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ONE-SIDED L^1-APPROXIMATION TO DIFFERENTIABLE FUNCTIONS

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§1. Introduction

A good deal of research has recently been devoted to the problem of characterizing those finite dimensional subspaces U_n for which one has uniqueness in the problem of determining the best L^1-approximation (one-or two-sided) from U_n to continuous functions. This paper is concerned with this problem in the case of one-sided L^1-approximation to continuously differentiable functions. Before relating the relevant results in this area, we first fix some notation.

K – a bounded, closed, convex subset of $\mathbb{R}^m$, with piecewise smooth boundary.

$C(K)$ – real-valued continuous functions defined on K.

$C^1(K)$ – continuously differentiable functions in C(K).

W – strictly positive (weight) functions in C(K).

U_n – a fixed n-dimensional subspace of $C^1(K)$.

For $f \in C(K)$, we set

$$Z(f) = \{x: f(x) = 0\} .$$

For $f \in C^1(K)$, we also set

$$Z_1(f) = \{x: f(x) = 0, \text{ and } |f(x)| \text{ is differentiable at } x\} .$$

For $f \in C(K)$ and $w \in W$,

$$\|f\|_w = \int_K |f(x)|w(x)dx .$$

We refer to $\|\cdot\|_w$ as the L^1_w-norm.

We shall concern ourselves with the problem of one-sided L^1_w-approximation (from below) to $f \in C^1(K)$ from U_n.

Definition 1.1 For given $w \in W$, we say that $u^* \in U_n$ is a best one-sided L^1_w-approximant to f if $u^*(x) \le f(x)$ all $x \in K$, and $\|f-u^*\|_w \le \|f-u\|_w$ for all $u \in U_n$ satisfying $u(x) \le f(x)$ for all $x \in K$.

In general it may be that for certain $f \in C^1(K)$ there exist no $u \in U_n$ satisfying $u \le f$. However we will always assume that U_n contains a strictly positive function. As such, to each $f \in C^1(K)$ there always exists a best one-sided L^1_w-approximant.

We are interested in the question of the uniqueness of the best approximant. As such we make the following definitions.

Definition 1.2 For $w \in W$, U_n is said to be a unicity space for w if to each $f \in C^1(K)$ there exists a unique best one-sided L^1_w-approximant from U_n.

Definition 1.3 U_n is said to be a unicity space for W if U_n is a unicity space for w for each $w \in W$.

The problem of characterizing unicity spaces for given $w \in W$, and for all W, has recently been intensively investigated. The reader interested in this problem for two-sided L^1-approximation should consult Kroo [3], [4], Pinkus [7], Strauss [11], and references therein. The one-sided L^1-approximation problem has also received considerable attention. When restricting ourselves only to continuous functions, most of the results are negative in character. Results in this direction for $K=[a,b]$ were obtained by DeVore [1], Pinkus [6], Strauss [10] and Sommer, Strauss [9]. It was finally shown by Nürnberger [5] and Pinkus, Totik [8] that if U_n, $n \ge 2$, contains a strictly positive function, then it is not a unicity space for any $w \in W$. Furthermore, Pinkus, Totik [8] proved that U_n is a unicity space for W if and only if U_n is "trivial". By this it is meant that U_n has a basis of non-negative functions with disjoint support.

The situation is radically altered if we demand that both the approximating subspace U_n, and the approximated functions f, be

all continuously differentiable. Here the results tend to be more positive in nature. DeVore [1] essentially proved that U_n is a unicity space for W if U_n is a Chebyshev system in $C^1[a,b]$ with the property that $Z^*(u) \le n-1$ for every non-trivial $u \in U_n$, where $Z^*(u)$ counts the number of zeros of u in $[a,b]$, where we count x as a double zero if $u(x) = u'(x) = 0$. Subspaces satisfying this property are referred to as ET_2-systems (extended Chebyshev of order 2). In Pinkus [6], Strauss [10] and Sommer, Strauss [9], further extensions may be found.

This paper contains two main results. In Section 2 we characterize unicity spaces for given $w \in W$ (Theorems 2.1 and 2.4). These results are essentially generalizations of results of Strauss [10] to the multidimensional setting. In Section 3 we consider the problem of characterizing those subspaces U_n which are unicity spaces for W (Theorem 3.1). Numerous examples are given in Section 4.

§2. Unicity Spaces for W

In this section we present theorems characterizing unicity spaces for given $w \in W$. As previously mentioned, we will always assume that $U_n \subset C^1(K)$, that we are approximating only functions in $C^1(K)$ (from below), and that U_n contains a strictly positive function.

