n-WIDTHS IN APPROXIMATION THEORY: A SURVEY

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0. Introduction

What we today refer to as the n-widths in the sense of Kolmogorov was introduced by Kolmogorov in 1936 [21]. Essentially little attention was paid to this concept until 1960 when Tichomirov initiated a series of papers which laid the foundation for and motivated much of the subsequent theory. At last count there were approximately 200 articles devoted to various aspects of n-widths and the subject has slowly drawn the attention of more and more approximation theorists as is evident, at the least, from the fact of this survey article.

The title is, of course, a misnomer. As with any survey I cannot and will not give a complete account of the subject matter. What I have tried to do is to give a sketch of some of the theoretical results concerned with n-widths and illustrate them by various examples. It is not possible or practical to reference all the papers concerned with the various aspects of the theory of n-widths. (As such apologies are due to the many authors whose works are not mentioned.) A fuller exposition of the theory is to appear in [40], where I hope to redress all wrongs.

This survey is organized as follows. In Section 1 we define the basic concepts and discuss various general properties of n-widths. n-Widths in a Hilbert space setting are discussed in Section 2. In Section 3 are considered exact n-widths for classes of functions generated by integral operators of total positivity type. A discussion of n-widths induced by diagonal matrices is given in Section 4. In Section 5 we

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review the recently proved asymptotic estimates for n-widths of Sobolev spaces. We conclude, in Section 6, with a discussion of some results on exact n-widths of classes of analytic functions.

1. Basic Results

Let X be a normed linear space (over $\mathbb{R}$ or $\mathbb{C}$) and A a subset of X. For an n-dimensional subspace X_n of X, define

$$E(A;X_n) = \sup_{a \in A} \inf_{x \in X_n} \|a-x\| .$$

The <u>n-width, in the sense of Kolmogorov</u>, of A in X is defined by

$$d_n(A;X) = \inf_{X_n} E(A;X_n) = \inf_{X_n} \sup_{a \in A} \inf_{x \in X_n} \|a-x\|$$

where the left-most infimum is taken over all subspaces X_n of X of dimension at most n. Equivalently, we may write

$$d_n(A;X) = \inf_{X_n} \inf\{\lambda : \lambda > 0 , A \subseteq X_n + \lambda S(X)\}$$

where S(X) denotes the unit ball of X. If $d_n(A;X) = E(A;X_n)$ for some subspace X_n with $\dim(X_n) \leq n$, then X_n is said to be an <u>optimal</u> subspace for $d_n(A;X)$. (Optimal subspaces need not exist, see e.g. Brown [4]).

$d_n(A;X)$ measures the extent to which A is approximable by n-dimensional subspaces of X. It is, in a certain sense, a measure of the "thickness" or "massivity" of A in X. It also provides a lower bound on the degree of approximation of A by any particular n-dimensional subspace of X.

For specific A and X, interest is generally directed to two questions: (1) Calculate $d_n(A;X)$ and determine (or characterize) optimal subspaces X_n for $d_n(A;X)$. (2) Determine the asymptotic behaviour of $d_n(A;X)$ as $n\uparrow\infty$. The latter question is of interest because generally the former is intractable, and even if solvable, too much computational effort is involved for its practical use. Often very simple n-dimensional subspaces may approximate A in an asymptotically optimal manner, and this fact is often worth knowing.

The following properties of d_n are fairly immediate consequences of the definition (see e.g. Lorentz [24], Pietsch [36], and Singer [43]).

THEOREM 1.1: <u>Let</u> X <u>be a normed linear space and</u> $A \subseteq X$.

(i) $d_n(\bar{A};X) = d_n(A;X)$, <u>where</u> $\bar{A}$ <u>is the closure of</u> A.

(ii) <u>For every scalar</u> α,

$$d_n(\alpha A;X) = |\alpha|\, d_n(A;X).$$

(iii) <u>For each</u> $A \subseteq X$, <u>let</u> $b(A) = \{\alpha x : x \in A\ ,\ |\alpha| \le 1\}$ <u>be the balanced hull of</u> A. <u>Then</u>

$$d_n(A;X) = d_n\big(b(A);X\big).$$

(iv) $d_n(\text{co } A;X) = d_n(A;X)$, <u>where</u> co A <u>is the convex hull of</u> A.

(v) <u>For any two sets</u> $A,B \subseteq X$, <u>set</u>

$$E(A;B) = \sup_{a \in A}\ \inf_{b \in B} \|a-b\|.$$

<u>Then if</u> $B \subseteq A$,

$$d_n(A;X) - E(A;B) \le d_n(B;X) \le d_n(A;X).$$

(vi) $d_n(A;X) \ge d_{n+1}(A;X)\ ,\ n = 0,1,\ldots$.

(vii) <u>Let</u> X <u>and</u> Y <u>be normed linear spaces</u>, $A \subseteq X \subseteq Y$. <u>Then</u>

$$d_n(A;X) \ge d_n(A;Y).$$

(viii) $\bar{A}$ <u>is compact if and only if</u> $d_n(A;X) \downarrow 0$ <u>and</u> A <u>is bounded</u>.

On the basis of properties (i), (iii) and (iv) we can and will henceforth assume that A is a convex, closed, centrally symmetric subset of X. Many examples exist of strict inequality in (vii), e.g., $d_n\big(S(\ell_1);\ell_\infty\big) = 1/2$, while $d_n\big(S(\ell_1);c_0\big) = 1$

for all $n \geq 1$ (Hutton [13]).

Garkavi (the proofs may also be found in Singer [43]) proved the following results concerning the existence of optimal subspaces.

THEOREM 1.2 (Garkavi [8]): *Let X be a normed linear space and X' the dual space of X. For every $A \subseteq X'$ there exists an optimal n-dimensional subspace for $d_n(A;X')$.*

THEOREM 1.3 (Garkavi [8]): *Let X be a Banach space. Assume that there exists a projection P of X" onto X of norm one. Then for every $A \subseteq X$ there exists an optimal n-dimensional subspace for $d_n(A;X)$.*

The proof of Theorem 1.2 is based on the compactness of the unit ball of X' in the weak*-topology. Theorem 1.3 is a fairly simple extension of Theorem 1.2.

If A is a subset of an (n+1)-dimensional subspace, then it always has an optimal n-dimensional subspace.

THEOREM 1.4 (Brown [4]): *Let A be a closed, convex, centrally symmetric subset of an (n+1)-dimensional subspace X_{n+1} of X. Then*

$$d_n(A;X) = \inf\{\|x\| : x \in \partial A\}.$$

Furthermore, there is an n-dimensional subspace of X_{n+1} which is optimal for $d_n(A;X)$.

The statement of Theorem 1.4 does *not* preclude the existence of optimal subspaces not in X_{n+1}.

The proof of Theorem 1.4, and more especially the lower bound, is based on a central result in the theory of n-widths. It is the following, often referred to as the Fundamental Theorem of Tichomirov.

