_i, y)\right] g^*(y) \ge 0 \quad \text{a.e.} \quad y \in [c,d]$

b) $\sum_{i=1}^{m+1} |a_i^*| = 1$

c) $(-1)^{i+m+1} a_i^* > 0, \qquad i = 1, \ldots, m+1 .$

From the above constructions, we get

$$(Kg^*, \mu^*) = \|Kg^*\|_\infty \|\mu^*\|_{T.V.}$$

and

$$(g^*, K^T\mu^*) = \|g^*\|_\infty \|K^T\mu^*\|_1 .$$

Again Proposition 3.9 yields the next result.

Proposition 5.7. *If $f = Kg^*$ where g^* is as above for some m, then in the problem $E_{\infty\infty}(\sigma)$,*

$$\tilde{\sigma} = \hat{\sigma} = \frac{\|Kg^*\|_\infty}{\|g^*\|_\infty} = \frac{\|K^T\mu^*\|_1}{\|\mu^*\|_{T.V.}} .$$

Remark 5.1. Let $p \in [1,\infty]$ and g^* be as above so that either Proposition 5.6 or 5.7 is satisfied. For $0 < \tau < \|g^*\|_\infty$, consider the problem

$$\min \left\{ \|f - Kg\|_p : \|g\|_\infty \le \tau \right\}.$$

For $p = \infty$ it follows from Theorem 5.2 that there is a unique solution g^τ. This g^τ is explicitly given by

$$g^\tau = \frac{\tau g^*}{\|g^*\|_\infty}$$

because (5.4) of Theorem 5.2 is satisfied. Moreover, this same g^τ is the solution for all other p as well since the associated h^* is a nontrivial linear functional on L^p which attains its norm on Kg^*.

Interchanging the roles of K and K^T in the analysis which led to Propositions 5.6 and 5.7, we get corresponding results for $E_{1q}(\sigma)$, $1 \le q \le \infty$. Explicitly, for $1 \le q \le \infty$, let g^* be the solution to

$$\min_t \|K^T g_t\|_{q'}$$

where $1/q + 1/q' = 1$, and g_t is as previously defined. For $1 < q \le \infty$, set

$$h^*(y) = \left|K^T g^*(y)\right|^{q'-1} \operatorname{sgn}\left(K^T g^*(y)\right),$$

and for $q = 1$ let ν^* play the corresponding role to the μ^* defined above. Then,

Proposition 5.8. *If $f = Kh^*$ where h^* is as above for some m and some $q \in (1,\infty]$, then in the problem $E_{1q}(\sigma)$,*

$$\tilde{\sigma} = \hat{\sigma} = \frac{\|Kh^*\|_1}{\|h^*\|_q} = \frac{\|K^T g^*\|_{q'}}{\|g^*\|_\infty}.$$

If $f = K\nu^*$ where ν^* is as above, then in the problem $E_{11}(\sigma)$,

$$\tilde{\sigma} = \hat{\sigma} = \frac{\|K\nu^*\|_1}{\|\nu^*\|_{T.V.}} = \frac{\|K^T g^*\|_\infty}{\|g^*\|_\infty}.$$

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