Theorem 2.1 U_n is a unicity space for given $w (\in W)$ if and only if to each $u \in U_n \setminus \{0\}$ there exists a $v \in U_n$ for which

$$\text{a)} \quad v(x) \le 0 \quad \text{for all} \quad x \in Z_1(u) ,$$
$$\text{b)} \quad \int_K v(x)w(x)dx > 0 . \tag{2.1}$$

Proof Assume that (2.1) holds. Since U_n contains a strictly positive function, we may assume, by perturbation, that there exists a $v \in U_n$ for which

$$\text{a)} \quad v(x) < 0 \quad \text{for all} \quad x \in Z_1(u)$$
$$\text{b)} \quad \int_K v(x)w(x)dx > 0 . \tag{2.2}$$

Assume U_n is not a unicity space for w. Thus there exists an $f \in C^1(K)$ which has two best one-sided L_w^1-approximants $u_1, u_2 \in U_n$, $u_1 \neq u_2$. Note that $\lambda u_1 + (1-\lambda)u_2$ are also best approximants for every $\lambda \in [0,1]$. Set

$$h(x) = f(x) - (u_1+u_2)(x)/2$$

and

$$\tilde{u}(x) = (u_1-u_2)(x) .$$

It now easily follows that $h(x) \geq |\tilde{u}(x)/2|$ for all $x \in K$, and that $0, \pm\tilde{u}/2$ are all best approximants to h. Since $h \in C^1(K)$ and $h \geq 0$, we have $Z(h) = Z_1(h)$. Furthermore since $h \geq |\tilde{u}/2|$, $Z_1(h) \subseteq Z_1(\tilde{u})$. From (2.2) there exists a $v \in U_n$ for which $v(x) < 0$ for all $x \in Z_1(h) = Z(h)$, and $\int_K v(x)w(x)dx > 0$. Thus $v(x) < h(x)$ on an open neighborhood of $Z(h)$. Since K is compact and $h \geq 0$, there exists a $c > 0$, small, such that $cv(x) < h(x)$ for all $x \in K$. Since $\int_K cv(x)w(x)dx > 0$, this contradicts the fact that 0 is a best approximant to h.

To prove the converse we assume that (2.1) does not hold. Thus there exists a $u^* \in U_n \setminus \{0\}$ such that if $v \in U_n$ satisfies $v(x) \leq 0$ for all $x \in Z_1(u^*)$, then $\int_K v(x)w(x)dx \leq 0$. The proof of the converse is rather lengthy. As such we divide it into a series of lemmas. In the first lemma we show by construction of a quadrature formula that it suffices to consider only a finite point set in $Z_1(u^*)$.

Lemma 2.2 Under the above assumptions there exist points $x_1, \cdots, x_r \in Z_1(u^*)$, $1 \leq r \leq n$, and $\{\lambda_i\}_{i=1}^r$, $\lambda_i > 0$, such that

$$\int_K u(x)w(x)dx = \sum_{i=1}^r \lambda_i u(x_i)$$

for all $u \in U_n$.

Proof Set

$$P = \{(u_1(x), \cdots, u_n(x)) : x \in Z_1(u^*)\}$$

where $u_1, \cdots, u_n$ is any basis for U_n. Let Q denote the closed, convex, positive cone generated by P. Set

$$\underline{c} = \left(\int_K u_1(x)w(x)dx , \cdots , \int_K u_n(x)w(x)dx\right) .$$

Assume $\underline{c} \notin Q$. There then exists a hyperplane passing through the origin separating $\underline{c}$ from Q. That is, there exists an $\underline{a} = (a_1, \cdots, a_n) \neq \underline{0}$ for which

$$\sum_{i=1}^n a_i \int_K u_i(x)w(x)dx > 0 \geq \sum_{i=1}^n a_i u_i(x)$$

for all $x \in Z_1(u^*)$. Set $v(x) = \sum_{i=1}^n a_i u_i(x)$. Then $v(x) \leq 0$ for all $x \in Z_1(v)$, and $\int_K v(x)w(x)dx > 0$, contradicting our assumption. Thus $\underline{c} \in Q$. Since Q is a closed convex cone in $\mathbb{R}^n$, there exists an r, $0 \leq r \leq n$, points $x_1, \cdots, x_r$ in $Z_1(u^*)$, and $\lambda_1, \cdots, \lambda_r$, $\lambda_i > 0$, such that

$$\int_K u_j(x)w(x)dx = \sum_{i=1}^r \lambda_i u_j(x_i)$$

for $j=1, \cdots, n$. Thus

$$\int_K u(x)w(x)dx = \sum_{i=1}^r \lambda_i u(x_i)$$

for all $u \in U_n$. Furthermore, since U_n contains a strictly positive function it follows that $r \geq 1$. Q.E.D.