THEOREM 1.5 (Tichomirov [48]): Let X_{n+1} be any (n+1)-dimensional subspace of a normed linear space X and let $S(X_{n+1})$ be the unit ball of X_{n+1}. Then

$$d_n\big(S(X_{n+1})\ ;\ X\big) = 1.$$

The main idea inherent in the proof of this theorem is that given X_{n+1} and any n-dimensional subspace X_n of X (and not necessarily of X_{n+1}), there exists an $x \in X_{n+1} \setminus \{0\}$ whose best approximation from X_n is the zero function, i.e., $E(x;X_n) = \|x\|$. This latter result was first proved by Krein, Krasnoselski, and Milman [23]. (It is an application of the Borsuk Antipodality Theorem.) For a proof see e.g. Lorentz [24]. Tichomirov first noted its application in obtaining lower bounds for n-widths. The idea is to find an (n+1)-dimensional subspace X_{n+1} for which $\lambda S(X_{n+1}) \subseteq A$. Then by properties (ii) and (v) of Theorem 1.1 and by Theorem 1.5,

$$d_n(A;X) \geq d_n\big(\lambda S(X_{n+1})\ ;\ X\big) = \lambda.$$

As an example of its application, consider the following (Tichomirov [48]):

Let $A_m(R)$, $R \geq 1$, denote the class of functions f analytic in $\Delta_R = \{z : |z| < R\}$ which satisfy

$$\|f^{(m)}\|_{L^\infty(\Delta_R)} = \sup\{|f^{(m)}(z)| : z \in \Delta_R\} \leq 1.$$

We will determine the n-widths of $A_m(R)$ as a subset of $L^\infty(\Delta_1)$. Babenko [1] proved that $E\big(A_m(R);\pi_{n-1}\big)_{L^\infty(\Delta_1)} = R^{m-n}(n-m)!/n!$ (for $n \geq m$), where $\pi_{n-1} = \mathrm{span}\{1,z,\ldots, z^{n-1}\}$. Thus $d_n\big(A_m(R);\ L^\infty(\Delta_1)\big) \leq R^{m-n}(n-m)!/n!$. To prove the lower bound, let $X_{n+1} = \pi_n$. By the above reasoning it therefore suffices to prove that for all $p \in \pi_n$ $(n \geq m)$,

$$\frac{R^{m-n}(n-m)!}{n!} \|p^{(m)}\|_{L^\infty(\Delta_R)} \le \|p\|_{L^\infty(\Delta_1)}$$

This is a well-known inequality which is a consequence of the inequalities

a) $$\|p\|_{L^\infty(\Delta_R)} \le R^n \|p\|_{L^\infty(\Delta_1)}$$

b) $$\|p'\|_{L^\infty(\Delta_1)} \le n\|p\|_{L^\infty(\Delta_1)}$$

The inequality (b) is implied by Bernstein's inequality $\|t'\|_\infty \le n\|t\|_\infty$ for all $t \in T_n$ (trigonometric polynomials of degree $\le n$). Thus $d_n(A_m(R); L^\infty(\Delta_1)) = R^{m-n}(n-m)!/n!$ for all $n \ge m$ (see also Theorem 6.1).

This method of obtaining lower bounds for $d_n(A;X)$ has been codified and we have:

DEFINITION 1.1: Let A be a convex, closed, centrally symmetric subset of a normed linear space X. The n-width, in the sense of Bernstein, of A in X, is given by

$$b_n(A;X) = \sup_{X_{n+1}} \sup\{\lambda : \lambda S(X_{n+1}) \subseteq A\}$$

$$= \sup_{X_{n+1}} \inf_{x \in \partial(X_{n+1} \cap A)} \|x\|,$$

where X_{n+1} is any subspace of X of dimension at least $n + 1$.

Thus from Theorem 1.5, $d_n(A;X) \ge b_n(A;X)$. There are also inequalities which go in the opposite direction. Mityagin and Henkin [34] proved that

$$d_n(A;X) \le (n+1)^2 b_n(A;X)$$

and if X = H is a Hilbert space, then

$$d_n(A;H) \le (n+1) b_n(A;H).$$

Pukhov [41] improved this latter result if H is a Hilbert space over the reals, to obtain

$$d_n(A;H) \leq \sqrt{e}\,\sqrt{n+1}\; b_n(A;H).$$

It is conjectured, again by Mityagin and Henkin, that the correct inequality should be

$$d_n(A;X) \leq (n+1)\; b_n(A;X)$$

for general X, while for a Hilbert space H it should be

$$d_n(A;H) \leq \sqrt{n+1}\; b_n(A;H).$$

These bounds can certainly not be bettered as it is known that $d_n(S(\ell_1);c_0) = 1$, $b_n(S(\ell_1);c_0) = (n+1)^{-1}$, while $d_n(S(\ell_1);\ell_2) = 1$, $b_n(S(\ell_1);\ell_2) = (n+1)^{-1/2}$.

One other general method of proving lower bounds for $d_n(A;X)$ is known when $X = C(B)$ (B a compact Hausdorff space).

THEOREM 1.6 (Lorentz [24]): *Let* A *be a subset of* C(B). *Assume that there exist* n+1 *points in* B, $x_1,\ldots,x_{n+1}$ *and a number* $\lambda > 0$ *with the following property: For every choice of signs* $\delta_i \in \{-1,1\}$, $i=1,\ldots,n+1$, *there is a function* $f \in A$ *such that*

$$\operatorname{sgn} f(x_i) = \delta_i \text{ and } |f(x_i)| \geq \lambda, \quad i=1,\ldots,n+1.$$

Then

$$d_n(A;C(B)) \geq \lambda.$$

This theorem follows from the fact that the set of points $(f(x_1),\ldots, f(x_{n+1}))$, as f ranges over a subspace of dimension n, must miss the interior of some quadrant in R^{n+1}.

Many other n-width concepts appear in the literature (see e.g.Pietsch [36],[37] and Tichomirov [50]). We shall discuss only two others. The first of these simply replaces best approximations by linear approximations, which are often more practical.

DEFINITION 1.2: *Let* X *be a normed linear space and* A *a subset of* X. *The linear n-width of* A *in* X *is given by*

$$\delta_n(A;X) = \inf_{P_n} \sup_{x \in A} \|x-P_n(x)\|$$

where the infimum is taken over all continuous linear operators P_n *of* X *into* X *of rank* n, i.e., *for which the range of* P_n *is of dimension* n.

P_n is said to be *optimal* if $\delta_n(A;X) = \sup\{\|x - P_n(x)\| : x \in A\}$. Properties (i), (ii), (iii), (iv) and (vi) of Theorem 1.1 hold with δ_n replacing d_n. In addition we also have, by definition, $\delta_n(A;X) \geq d_n(A;X)$, and the easily proven

PROPOSITION 1.7: *Assume that* $A \subseteq X \subseteq Y$ *and* X *is a subspace of the normed linear space* Y. *Then*

$$\delta_n(A;X) \geq \delta_n(A;Y).$$

The statement of Theorem 1.4 may also be modified so that it holds for δ_n (as does Theorem 1.5).

We now introduce another n-width concept, the motivation for which will become evident later.

DEFINITION 1.3: *Let* X *be a normed linear space and* $A \subseteq X$. *The* n-*width, in the sense of Gel'fand, of* A *in* X, *is given by*

$$d^n(A;X) = \inf_{L^n} \sup_{x \in A \cap L^n} \|x\|$$

where the infimum is taken over all closed subspaces L^n *of* X *of codimension* n.

L^n is said to be *optimal* if $d^n(A;X) = \sup\{\|x\| : x \in A \cap L^n\}$. It is *not* in general true that $d^n(A;X) = d^n(\bar{A};X)$ or that $d^n(A;X) = d^n(\mathrm{co}A;X)$. However, we always assume that A is closed, convex and centrally symmetric.

Unlike the case for d_n and δ_n, we have

PROPOSITION 1.8 (Helfrich [11]): *Let* X *and* Y *be normed linear spaces and assume* X *is a subspace of* Y. *Then for every* $A \subseteq X$,

$$d^n(A;X) = d^n(A;Y).$$

This implies that it suffices, in the calculation of $d^n(A;X)$, to take $X = \mathrm{span}(A)$. If X is only a subset and not a subspace of Y, e.g. $X = \ell_p^m(\mathbb{R})$, $Y = \ell_p^m(\mathbb{C})$, then strict inequality may, and sometimes does, occur. d^n also satisfies Theorems 1.4 and 1.5, with the obvious modifications, and we have

$$\delta_n(A;X) \geq d^n(A;X) \geq b_n(A;X)$$

as was the case with $d_n(A;X)$. No a priori relationship exists between d_n and d^n. There are examples for which $d_n > d^n$ and also examples for which $d_n < d^n$.

It is only rarely that one deals with a truly arbitrary subset A of X. It is most often the case that A is the image of the unit ball of some normed linear space X under a continuous linear mapping T of X to Y, or some variant thereof. (Note that here A is a subset of Y. We apologize to the reader for this altered notation, but it is the common one used.) We write

$$d_n(T(X);Y) = \inf_{X_n} \sup_{\|x\|_X \leq 1} \inf_{y \in X_n} \|Tx-y\|_Y$$

the infimum being taken over all n-dimensional subspaces X_n of Y.

When we are considering n-widths in this explicit form, then different definitions of δ_n and d^n are given, which could not possibly have been previously given. We set

$$\delta_n(T(X);Y) = \inf_{P_n} \sup_{\|x\| \leq 1} \|Tx-P_nx\|$$

where P_n is any continuous linear operator of X into Y of rank n, and

$$d^n(T(X);Y) = \inf_{L^n} \sup_{\substack{\|x\| \leq 1 \\ x \in L^n}} \|Tx\|$$

where L^n is any subspace of X of codimension n.

Again we emphasize that these definitions differ from the previous ones.

For ease of notation, let $L(X,Y)$ denote the class of continuous linear operators from X to Y, and $K(X,Y)$ the class of compact operators in $L(X,Y)$ (often referred to as the class of completely continuous operators). For $T \in L(X,Y)$, let $T' \in L(Y', X')$ denote its adjoint. Then we have

PROPOSITION 1.9 (Ha [10], Pietsch [36]): _For every_ $T \in L(X,Y)$,

$$d^n(T(X);Y) = d_n(T'(Y');X')$$

From Schauder's Theorem and property (viii) of Theorem 1.1:

PROPOSITION 1.10: $d^n(T(X);Y) \downarrow 0$ _iff_ $T \in K(X,Y)$.

This statement is not true for $\delta_n(T(X);Y)$.

The reverse inequality to that of Proposition 1.9 is not valid without further restrictions, e.g. $d_n(S(\ell_1);c_0) = 1$, $d^n(S(\ell_1);\ell_\infty) = 1/2$, for $n \geq 1$ (Hutton [13]).

THEOREM 1.11 (Pietsch [36]): _If_ $T \in K(X,Y)$ _or if_ Y _is a reflexive Banach space, then_

$$d_n(T(X);Y) = d^n(T'(Y');X').$$

And analogously

THEOREM 1.12 (Hutton [13]): _If_ $T \in K(X,Y)$ _or if_ Y _is a reflexive Banach space, then_

$$\delta_n(T(X);Y) = \delta_n(T'(Y');X').$$

2. Hilbert Spaces

Kolmogorov in his 1936 paper considered two specific examples which almost fall within the framework of the following basic general result.

Let H_1, H_2 be two Hilbert spaces and let $T \in K(H_1,H_2)$. The operators $T'T$ and TT' (T' is the adjoint of T) are self-adjoint, non-negative, compact operators. Let $\{\lambda_k(TT')\}_{k=1}$

denote the sequence of positive eigenvalues of TT', listed to their algebraic multiplicity and given in non-increasing order of magnitude. Let $\{\psi_k\}_{k=1}$ denote the orthonormal eigenvectors of TT', which satisfy $TT'\psi_k = \lambda_k\psi_k$. Then $\{\psi_k\}_{k=1}$ is complete in the range of TT'. The s-numbers (singular values) of T are defined by $s_k(T) = [\lambda_k(TT')]^{1/2}(= [\lambda_k(T'T)]^{1/2})$. Let $\phi_k = T'\psi_k$, $k = 1,2,...$. Then $T'T\phi_k = \lambda_k\phi_k$.

THEOREM 2.1: Let $T \in K(H_1,H_2)$, and let $\{s_k(T)\}$, $\{\psi_k\}$ and $\{\phi_k\}$ be as defined above. Then

$$d_n(T(H_1);H_2) = d^n(T(H_1);H_2) = \delta_n(T(H_1);H_2) =$$
$$b_n(T(H_1);H_2) = s_{n+1}(T) \ , \ n = 0,1,2...$$

Furthermore,

1) $X_n = \mathrm{span}\{\psi_1,...,\psi_n\}$ is optimal for d_n.
2) $L^n = \{\phi : \phi \in H_1 , (\phi,\phi_k) = 0, k = 1,...,n\}$ is optimal for d^n.
3) $P_n = \Sigma_{i=1}^n(\cdot,\phi_i)\psi_i$ is optimal for δ_n.
4) $X_{n+1} = \mathrm{span}\{\psi_1,...,\psi_{n+1}\}$ is optimal for b_n.

The proof of this result is via the max-min and min-max characterizations of the (n+1)st eigenvalue of a compact, self-adjoint, non-negative operator.

Remark: These optimal subspaces are not unique.

As a simple application of Theorem 2.1, consider the following:

EXAMPLE 2.1: Let $L^2 = L^2[0,2\pi]$ with usual norm $\|f\|_2^2 = 1/2\pi \int_0^{2\pi} |f(x)|^2dx$. Let $k \in L^2$ be real, 2π-periodic and $k(x) \sim \Sigma_{-\infty}^{\infty}a_m e^{imx}$, where $|a_0| \geq |a_1| \geq |a_2| \geq \cdots$. Set

$$\mathcal{K}_2 = \{k*\phi : \|\phi\|_2 \leq 1\}.$$

Then $d_0(\tilde{K}_2;L^2) = |a_0|, d_{2n-1}(\tilde{K}_2;L^2) = d_{2n}(\tilde{K}_2;L^2) = |a_n|$, $n = 1,2,\ldots$, and T_{n-1} (trigonometric polynomials of degree $\leq n-1$) is an optimal $(2n-1)$-dimensional subspace for $d_{2n-1}(\tilde{K}_2;L^2)$ (and hence also for $d_{2n}(\tilde{K}_2;L^2)$).

Before discussing other examples, we list some generalizations of Theorem 2.1, and also some results which are not, in fact, direct generalizations.

I) Let $T \in L(H_1,H_2)$. Assume that $TT'\psi_k = \lambda_k\psi_k$, $k = 1,2,\ldots$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1} \geq \lambda_i > 0$, $i > n+1$, and that the $\{\psi_k\}$ form an orthonormal basis for the range of TT'. Then $d_n(T(H_1);H_2) = \lambda_{n+1}^{1/2}$.

II) Let $T \in K(H_1,H_2)$ and let $\{s_k(T)\}$ be as above. For $q \in [1,\infty]$, set

$$B = \{T\phi : \|T\phi\|_{H_2}^q + \|\phi\|_{H_1}^q \leq 1\}.$$

Then $d_n(B;H_2) = s_{n+1}(T)/(s_{n+1}^q(T) + 1)^{1/q}$.

III) Let U_r be a fixed r-dimensional subspace of H_2, and set $C_r = \{T\phi+u : \|\phi\|_{H_1} \leq 1, u \in U_r\}$. Let Q_r denote the orthogonal projection of H_2 onto U_r and $T_r = T - Q_rT$. Then

$$d_n(C_r;H_2) = \begin{cases} \infty, & n < r \\ d_{n-r}(T_r(H_1),H_2), & n \geq r \end{cases}$$

and for $n \geq r$, $\text{span}\{U_r,\psi_1,\ldots,\psi_{n-r}\}$ is optimal for $d_n(C_r;H_2)$ whenever $\text{span}\{\psi_1,\ldots,\psi_{n-r}\}$ is optimal for $d_{n-r}(T_r(H_1);H_2)$.