If $F \in C^1(K)$, $F \geq 0$ and $F(x_i) = 0$, $i=1, \cdots, r$, then it follows from Lemma 2.2 that the zero function is a best approximant to F. If we can construct such an F with the further property that $F(x) \geq |u^*(x)|$ for all $x \in K$, then $0, \pm u^*$ are all best approximants to F, and U_n is not a unicity space for w. We now construct such an F.

For convenience we assume that $K \subset \mathbb{R}^2$ with piecewise smooth boundary ∂K. For $u \in U_n$, we consider, in greater detail, points in $Z_1(u)$. Recall that K is convex. Assume $(x_0, y_0) \in Z_1(u)$. Then $u(x_0, y_0) = 0$. In addition we have:

(i) If $(x_0, y_0) \in \text{int } K$, then $u_x(x_0, y_0) = u_y(x_0, y_0) = 0$.

(ii) If $(x_0, y_0) \in \partial K$ and a tangent to K exists at (x_0, y_0), then the derivative of u in the tangent direction is zero.

(iii) If $(x_0,y_0) \in \partial K$ and a tangent to K does not exist there, then there is no additional assumption.

Lemma 2.3 Let the above assumptions hold. Assume $\{(x_i,y_i)\}_{i=1}^{r} \subseteq Z_1(u^*)$, r finite. Then there exists an $F \in C^1(K)$ such that $F(x_i,y_i)=0$, $i=1,\cdots,r$, and $F(x,y) \geq |u^*(x,y)|$ for all $(x,y) \in K$.

Proof It suffices to construct, in a neighborhood of each (x_i,y_i), a continuously differentiable function f which satisfies $f(x,y) \geq |u^*(x,y)|$ for all (x,y) in this neighborhood. We then set $g(x,y) = f(x,y) + M((x-x_i)^2+(y-y_i)^2)$, some $M>0$. g satisfies the above inequality, vanishes at (x_i,y_i), and is strictly positive on the boundary of the neighborhood. It is then easily seen that there exists an F as in the statement of the lemma which agrees with g on a neighborhood of (x_i,y_i). Thus we must construct the requisite f. We construct f as follows:

(i) $(x_i,y_i) \in \text{int } K$. Assume without loss of generality that $(x_i,y_i) = (0,0)$. Thus $u^*(0,0) = u_x^*(0,0) = u_y^*(0,0) = 0$. For $0 \leq r \leq R$, some $R \geq 0$ fixed, set

$$h(r) = \max_{0\leq\theta\leq 2\pi} \left|\frac{\partial}{\partial r} u^*(r\cos\theta, r\sin\theta)\right| .$$

Since $u^* \in C^1(K)$, $h \in C[0,R]$, and $h(0)=0$. For (x_0,y_0) satisfying $x_0 = r_0\cos\theta_0$, $y_0 = r_0\sin\theta_0$, $0 \leq r_0 \leq R$, set

$$f(x_0,y_0) = f(r_0\cos\theta_0, r_0\sin\theta_0) = \int_0^{r_0} h(r)dr .$$

$f(0,0)=0$, f is non-negative, and f is continuously differentiable on $\{(x,y): x^2+y^2<R^2\}$. Furthermore

$$f(x_0,y_0) = f(r_0\cos\theta_0, r_0\sin\theta_0) \geq \int_0^{r_0} \left|\frac{\partial}{\partial r} u^*(r\cos\theta_0, r\sin\theta_0)\right| dr$$

$$\geq \left|\int_0^{r_0} \frac{\partial}{\partial r} u^*(r\cos\theta_0, r\sin\theta_0) dr\right|$$

$$= |u^*(r_0\cos\theta_0, r_0\sin\theta_0)| = |u^*(x_0,y_0)| .$$

(ii) $(x_i,y_i) \in \partial K$ and a tangent to K exists at (x_i,y_i). Without loss of generality we assume that $(x_i,y_i) = (0,0)$, $K \subset \{(x,y): x \geq 0\}$ and the tangent to ∂K at $(0,0)$ is along the y-axis. Thus $u(0,0) = u_y(0,0) = 0$. By Whitney's Extension Theorem we may assume that $u^* \in C^1(H)$, where $H = \{(x,y): x \geq 0\}$. Now set

$$\tilde{f}(y) = \int_0^y \left|\frac{\partial}{\partial y} u^*(0,t)\right| dt + Ny^2 ,$$