IV) Let $v_1,\ldots,v_r$ be r linearly independent functions in H_1, and set

$D^r = \{T\phi : \|\phi\|_{H_1} \leq 1, (\phi,v_i) = 0, i = 1,\ldots,r\}$.

Let P_r denote the orthogonal projection of H_1 onto $V_r = \text{span}\{v_1,\ldots,v_r\}$, and $T^r = T - TP_r$. Then

$$d_n(D^r;H_2) = d_n(T^r(H_1);H_2),$$

and the optimal subspaces agree.

V) (Melkman and Micchelli [32]). In place of the H_1 considered above, let H_0 and H_1 be two Hilbert semi-normed spaces over the same linear space M. Let $\|\cdot\|_0$, $\|\cdot\|_1$ denote their respective semi-norms. Then $\|\cdot\|_*$ $= \max\{\|\cdot\|_0, \|\cdot\|_1\}$ defines a semi-norm on M and we denote this semi-normed linear space by E. Set

$$F = \{T\phi : \|\phi\|_i \le 1, i = 0,1\}$$

where $T \in L(E,H_2)$. For each $\lambda \in [0,1]$, set $\|\cdot\|_\lambda =$ $(\lambda\|\cdot\|_0^2 + (1-\lambda)\|\cdot\|_1^2)^{1/2}$, and $F_\lambda = \{T\phi : \|\phi\|_\lambda \le 1\}$. Then

$$d_n(F;H_2) = \inf\{d_n(F_\lambda;H_2) : 0 \le \lambda \le 1\}$$

and if $d_n(F;H_2) = d_n(F_\mu;H_2)$ and X_n is optimal for $d_n(F_\mu;H_2)$, then it is also optimal for $d_n(F;H_2)$.

VI) (Ismagilov [14]). Let $K \in C([0,1]\times[0,1])$ and set

$$K_1 = \{f : f(x) = \int_0^1 K(x,y)h(y)dy, \|h\|_1 \le 1\}$$

where $\|h\|_p = (\int_0^1 |h(y)|^p dy)^{1/p}$. Let $\lambda_1 \ge \lambda_2 \ge \dots$ and $\phi_1, \phi_2, \dots$ denote the eigenvalues and orthonormal eigenfunctions of the compact, self-adjoint, non-negative integral operator with kernel $K'K(x,y) = \int_0^1 K(z,x)K(z,y)dz$. Then

$$\left(\sum_{j=n+1}^{\infty} \lambda_j\right)^{1/2} \le d_n(K_1;L^2) \le \max_{0\le x\le 1} \left(\sum_{j=n+1}^{\infty} \lambda_j|\phi_j(x)|^2\right)^{1/2}.$$

Before presenting further theoretical results, let us consider some examples.

EXAMPLE 2.2 (Kolmogorov [21]): Let $\tilde{W}_p^{(r)}$ $(= \tilde{W}_p^{(r)}[0,2\pi])$ denote the L^p Sobolev space of periodic functions on $[0,2\pi]$. Thus $\tilde{W}_p^{(r)} = \{f : f$ is 2π-periodic, $f^{(r-1)}$ is abs.cont. and $f^{(r)} \in L^p[0,2\pi]\}$. Set $\tilde{B}_p^{(r)} = \{f : f \in \tilde{W}_p^{(r)}, \|f^{(r)}\|_p \le 1\}$ and let us calculate $d_n(\tilde{B}_2^{(r)};L^2)$.

Every $f \in \widetilde{W}_2^{(r)}$ may be written in the form

$$f(x) = a_0 + \frac{1}{\pi}\int_0^{2\pi} D_r(x-t)\, f^{(r)}(t)\, dt$$

with the stipulation that $\int_0^{2\pi} f^{(r)}(t)dt = 0$, and $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$. The kernel $D_r(x)$ is given by

$$D_r(x) = \sum_{k=1}^{\infty} \frac{\cos(kx - \frac{r\pi}{2})}{k^r}, \quad r = 1,2,\ldots \quad .$$

Thus from a combination of (III) and (IV) (see also Example 2.1) we obtain $d_0(\widetilde{B}_2^{(r)};L^2) = \infty$, $d_{2n-1}(\widetilde{B}^{(r)};L^2) = d_{2n}(\widetilde{B}_2^{(r)};L^2) = n^{-r}$, $n \geq 1$, and T_{n-1} is an optimal subspace for d_{2n-1} (and hence for d_{2n}).

EXAMPLE 2.3 (Kolmogorov [21]): $W_p^{(r)}$ $(= W_p^{(r)}[0,1])$ will denote the L^p Sobolev space of functions on $[0,1]$, given by $W_p^{(r)} = \{f : f^{(r-1)}$ is abs. cont., $f^{(r)} \in L^p[0,1]\}$. Set $B_p^{(r)} = \{f : f \in W_p^{(r)},\ \| f^{(r)}\|_p \leq 1\}$. We wish to calculate $d_n(B_2^{(r)};L^2)$. Every function $f \in W_2^{(r)}$ has the representation

$$f(x) = \sum_{i=0}^{r-1} f^{(i)}(0)\, x^i/i! + 1/(r-1)!\int_0^1 (x-y)_+^{r-1} f^{(r)}(y)dy.$$

An application of (III) gives

$$d_n(B_2^{(r)};L^2) = \begin{cases} \infty & , \quad n < r \\ \lambda_{r,\, n-r+1}^{1/2} & , \quad n \geq r \end{cases}$$

where $\lambda_{r,k}$ is the k th eigenvalue (arranged in decreasing order of magnitude) of the eigenvalue problem

$$(-1)^r \lambda y^{(2r)}(x) = y(x)$$
$$y^{(i)}(0) = y^{(i)}(1) = 0,\ i = 0,1,\ldots, r-1,$$

and span $\{1,x,\ldots,x^{r-1}, y_{r,1}(x),\ldots, y_{r,n-r}(x)\}$ is an optimal subspace where $y_{r,k}$ is the eigenfunction associated with the eigenvalue $\lambda_{r,k}$.

EXAMPLE 2.4 (Melkman [27]): Let $L^2(\mathbb{R})$ denote the space of complex-valued square integrable functions on $\mathbb{R}$ with the usual L^2-norm. For $T,\sigma > 0$, let

$$\mathcal{D}_T = \{f : f \in L^2(\mathbb{R}), f(t) = 0 \text{ for } |t| \geq T\}$$

and

$$\mathcal{B}_\sigma = \{f : f \in L^2(\mathbb{R}), \hat{f}(x) = 0 \text{ for } |x| \geq \sigma\}.$$

$\mathcal{D}_T$ is the class of functions "time-limited" to $(-T,T)$, and $\mathcal{B}_\sigma$ the class of functions "band-limited" $(-\sigma,\sigma)$. Let D and B denote the projections of $L^2(\mathbb{R})$ onto $\mathcal{D}_T$ and $\mathcal{B}_\sigma$, respectively, given by

$$(Df)(t) = \begin{cases} f(t), & |t| < T \\ 0, & |t| \geq T \end{cases}$$

$$(\widehat{Bf})(x) = \begin{cases} \hat{f}(x), & |x| < \sigma \\ 0, & |x| \geq \sigma, \end{cases}$$

and set

$$G = \{f : f \in L^2(\mathbb{R}), \|(I-D)f\| \leq \varepsilon, \|(I-B)f\| \leq \eta\} .$$

BD is a self-adjoint, non-negative, compact operator. It has eigenvalues $\{\lambda_k\}_{k=1}^\infty$, where $1 > \lambda_1 > \lambda_2 > \dots$. We have

$$d_n(G;L^2(\mathbb{R})) = \left(\frac{\varepsilon^2 + \eta^2 + 2\varepsilon\eta\sqrt{\lambda_{n+1}}}{1 - \lambda_{n+1}}\right)^{1/2},$$

and

$$d_n(G;L^2(-T,T)) = \frac{\eta + \varepsilon\sqrt{\lambda_{n+1}}}{\sqrt{1 - \lambda_{n+1}}}.$$

(These n-widths do not tend to zero.)