$N>0$, for all $|y| \leq R$ (some $R \geq 0$). $\tilde{f} \in C^1[-R,R]$, $\tilde{f}(0)=0$, $\tilde{f}$ is non-negative, and $\tilde{f}(y) \geq |u^*(0,y)|$ for all $|y| \leq R$. We define $f(x,y) = \tilde{f}(y)+Mx$, where $M \geq \|u_x\|_\infty$. For $x \geq 0$ and any fixed y we have

$$u^*(x,y) = u^*(0,y) + xu_x^*(\tilde{x},y)$$

where $0 \leq \tilde{x} \leq x$. Thus for $|y| \leq R$

$$|u^*(x,y)| \leq |u^*(0,y)| + x|u^*(\tilde{x},y)| \leq \tilde{f}(y) + Mx = f(x,y).$$

(iii) $(x_i,y_i) \in \partial K$ and no tangent to K at (x_i,y_i) exists. Since ∂K is piecewise smooth, one-sided tangents to K exist at all $(x,y) \in \partial K$. Since K is convex, the point (x_i,y_i) is a "corner" with angle $<\pi$. Thus we may assume that $(x_i,y_i) = (0,0)$, and $K \subset \{(x,y): x>0\} \cup \{0,0\}$. Set $f(x,y) = Mx$ for $M>0$. For M sufficiently large it is easily seen that $|u^*(x,y)| \leq f(x,y)$ and $f(0,0)=0$. This proves the lemma.

The proof of Theorem 2.1 is complete.

Theorem 2.1 and the proof thereof give the following additional characterizations of unicity spaces for w.

Theorem 2.4 Let $w \in W$. The following are equivalent.

a) U_n is a unicity space for w.

b) For each $f \in C^1(K)$ and any best one-sided L_w^1-approximant $\bar{u}$ to f from U_n, there does not exist a $u \in U_n \setminus \{0\}$ for which $Z(f-\bar{u}) \subseteq Z_1(u)$.

c) For any quadrature formula

$$\int_K u(x)w(x)dx = \sum_{i=1}^{r} \lambda_i u(x_i) . \qquad (2.3)$$

exact for all $u \in U_n$ where $\lambda_i > 0$, $i=1,\cdots,r$ and r finite, there does not exist a $u \in U_n\setminus\{0\}$ with $\{x_i\}_{i=1}^r \subseteq Z_1(u)$.

Proof (a)→(b). Assume, contrary to (b) that there exists an $f \in C^1(K)$ and a best one-sided L^1_w-approximant $\bar{u}$ to f, and a $\tilde{u} \in U_n/\{0\}$ such that $Z(f-\bar{u}) \in Z_1(\tilde{u})$. From (2.1) (see also (2.2)) there exists a $v \in U_n$ for which $v(x) < 0$ for all $x \in Z_1(\tilde{u})$ and $\int_K v(x)w(x)dx > 0$. Thus for $\alpha > 0$ sufficiently small $f(x) \geq \bar{u}(x)+\alpha v(x)$ for all $x \in K$ and $\int_K(\bar{u}(x)+\alpha v(x))w(x)dx > \int_K \bar{u}(x)w(x)dx$. This contradicts the fact that $\bar{u}$ is a best approximant to f. Thus (b) holds.

(b)→(c). Assume, contrary to (c), that there exists a quadrature formula of the form (2.3) and a $u \in U_n\setminus\{0\}$ with $\{x_i\}_{i=1}^r \subseteq Z_1(u)$. From Lemma 2.3 and the construction therein, there exists an $F \in C^1(K)$ satisfying $F(x_i)=0$, $i=1,\cdots,r$, $F(x) \geq |u(x)|$ for all $x \in K$, and $F(x) \neq 0$, $x \notin \{x_i\}_i^r$. From (2.3) it follows that 0 is a best approximant to F. Furthermore $Z_1(F) = Z(F) = \{x_i\}_{i=1}^r \subseteq Z_1(u)$, contradicting (b).

(c)→(a). If U_n is not a unicity space then (2.1) does not hold and the existence of the quadrature formula (2.3) and the u as in (c) is the content of Lemma 2.2. Q.E.D.

§3. Unicity Spaces for W

The Characterization Theorems 2.1 and 2.4 depend on w. Since these characterizations are difficult to verify for specific w, it is natural to ask for necessary and sufficient conditions guaranteeing that U_n is a unicity space for every $w \in W$. To this end we introduce the following condition, which is due to DeVore [2] in the case $K=[a,b]$.