EXAMPLE 2.5: Let k be continuous and 2π-periodic, $k(x) \sim \sum_{-\infty}^{\infty} a_m e^{imx}$ where $|a_0| \geq |a_1| \geq \dots$. Set $\mathcal{K}_1 = \{k*\phi : \|\phi\|_1 \leq 1\}$. Then (from (VI) and Example 2.1), $d_{2n-1}(\mathcal{K}_1;L^2) = (\sum_{|k|\geq n} |a_k|^2)^{1/2}$, while $d_{2n}(\mathcal{K}_1;L^2) = (\sum_{|k|\geq n+1} |a_k|^2 + |a_n|^2)^{1/2}$. T_{n-1} is optimal for d_{2n-1}, and span$\{T_{n-1}$, $\cos(nx+\alpha)\}$, for any real α, is optimal for d_{2n}.

We would like to apply this result to the calculation of $d_n(\tilde{B}_1^{(r)};L^2)$. However we cannot due to the orthogonality condition $\int_0^{2\pi} f^{(r)}(t)dt = 0$. From the method of proof of (VI) one may obtain (Taikov [47])

$$\sup_{0\le y\le \frac{\pi}{n-1}} \left(\sum_{k=n}^{\infty} k^{-2r}(1-\cos ky)\right)^{1/2} \le d_n(\tilde{B}_1^{(r)};L^2) \le \sup_{0\le y\le 2\pi} \left(\sum_{k=n}^{\infty} k^{-2r}(1-\cos ky)\right)^{1/2}.$$

(The upper bound is the exact error in approximation from T_{n-1}.)

As previously mentioned, the eigenfunctions are not the only subspaces which are optimal for $d_n(T(H_1);H_2)$. This fact was noted by Karlovitz [16], [17]. We shall not deal with the general question of when a subspace is optimal, but will construct explicit optimal subspaces for certain classes of functions. We consider integral operators whose kernels possess total positivity properties. Our first class of kernels is the following:

DEFINITION 2.1: <u>We say that the function</u> $K \in C([0,1]\times[0,1])$ <u>is nondegenerate totally positive</u> (NTP) <u>if for every choice of</u> $0 < x_1 < \cdots < x_n < 1$; $0 < y_1 < \cdots < y_n < 1$, <u>and all n</u>

1) $K\begin{pmatrix} x_1,\ldots,x_n \\ y_1,\ldots,y_n \end{pmatrix} \ge 0$

2) $\dim \operatorname{span}\{K(x_1,\cdot),\ldots,K(x_n,\cdot)\}$
$= \dim \operatorname{span}\{K(\cdot,y_1),\ldots,K(\cdot,y_n)\} = n.$

The condition of nondegenerate total positivity implies that the kernel $KK'(x,y)$ is totally positive (TP) and

$$KK'\begin{pmatrix} x_1,\ldots,x_n \\ x_1,\ldots,x_n \end{pmatrix} > 0$$

for all $0 < x_1 < \ldots < x_n < 1$. This in turn implies that the eigenvalue problem $KK'\psi(x) \left(= \int_0^1 KK'(x,y)\psi(y)dy\right) = \lambda\psi(x)$ possesses an infinite number of eigenvalues, all of which are positive and simple, i.e., $\lambda_1 > \lambda_2 > \ldots > 0$, and if ψ_i denotes the eigenfunction (unique up to multiplication by a constant) associated with eigenvalue λ_i, then on $(0,1)$ ψ_i has exactly $(i-1)$ sign changes and no other zeros. Set $\phi_i = K'\psi_i$. Then $K'K\phi_i = \lambda_i\phi_i$ and ϕ_i also has exactly $(i-1)$ zeros (sign changes) on $(0,1)$. Let

$$0 < \xi_1 < \ldots < \xi_n < 1 \; ; \; 0 < \eta_1 < \ldots < \eta_n < 1$$

denote the n zeros of ϕ_{n+1} and ψ_{n+1}, respectively, on $(0,1)$.

THEOREM 2.2 (Melkman, Micchelli [31]): Let K be an NTP kernel, and $K_2 = \{K\phi : (K\phi)(x) = \int_0^1 K(x,y)\phi(y)dy, \; \|\phi\|_2 \leq 1\}$. Then

$$d_n(K_2;L^2) = \lambda_{n+1}^{1/2},$$

and the following three subspaces are optimal for d_n:

1) $X_n^1 = \text{span}\{\psi_1,\ldots,\psi_n\}$,

2) $X_n^2 = \text{span}\{K(\cdot,\xi_1),\ldots,K(\cdot,\xi_n)\}$.

3) $X_n^3 = \text{span}\{KK'(\cdot,\eta_1),\ldots,KK'(\cdot,\eta_n)\}$.

The following three subspaces are optimal for $d^n(K_2;L^2)(= \lambda_{n+1}^{1/2})$:

1) $L_1^n = \{h : (h,\phi_i) = 0, i = 1,\ldots,n\}$,

2) $L_2^n = \{h : (Kh)(\eta_i) = 0, i = 1,\ldots,n\}$,

3) $L_3^n = \{h : (K'Kh)(\xi_i) = 0, i = 1,\ldots,n\}$.

This theorem, as stated, cannot be directly applied to the set $B_2^{(r)}$ (see Example 2.3) due to the presence of the free r-dimensional subspace π_{r-1}. However, the above result was extended to include this class. We state not the full extension, but only the result for $B_2^{(r)}$.

There exist points $0<\xi_1 < \ldots < \xi_{n-r} < 1$ and $0<\eta_1 < \ldots < \eta_n < 1$ such that the following subspaces are also optimal for $d_n(B_2^{(r)};L^2)$.

1) $X_n^2 = \text{span}\{1,x,\ldots,x^{r-1},(x-\xi_1)_+^{r-1},\ldots,(x-\xi_{n-r})_+^{r-1}\}$

2) $X_n^3 = \{s : s \in \text{span}\{1,x,\ldots,x^{2r-1},(x-\eta_1)_+^{2r-1},\ldots,(x-\eta_n)_+^{2r-1}\},$
$s^{(i)}(0) = s^{(i)}(1) = 0, i = r,\ldots, 2r-1\}.$

Furthermore, interpolation from X_n^2 or X_n^3 at the points $\{\eta_i\}_{i=1}^n$ is also an optimal method of approximation. (Note that X_n^3 is a class of "natural splines".)