Definition 3.1 An n-dimensional subspace U_n of $C^1(K)$ (which contains a strictly positive function) is said to satisfy Condition B if to each $u \in U_n\setminus\{0\}$ there exists a $v \in U_n\setminus\{0\}$ satisfying

$$\text{(a)} \quad Z_1(u) \subseteq Z_1(v)$$

$$\text{(b)} \quad v(x) \geq 0 \quad \text{for all} \quad x \in K. \tag{3.1}$$

DeVore showed that if U_n satisfies Condition B, then U_n is a unicity space for every $w \in W$ in the case where K is an interval. Here we extend this result to general K, and also prove the more difficult converse result.

Theorem 3.1 U_n is a unicity space for every $w \in W$ if and only if U_n satisfies Condition B.

Proof Assume U_n satisfies Condition B. (3.1) implies that (2.1) holds for every $w \in W$. Thus U_n is a unicity space for every $w \in W$.

To prove the converse assume that U_n does not satisfy Condition B. There therefore exists a $u^* \in U_n\setminus\{0\}$ such that if $u \in U_n$ satisfies $Z_1(u^*) \subseteq Z_1(u)$ and $u(x) \geq 0$ on K, then $u \equiv 0$. Set

$$P = \{u:\ u \in U_n,\ u(x) \leq 0 \quad \text{for all} \quad x \in Z_1(u^*)\} .$$

Since U_n contains a strictly positive function, $P \neq \{0\}$. Furthermore P contains no non-negative non-trivial function. We shall prove the following result.

Lemma 3.2 Let P be as above. There exists a $w \in W$ such that

$$\int_K u(x)w(x)dx \leq 0$$

for all $u \in P$.

Before proving Lemma 3.2, let us note its consequence. Let w be as in Lemma 3.2 and u^* be as above. Then U_n does not satisfy (2.1) for this w, i.e., if $v \in U_n\setminus\{0\}$, $v(x) \leq 0$ for all $x \in Z_1(u^*)$, then $\int_K v(x)w(x)dx \leq 0$. Therefore, by Theorem 2.1 U_n is not a unicity space for this particular w, and Theorem 3.1 is proved.

Proof of Lemma 3.2 Set

$$\tilde{P} = \{u(x)dx : u \in P\} .$$

Consider $\tilde{P}$ as a subset of $M(K)$, the set of Borel measures of bounded total variation on K, endowed with the weak*-topology induced by $C(K)$. From the definition of P, it follows that $\tilde{P}$

is a closed, convex cone in some finite-dimensional subspace of $M(K)$. Set

$$Q = \{\mu : \mu \in M(K),\ d\mu \geq 0,\ \int_K d\mu = 1\}.$$

Thus Q is the set of probability densities. Q is convex and compact in this same topology. By assumption, $Q \cap \tilde{P} = \phi$. Thus there exists a continuous linear functional w on $M(K)$ such that $w(u) \leq 0$ for all $u \in \tilde{P}$ and $w(\mu) > 0$ for all $\mu \in Q$. Every continuous linear functional on this space may be represented as integration over K against a continuous function which we again denote by w. Thus $w \in C(K)$,

$$\int_K u(x)w(x)dx \leq 0 \qquad \text{for all } u \in P$$

and

$$\int_K w(x)d\mu(x) > 0 \qquad \text{for all } \mu \in Q.$$

Since the point functionals are in Q, it follows that $w(x) > 0$ for all $x \in K$. Thus $w \in W$.

§4. Examples

Let us consider some simple examples of spaces of functions and check as to whether or not they satisfy Condition B.

Example 1 $U_3 = \text{span}\{1,x,y\}$, $K = \{(x,y): x^2+y^2 \leq 1\}$.

It is easily seen that if $u \in U_3\backslash\{0\}$, and $(x_0,y_0) \in Z_1(u)$, then $(x_0,y_0) \in \partial K$, i.e., $x_0^2+y_0^2=1$, and

$$u(x,y) = a(1-x_0x-y_0y)$$

for some $a \in \mathbb{R}$, $a \neq 0$. Now, $v(x,y) = 1-x_0x-y_0y \geq 0$ on K since $x_0x+y_0y \leq (x_0^2+y_0^2)^{\frac{1}{2}}(x^2+y^2)^{\frac{1}{2}} = (x^2+y^2)^{\frac{1}{2}} \leq 1$ for all $(x,y) \in K$. Thus U_3 satisfies Condition B on K.

Example 2 $U_3 = \text{span}\{1,x,y\}$, $K = \{(x,y): 0 \leq x,y \leq 1\}$.

The function $u(x,y)=x-y$ is such that $Z_1(u) = \{(0,0),(1,1)\}$. No $v \in U_3\backslash\{0\}$ vanishes at $(0,0)$ and $(1,1)$ and is non-negative on K. Thus U_3 does not satisfy Condition B on K.

Example 3 $U_4 = \text{span}\{1,x,y,xy\}$, K has interior in $\mathbb{R}^2$.