Analogues of the results of Theorem 2.2 exist for classes of periodic functions. However the kernel $K(x,y) = k(x-y)$ of such a class cannot possibly be totally positive, since the even order determinants (as given in Definition 2.1) must change sign. Nonetheless we can obtain analogues of Theorem 2.2 for even integers n when k is a cyclic variation diminishing kernel. We prefer, however, to present a slightly altered class of functions, since they include, as a special case, $\tilde{B}_2^{(r)}$ (see Example 2.2). Set

$$\tilde{\mathcal{B}}_2 = \{a + (G*\phi)(x) \ : \ \|\phi\|_2 \le 1 \ , \ \phi \perp 1 \ , \ a \in \mathbb{R}\}$$

where by $\phi \perp 1$ we mean $\int_0^{2\pi}\phi(y)\,dy = 0$. Assume that G is a real, 2π-periodic, continuous function such that for every choice of $0 \le y_1 < \ldots < y_m < 2\pi$, and each m, the subspace

$$X_m = \left\{b + \sum_{i=1}^m b_i \ G(\cdot - y_i) \ : \ \sum_{i=1}^m b_i = 0\right\}$$

is of dimension m, and is a weak Tchebycheff system for m odd. In this case we say that G satisfies Property B. If $G(x) \sim \sum_{-\infty}^{\infty} a_m e^{imx}$ and G satisfies Property B, then $|a_m| \ge |a_{m+1}|$, $m = 1,2,\ldots$.

THEOREM 2.3: Let G satisfy Property B. Then, $d_n(\tilde{\mathcal{B}}_2;L^2) = \infty$, and $d_{2n-1}(\tilde{\mathcal{B}}_2;L^2) = d_{2n}(\tilde{\mathcal{B}}_2;L^2) = |a_n|$, $n = 1,2,\ldots$. Furthermore,

1) T_{n-1} is optimal for d_{2n-1} .

2) $X_{2n}^2 = \left\{ b + \sum_{i=0}^{2n-1} b_i \, G\left(\cdot - \frac{i\pi}{n}\right) : \sum_{i=0}^{2n-1} b_i = 0 \right\}$ is an optimal 2n-dimensional subspace for d_{2n}.

In addition, there exists a $\beta \in [0, \pi/n)$ for which interpolation from X_{2n}^2 at $\{\beta + \frac{i\pi}{n}\}_{i=0}^{2n-1}$ is an optimal method of approximation.

3. Exact n-Widths for Integral Operators

One of the first major results in a non-Hilbert space setting was due to Tichomirov [48]. He calculated $d_n(\tilde{B}_\infty^{(r)}; L^\infty)$ and proved the optimality of T_{n-1} for d_{2n-1} $(= d_{2n})$. (The upper bound is the Favard, Krein, Achieser Theorem.) More striking was a later result of Tichomirov [49], where he calculated $d_n(B_\infty^{(r)}; L^\infty)$ (the non-periodic case) and proved the existence of an optimal subspace of the form $\text{span}\{1, x, \ldots, x^{r-1}, (x-\xi_1)_+^{r-1}, \ldots, (x-\xi_{n-r})_+^{n-1}\}$ for some choice of $0 < \xi_1 < \ldots < \xi_{n-r} < 1$.

To explain this latter result, recall that every $f \in B_\infty^{(r)}$ is of the form

$$f(x) = \sum_{i=0}^{r-1} a_i x^i + \frac{1}{(r-1)!} \int_0^1 (x-y)_+^{r-1} f^{(r)}(y)\,dy$$

where $\|f^{(r)}\|_\infty \le 1$. The optimal subspace obtained is spanned by the kernel of the integral operator evaluated at n-r distinct points and, of necessity, the fixed r-dimensional subspace spanned by $1, x, \ldots, x^{r-1}$. The underlying property which permits us to calculate the n-widths, and determine optimal subspaces for this class, is the total positivity of the kernel $(x-y)_+^{r-1}$.

To ease our exposition we will consider sets of the form

$$K_p = \left\{ Kh(x) : Kh(x) = \int_0^1 K(x,y)h(y)\,dy \,,\ \|h\|_p \le 1 \right\}$$

and concern ourselves with the evaluation of the n-widths of

K_p in $L^q(p,q \in [1,\infty])$, i.e., we disregard the complications due to the presence of a fixed finite-dimensional subspace.
To state the main result, we define

$$\Lambda_n = \{\underline{\xi} : \underline{\xi} = (\xi_1,\ldots,\xi_n) \ , \ 0 < \xi_1 < \ldots < \xi_n < 1\}.$$

For each $\underline{\xi} \in \Lambda_n$, set $h_{\underline{\xi}}(x) = (-1)^i$, $x \in (\xi_{i-1},\xi_i)$, $i = 1,\ldots, n+1$, where $\xi_0 = 0$, $\xi_{n+1} = 1$.

THEOREM 3.1 (Micchelli, Pinkus [33]): Let K be an NTP kernel, and $q \in [1,\infty]$. There exists an $\underline{\xi}^* \in \Lambda_n$ (dependent on q) such that

$$\| Kh_{\underline{\xi}^*}\|_q = \inf\{\| Kh_{\underline{\xi}}\|_q : \underline{\xi} \in \Lambda_n\}.$$

Furthermore,

$$d_n(K_\infty;L^q) = d^n(K_\infty;L^q) = \delta_n(K_\infty;L^q) = \| Kh_{\underline{\xi}^*}\|_q ,$$

and

1) $X_n = \text{span}\{K(\cdot,\xi_1^*),\ldots, K(\cdot,\xi_n^*)\}$ is optimal for X_n.
2) $L^n = \{h : Kh(\eta_i^*) = 0 , \ i = 1,\ldots,n\}$ is optimal for d^n, where the $\{\eta_i^*\}_{i=1}^n$ are the n unique zeros of $Kh_{\underline{\xi}^*}$ in (0,1).
3) The unique operator of interpolation from X_n at the points $\{\eta_i^*\}_{i=1}^n$ is optimal for δ_n.

From duality considerations (Proposition 1.9 and Theorem 1.11), we also have

PROPOSITION 3.2 (Micchelli, Pinkus [33]): Let K be an NTP kernel, $p \in [1,\infty]$, $1/p + 1/p' = 1$. Then
$d_n(K_p;L^1) = d^n(K_p;L^1) = \delta_n(K_p;L^1) = \inf\{\|K'h_{\underline{\xi}}\|_{p'} : \underline{\xi} \in \Lambda_n\}$.

A statement concerning optimal subspaces, as in Theorem 3.1, is also valid for Proposition 3.2. (The above quantities equal the associated Bernstein n-widths when $q = \infty$ (in Theorem 3.1), and $p = 1$ (in Proposition 3.2).)

It is conjectured that for all $1 \le q \le p \le \infty$, $d_n(K_p;L^q) = d^n(K_p;L^q) = \delta_n(K_p;L^q)$, and that there exists an optimal subspace for d_n of the form $X_n = \text{span}\{K(\cdot,\xi_1),\ldots, K(\cdot,\xi_n)\}$, with the analogous statements for d^n and δ_n. However this has only been verified when $q=\infty$; $p=1$; and $p=q=2$.

When dealing with periodic functions the situation is somewhat different, but still analogous. Set

$$\mathcal{B}_p = \{a + (G*h)(x) \;:\; \|h\|_p \le 1\,,\; h \perp 1\,,\; a \in \mathbb{R}\}$$

and $h_n(x) = (-1)^i\,,\ \frac{\pi(i-1)}{n} < x \le \frac{\pi i}{n}\,,\ i = 1,\ldots, 2n.$

THEOREM 3.3 (Pinkus [40]): Let G satisfy Property B (see Section 2). Then

(A) $d_{2n-1}(\mathcal{B}_\infty;L^\infty) = d^{2n-1}(\mathcal{B}_\infty;L^\infty) = \delta_{2n-1}(\mathcal{B}_\infty;L^\infty) = \|G*h_n\|_\infty$.

1) T_{n-1} is optimal for d_{2n-1}.

2) $L^{2n-1} = \{f : f \in \mathcal{B}_\infty\,,\ f \perp T_{n-1}\}$ is optimal for d^{2n-1}.