Let $(x_0,y_0) \in \text{int } K$, and $u(x,y) = (x-x_0)(y-y_0) \in U_4$. Then $(x_0,y_0) \in Z_1(u)$ and every $v \in U_4$ for which $(x_0,y_0) \in Z_1(v)$ is necessarily of the form $v = au$, $a \in \mathbb{R}$. Thus U_4 does not satisfy Condition B on K.

Example 4 $U_5 = \text{span}\{1,x,y,x^2,y^2\}$, $K = \{(x,y): x^2+y^2 \leq 1\}$.

We will show that U_5 satisfies Condition B on K. Set $v(x,y) = 1-(x^2+y^2)$. v is non-negative on K and $Z_1(v) = \{(x,y): x^2+y^2 = 1\}$. Let $u \in U_5\backslash\{0\}$. If $Z_1(u)$ contains no interior points of K, then the above v can be used in Condition B. We therefore assume that $u \in U_5\backslash\{0\}$ and $(x_0,y_0) \in Z_1(u) \cap \text{int } K$. It is easily checked that

$$u(x,y) = a(x-x_0)^2+b(y-y_0)^2$$

for some constants a,b. From the above form of u, we see that $Z_1(u)$ contains no other interior points of K. If $Z_1(u) = \{(x_0, y_0)\}$, set $v(x,y) = (x-x_0)^2+(y-y_0)^2$ in Condition B. Assume $(x_1,y_1) \in Z_1(u) \cap K$, $(x_1,y_1) \neq (x_0,y_0)$. Thus $x_1^2+y_1^2 = 1$. Since $(x_1,y_1) \in Z_1(u)$ we have $u(x_1,y_1) = 0$ and $y_1u_x(x_1,y_1)-x_1u_y(x_1,y_1)=0$. We set up the linear equations implied by the above form of u from which it necessarily follows that

$$(x_1-x_0)(y_1-y_0)(1-x_1x_0-y_1y_0) = 0.$$

If $x_1=x_0$, then it follows that $u(x,y) = a(x-x_0)^2$, while if $y_1=y_0$, then $u(x,y) = b(y-y_0)^2$. In each of these cases the choice of v, as in Condition B, is obvious. Furthermore one of the above two cases must hold since $1-x_1x_0-y_1y_0>0$ for any $(x_0,y_0) \in \text{int } K$ and $(x_1,y_1) \in \partial K$.

A general negative result is the following.

Let $U_n,V_m \subseteq C^1[0,1]$; $\dim U_n = n$, $\dim V_m = m$; $n,m \geq 2$. Let P denote the tensor product of U_n and V_m on $[0,1] \times [0,1]$. Thus

$$P = \{p(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}u_i(x)v_j(y)\}$$

where $u_1,\cdots,u_n$ and $v_1,\cdots,v_m$ are any bases for U_n and V_m, respectively.

Proposition 4.1 If P as above, contains a strictly positive function, then P does not satisfy Condition B on $[0,1]\times[0,1]$.

We use the following lemma in the proof of the above proposition.

Lemma 4.2 $U_n \subseteq C[0,1]$, $\dim U_n = n, n \geq 2$, and U_n contains a strictly positive function. Then there exist points $0 < x_1 < \cdots < x_{n-1} < 1$ and a function $u^* \in U_n$ for which

(i) $u^*(x_i)=0$, $i=1,\cdots,n-1$

(ii) u^* changes sign on $[0,1]$

(iii) if $u \in U_n$ and $u(x_i)=0$, $i=1,\cdots,n-1$, then $u=\alpha u^*$ for some $\alpha \in \mathbb{R}$.

Proof Let $u_1,\cdots,u_n$ be any basis for U_n. Since $U_n \subseteq C[0,1]$, there exist points $0 < z_1 < \cdots < z_n < 1$ such that $\det(u_i(z_j))_{i,j=1}^n \neq 0$. Let $u^i \in U_n$ satisfy $u^i(z_j) = \delta_{ij}$, $i,j=1,\cdots,n$. If u^i changes sign for some i, set $u^*=u^i$, $\{x_1,\cdots,x_{n-1}\} = \{z_1,\cdots,z_{i-1},z_{i+1},\cdots,z_n\}$ and we are finished. We therefore assume that $u^i \geq 0$ for all $i=1,\cdots,n$. Since U_n contains a strictly positive function, it follows that for each $z \in [0,1]$ there exists an $i \in \{1,\cdots,n\}$ for which $u^i(z) \neq 0$. From continuity considerations and since $n \geq 2$, it follows that there exists a point $z^* \in (0,1)$ and $i,j \in \{1,\cdots,n\}$, $i \neq j$, such that $u^i(z^*), u^j(z^*) > 0$. Thus $z^* \notin \{z_1,\cdots,z_n\}$. Set

$$u^* = u^j(z^*)u^i - u^i(z^*)u^j .$$

u^* vanishes at $\{x_1,\cdots,x_{n-1}\} = \{z_1,\cdots,z_n,z^*\}\setminus\{z_i,z_j\}$, and changes sign on $[0,1]$ $(u^*(z_i) > 0,\ u^*(z_j) < 0)$. Furthermore, if $u \in U_n$ satisfies $u(x_i)=0$, $i=1,\cdots,n-1$, then it easily follows that $u = \alpha u^*$ for some $\alpha \in \mathbb{R}$.

Proof of Proposition 4.1 Assume P satisfies Condition B. Since P contains a strictly positive function, it follows that both U_n and V_m contain strictly positive functions. The subspaces U_n and V_m satisfy the conditions of Lemma 4.2 and there therefore exist points $0 < x_1 < \cdots < x_{n-1} < 1$ and $u^* \in U_n$, and points $0 < y_1 < \cdots < y_{m-1} < 1$ and $v^* \in V_m$ satisfying the statement

of Lemma 4.2.

Set $p^*(x,y) = u^*(x)v^*(y) \in P$. It is easily seen that $p^*(x_r, y_s) = p_x^*(x_r,y_s) = p_y^*(x_r,x_s) = 0$ for each $r=1,\cdots,n-1$; $s=1,\cdots,m-1$. Thus $(x_r,y_s) \in Z_1(p^*)$. Since P satisfies Condition B, there exists a $p \in P\setminus\{0\}$ for which $p \geq 0$ and $p(x_r,y_s)=0$, $r=1,\cdots,n-1$; $s=1,\cdots,m-1$. It now follows from the properties of u^* and v^* that $p=\alpha p^*$ for some $\alpha \in \mathbb{R}$. But p^* changes sign on $[0,1]\times[0,1]$. Thus $\alpha=0$ and $p \equiv 0$ which contradicts Condition B. Q.E.D.

Remark If n or $m=1$ then the above result is no longer valid.

In the remaining discussion, we assume that $K=[a,b]\subset\mathbb{R}$.

As was noted in Section 1, DeVore [1] proved that if U_n is an ET_2-system then it is a unicity space for W. In fact an ET_2-system is readily seen to satisfy Condition B. However the Chebyshev space property is insufficient in and of itself to guarantee Condition B. The space $U_2=\text{span}\{1,x^3\}$ is a T-system on $[-1,1]$, but does not satisfy Condition B since $u(x)=x^3$ is such that $Z_1(u)=\{0\}$ and U_2 contains no non-negative function which vanishes at 0. On the other hand, nothing like the ET_2-property is necessary for Condition B is hold. If $U_2=\text{span}\{1,u_2\}$ then U_2 satisfies Condition B if and only if for any point x which is either an endpoint or a point at which $u_2'(x)=0$, it follows that $u_2(x)$ is either the maximum or minimum value of u_2 on $[a,b]$. Thus u_2 may well equioscillate many times. This may be regarded as a special case of our next result. In dealing with this next result, we introduce the following notation.

Definition 4.1 Let A be a subset of $[a,b]$. A is said to have index $I(A)$ where $I(A)$ counts the number of points in A under the convention that points in $A\cap(a,b)$ are counted twice and points in $A\cap\{a,b\}$ once.

Proposition 4.3 Let U_{n-1} be an $(n-1)$-dimensional ET_2-system on $C^1[a,b]$. Let $u_n \in C^1[a,b]$ and $U_n=\text{span}\{U_{n-1},u_n\}$. Then U_n satisfies Condition B if and only if there does not exist a function $u \in U_n$ which has a sign change and satisfies $I(Z_1(u)) \geq n-1$.

Proof Assume that U_n satisfies Condition B and there exists a $u^* \in U_n$ which has a sign change and satisfies $I(Z_1(u^*)) \geq n-1$. Thus there exists a $v \in U_n\setminus\{0\}$ satisfying $v \geq 0$ and $Z_1(u^*) \subseteq$

$Z_1(v)$. From our assumption on U_{n-1}, we see that $u^*, v \in U_{n-1}$. Thus $u^*=cu_n+u_1$, $v=du_n+u_2$ where $c,d \neq 0$ and $u_1,u_2 \in U_{n-1}$. Now $du^*-cv = du_1-cu_2 \in U_{n-1}$ and thus $I(Z_1(du^*-cv)) < n-1$ (U_{n-1} is an ET_2-system). However $du^*-cv \not\equiv 0$ since u^* changes sign on $[a,b]$ and v does not, and $Z_1(u^*) \subseteq Z_1(du^*-cv)$. Thus $I(Z_1(du^*-cv)) \geq n-1$. This is a contradiction.