3) For $f = a + G*h$, set $S_{2n-1}f = a + t^* * h$, where t^* is the unique best L^1-approximant to G from T_{n-1}. Then S_{2n-1} is optimal for δ_{2n-1}.

(B) $d_{2n}(\mathcal{B}_\infty;L^q) = d^{2n}(\mathcal{B}_\infty;L^q) = \delta_{2n}(\mathcal{B}_\infty;L^q) = \|G*h_n\|_q$ for all $q \in [1,\infty]$.

1) $X_{2n} = \left\{b + \sum_{i=0}^{2n-1} b_i G(\cdot - \frac{\pi i}{n}) : \sum_{i=0}^{2n-1} b_i = 0\right\}$ is optimal for d_{2n-1}.

2) $L^{2n} = \{f : f \in \mathcal{B}_\infty\,,\ f(\frac{\pi i}{n}) = 0\,,\ i = 0,1,\ldots,2n-1\}$ is optimal for d^{2n-1}.

3) There exists a $\beta \in [0,\pi/n)$ such that interpolation from X_{2n} at the points $\{\beta + \frac{\pi i}{n}\}_{i=0}^{2n-1}$ is optimal for δ_{2n}.

(C) $d_{2n-1}(\mathcal{B}_1;L^1) = d^{2n-1}(\mathcal{B}_1;L^1) = \delta_{2n-1}(\mathcal{B}_1;L^1) = \|G*h_n\|_\infty$ and the optimal subspaces of (A) are optimal here.

(D) $d_{2n}(\mathcal{B}_p;L^1) = d^{2n}(\mathcal{B}_p;L^1) = \delta_{2n}(\mathcal{B}_p;L^1) = \|G*h_n\|_{p'}$, for all $p \in [1,\infty]$, where $1/p + 1/p' = 1$, and the optimal subspaces of (B) are optimal here.

It is conjectured that the following hold for $1 \le q \le p \le \infty$.

CONJECTURE 1: T_{n-1} is optimal for $d_{2n-1}(\mathcal{B}_p; L^q)$.

CONJECTURE 2: X_{2n}, as above, is optimal for $d_{2n}(\mathcal{B}_p; L^q)$.

CONJECTURE 3: $d_{2n-1}(\mathcal{B}_p; L^q) = d_{2n}(\mathcal{B}_p; L^q)$.

We have seen that Conjectures 1 and 3 are valid if $p = q \in \{1, 2, \infty\}$, and Conjecture 2 holds for $p = q = 2$; $p = \infty$, all q ; $q = 1$, all p.

The following generalization of a result of Taikov [45] proves Conjectures 1 and 3 for $q = 1$, all p.

THEOREM 3.4: Let G satisfy Property B. Then for $p \in [1, \infty]$, $1/p + 1/p' = 1$,

$$d_{2n-1}(\mathcal{B}_p; L^1) = d^{2n-1}(\mathcal{B}_\infty; L^{p'}) = \| G * h_n \|_{p'} .$$

1) T_{n-1} is optimal for d_{2n-1}.

2) $L^{2n-1} = \{f : f \in \mathcal{B}_\infty , f \perp T_{n-1}\}$ is optimal for d^{2n-1}.

Analogues of the above results, i.e., Theorems 3.3 and 3.4, are valid when we replace $\mathcal{B}_p$ by

$$\mathcal{K}_p = \{k * h : \|h\|_p \le 1\}$$

where k is a cyclic variation diminishing kernel (see Pinkus [38]).

For the special case of the Sobolev class of periodic functions, there has been some very nice work done by Korneichuk and associates (see e.g. Korneichuk [22], Motornyi and Ruban [35], Ruban [42], and references therein). Their interest is with n-widths of the class of functions $f \in W_\infty^{(r)}$ for which $\omega(f^{(r-1)}; t) \le \omega(t)$, where $\omega(t)$ is a given concave modulus of continuity.

4. Matrices and n-Widths

When dealing with the n-widths of the class $A = \{Tx : \|x\| \le 1\}$, $T \in L(X,Y)$, it is natural to first consider simple linear operators. We will therefore assume that T is given by a matrix $D \in \mathbb{C}^{m\times m}$ and $X = \ell_p(\mathbb{C}^m)\ (=\ell_p^m)$, $Y = \ell_q(\mathbb{C}^m)\ (=\ell_q^m)$. Except when $p = q = 2$ (and $n = 0$, $n = m-1$), very little is known about these n-widths for a general matrix. When D is totally positive then analogues of the results of Section 3 (and Section 2) hold, but the matrix setting is, in some sense, incidental.

We will assume that D is an $m\times m$ diagonal matrix, $D = \mathrm{diag}\{D_1,\ldots,D_m\}$, where without loss of generality, we assume that $D_1 \geq D_2 \geq \ldots \geq D_m > 0$. Set $\mathcal{D}_p = \{D\underline{x} : \|\underline{x}\|_{\ell_p^m} \leq 1\}$.

THEOREM 4.1 (Pietsch [36]): Given $1 \leq q \leq p \leq \infty$, let $1/r = 1/q - 1/p$. Then for $0 \leq n < m$,

$$d_n(\mathcal{D}_p;\ell_q^m) = d^n(\mathcal{D}_p;\ell_q^m) = \delta_n(\mathcal{D}_p;\ell_q^m) = \Big(\sum_{k=n+1}^{m} D_k^r\Big)^{1/r}.$$

Furthermore,

1) $X_n = \mathrm{span}\{\underline{e}^1,\ldots,\underline{e}^n\}$ is optimal for d_n, where $\underline{e}^i$ is the i^{th} unit vector.
2) $L^n = \{\underline{x} : x_i = 0,\ i = 1,\ldots,n\}$ is optimal for d^n.
3) $P_n = \mathrm{diag}\{D_1,\ldots, D_n, 0,\ldots, 0\}$ is optimal for δ_n.

While this is hardly a surprising result, it does not remain true if $q > p$, nor do these n-widths equal the Bernstein n-widths $b_n(\mathcal{D}_p;\ell_q^m)$ except when $q = p$.

When $q > p$, we have the following two results.

PROPOSITION 4.2: If $1 \leq p < q \leq \infty$, $1/r = 1/q-1/p\ (<0)$, and $D_{n+2} \leq \Big(\sum_{k=1}^{n+1} D_k^r\Big)^{1/r}$, then

$$d_n(\mathcal{D}_p;\ell_q^m) = d^n(\mathcal{D}_p;\ell_q^m) = \delta_n(\mathcal{D}_p;\ell_q^m) = b_n(\mathcal{D}_p;\ell_q^m) = \Big(\sum_{k=1}^{n+1} D_k^r\Big)^{1/r}.$$

THEOREM 4.3 (Sofman [44]):

$$d_n(\mathcal{D}_1;\ell_2^m) = d^n(\mathcal{D}_2;\ell_\infty^m) = \delta_n(\mathcal{D}_1;\ell_2^m) = \delta_n(\mathcal{D}_2;\ell_\infty^m) = \max_{n<r\leq m}\left(\frac{r-n}{\sum_{k=1}^{r} D_k^{-2}}\right)^{1/2}.$$

(Note that $r = n+1$ in Theorem 4.3 if and only if the inequality of Proposition 4.2 holds.)

Because of their importance in the asymptotic estimates of n-widths of Sobolev spaces, much effort has been devoted to estimates of $d_n(\mathcal{D}_1;\ell_\infty^m)$ $(= d^n(\mathcal{D}_1;\ell_\infty^m) = \delta_n(\mathcal{D}_1;\ell_\infty^m))$ and $d_n(\mathcal{D}_2;\ell_\infty^m)$ $(= d^n(\mathcal{D}_1;\ell_2^m))$ when $D = I$, the identity matrix. Set $I_p = \{\underline{x} : \|\underline{x}\|_{\ell_p^m} \le 1\}$. The following lower bound holds for $d_n(\mathcal{D}_1;\ell_\infty^m)$.