To prove the converse we need only consider functions $u \in U_n \setminus \{0\}$ which change sign on $[a,b]$: By assumption such u satisfy $I(Z_1(u)) < n-1$. Every ET_2-system U_{n-1} of dimension $n-1$ contains a $v \geq 0$ ($v \not\equiv 0$) such that $Z_1(u) \subseteq Z_1(v)$. Thus U_n satisfies Condition B. Q.E.D.

Remark In DeVore's result and in the above proposition it is not necessary that U_{n-1} be an ET_2-system. It suffices to assume that U_{n-1} is a T-system and $I(Z_1(u)) \leq n-2$ for all $u \in U_{n-1} \setminus \{0\}$. The only difference here is that we always count zeros at the end-points once and never twice.

Proposition 4.4 If n is odd, $n \geq 3$, then the conditions of Proposition 4.3 imply that u_n lies in the convexity cone induced by U_{n-1}, i.e. $\text{span}\{U_{n-1}, u_n\}$ forms a weak Chebyshev subspace.

Proof Assume that u_n is not contained in the convexity cone of U_{n-1}. Then there exists a $u \in U_{n-1}$ such that $\tilde{u}=u_n+u$ has at least n sign changes. Assume that the points $a < t_1 < \cdots < t_n < b$ are "sign changes" of $\tilde{u}$ and

$$\tilde{u}(t_{2i}) = 0\,, \quad \tilde{u}'(t_{2i}) \leq 0 \quad \text{for } i=1,\cdots,(n-1)/2.$$

Since U_{n-1} is an ET_2-system there exists a (unique) $\bar{u}$ in U_{n-1} satisfying

$$\left.\begin{aligned} \bar{u}(t_{2i}) &= \tilde{u}(t_{2i}) \\ \bar{u}'(t_{2i}) &= \tilde{u}'(t_{2i}) \end{aligned}\right\}, \quad i=1,\cdots,(n-1)/2\,.$$

We shall represent $\bar{u}$ by fundamental polynomials $l_{\mu\nu}$ in U_{n-1}, $\mu=1,\cdots,(n-1)/2$ and $\nu=0,1$, based on the points $\{t_{2i}\}$. Set

$$l^{(j)}_{\mu\nu}(t_{2i}) = \begin{cases} 1\,, & \text{if } \mu=i \text{ and } \nu=j \\ 0 & \text{elsewhere} \end{cases}$$

for all $i=1,\cdots,(n-1)/2$ and $j=0,1$. Then

$$\bar{u}(t) = \sum_{i=1}^{(n-1)/2} (\tilde{u}(t_{2i})l_{i0}(t) + \tilde{u}'(t_{2i})l_{i1}(t)) = \sum_{i=1}^{(n-1)/2} \tilde{u}'(t_{2i})l_{i1}(t)\,.$$

Suppose that $\tilde{u}'(t_{2i})=0$ for all $i=1,\cdots,(n-1)/2$ then $\{t_{2i}\}_{i=1}^{(n-1)/2} \subset Z_1(\tilde{u})$, i.e. $I(Z_1(\tilde{u})) \geq n-1$. Moreover $\tilde{u}$ has a sign change. This contradicts the assumptions. Hence we assume that $\tilde{u}'(t_{2i_0}) < 0$ for some i_0. Since U_{n-1} is an ET_2-system it follows from the construction of l_{i1} that $l_{i1}(t) < 0$ for $t < t_2$ and $l_{i1}(t) > 0$ for $t > t_{n-1}$. Then $\tilde{u}'(t_{2i}) \leq 0$, $i=1,\cdots,(n-1)/2$ implies $\bar{u}(t) > 0$ for $t < t_2$ and $\bar{u}(t) < 0$ for $t > t_{n-1}$. Hence $(\tilde{u}-\bar{u})(t_1) = -\bar{u}(t_1) < 0$ and $(\tilde{u}-\bar{u})(t_n) = -\bar{u}(t_n) > 0$ and therefore $\tilde{u}-\bar{u}$ has a sign change. Moreover $I(Z_1(\tilde{u}-u)) \geq n-1$. This contradiction completes the proof. Q.E.D.

Other examples of B-spaces have been already given. It was shown in [6] that all subspaces of differentiable polynomial spline functions with fixed knots are B-spaces. In [9] it was proved that all subspaces which are differentiably composed from certain Chebyshev subspaces are also B-spaces.

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