PROPOSITION 4.4 (Pinkus [39]):

$$d_n(I_1;\ell_\infty^m) \ge \left(1 + [(m-1)n/(m-n)]\right)^{-1/2}.$$

This lower bound is sometimes attained and sometimes not. It can only be attained if $m \le \min\{n^2, (m-n)^2\}$ (Melkman [29]), and a full discussion of the lower bound, and when it is attained (or is not attained) may be found in Melkman [29].

Various estimates have been given for the more important upper bound. We list these below:

(1) Ismagilov [15], Glushkin [9]: $d_n(I_1;\ell_\infty^m) \le c\sqrt{m}/n$, c a constant independent of m and n.

(2) Kashin [18]: $d_n(I_1;\ell_\infty^m) \le 2((\ln m)/n)^{1/2}$.

(3) Kashin [19]: For $n < m < n^\lambda$, $d_n(I_1;\ell_\infty^m) \le c_\lambda/n^{1/2}$, c_λ depends only on λ.

(4) Kashin [20]: $d_n(I_1;\ell_\infty^m) \le \frac{c}{\sqrt{n}} \left(1 + \ln\left(\frac{m}{n}\right)\right)^{1/2}$, c independent of m and n.

(5) Hollig [12]: $d_n(I_1;\ell_\infty^m) \le \frac{8}{\sqrt{n}} \left(\frac{\ln m}{\ln n}\right)$

Different techniques enter into these different estimates. Estimate (5), for example, uses algebra to obtain estimates on a combinatorial problem. Estimate (2) is based on an inequality from probability theory. A totally different approach related to packing on the unit sphere is applied by Melkman [28].

We also have:

THEOREM 4.5 (Kashin [20]):

$$d_n(I_2;\ell_\infty^m) \le \frac{c}{\sqrt{n}}\left(1 + \ell n(m/n)\right)^{3/2},$$

where c *is a constant independent of* m *and* n.

This is a much more difficult problem than the previous one. The proof of Theorem 4.5 is probabilistic in nature and very complicated. A simpler and constructive proof of this inequality would be of interest.

5. Asymptotic Estimates of n-Widths of Sobolev Spaces

One problem which was the subject of considerable interest for a number of years was the determination of the asymptotic behaviour of the values $d_n(B_p^{(r)};L^q)$, $d^n(B_p^{(r)};L^q)$ and $\delta_n(B_p^{(r)};L^q)$ for $p,q \in [1,\infty]$. This problem has been essentially solved. (There are some open problems when r is small.)

We write $d_n(B_p^{(r)};L^q) \asymp n^s$ to mean that there exist positive constants C and D, independent of n, although they may and generally do depend on p,q and r, for which
$Dn^s \le d_n(B_p^{(r)};L^q) \le Cn^s$.

THEOREM 5.1: *For* $r \ge 2$, $1/p + 1/p' = 1$,

1) $$d_n(B_p^{(r)};L^q) \asymp \begin{cases} n^{-r} & , 1 \le q \le p \le \infty \\ n^{-r} & , 2 \le p \le q \le \infty \\ n^{-r+1/p-1/2} & , 1 \le p \le 2 \le q \le \infty \\ n^{-r+1/p-1/q} & , 1 \le p \le q \le 2 \end{cases}$$

2) $$d_n(B_p^{(r)};L^q) \asymp \begin{cases} n^{-r} & , 1 \le q \le p \le \infty \\ n^{-r} & , 1 \le p \le q \le 2 \\ n^{-r+1/2-1/q} & , 1 \le p \le 2 \le q \le \infty \\ n^{-r+1/p-1/q} & , 2 \le p \le q \le \infty \end{cases}$$

$$3)\qquad \delta_n(B_p^{(r)};L^q) \asymp \begin{cases} n^{-r} & , 1 \le q \le p \le \infty \\ n^{-r+1/p-1/q} & , 1 \le p \le q \le 2 \\ n^{-r+1/p-1/q} & , 2 \le p \le q \le \infty \\ n^{-r+1/p-1/2} & , 1 \le p \le 2 \le q \le \infty,\ p' \ge q \\ n^{-r+1/2-1/q} & , 1 \le p \le 2 \le q \le \infty,\ p' \le q . \end{cases}$$

These results extend, with minor modifications, to the unit cube in $\mathbb{R}^N$. In this case we replace r by r/N, and it is then necessary that the associated $B_p^{(r)}$ lies in L^q, i.e., the Sobolev Embedding Theorem obtains, or in other words $r/N-1/p+1/q > 0$. It may also be extended to more general compact domains in $\mathbb{R}^N$. Furthermore, interpolation theory techniques allow for extensions of these results to Besov spaces.

It transpires that the lower bounds are more easily obtained than the upper bounds. The lower bounds are calculated by a discretization technique.

THEOREM 5.2 (Maiorov [25]): <u>For</u> $p,q \in [1,\infty]$

$$d_n(B_p^{(r)};L^q) \ge Cm^{-r+1/p-1/q}\, d_n(I_p;\ell_q^m)$$
$$d^n(B_p^{(r)};L^q) \ge Cm^{-r+1/p-1/q}\, d^n(I_p;\ell_q^m),$$

<u>where</u> C <u>is a constant independent of</u> n <u>and</u> m, <u>and</u> m <u>is any positive integer</u>.

The lower bounds of Theorem 5.1 are then proven by an application of Theorem 5.2 with m = 2n, and using the results of Section 4.

The upper bound is more interesting. From Birman and Solomjak [2] it follows that

$$\delta_n(B_p^{(r)};L^q) \le \begin{cases} Cn^{-r} & , 1 \le q \le p \le \infty \\ Cn^{-r+1/p-1/q} & , 1 \le p \le q \le \infty . \end{cases}$$

The crux of the problem is therefore the upper bound for $d_n(B_p^{(r)};L^q)$ when $p < q$, $q > 2$; for $d^n(B_p^{(r)};L^q)$ when $p < q$, $p < 2$;

and for $\delta_n(B_p^{(r)};L^q)$ when $p<2<q$. It was for some time generally thought that the upper bound $n^{-r+1/p-1/q}$ for $p\le q$ was asymptotically optimal, and that the lower bound needed to be improved. The converse turned out to be true. The upper bound is based on the following inequality which results from factorization and telescoping.

THEOREM 5.3 (Maiorov [25]): Let s_n denote any of the three quantities d_n, d^n or δ_n. Then

$$s_n(B_p^{(r)};L^q) \le C\sum_{k=0}^{\infty} 2^{-k(r-1/p+1/q)} s_{n_k}(I_p;\ell_q^{b_k})$$

where $b_k = r+2^k$, and the n_k are non-negative integers for which $\sum_{k=0}^{\infty} n_k \le n$.

It transpires that it suffices to prove $d_n(B_p^{(r)};L^\infty) \le Cn^{-r+1/p-1/2}$, for $1\le p\le 2$, to complete the picture for d_n. It is here that the estimate of Kashin, Theorem 4.5, is used. For δ_n, one must prove that $\delta_n(B_p^{(r)};L^{p'}) \le Cn^{-r+1/p-1/2}$ for $1\le p\le 2$, and here an estimate of Hollig [12] is used in Theorem 5.3 (see also Maiorov [26]). A totally different and interesting proof of the upper bound for $\delta_n(B_1^{(r)};L^\infty)$ is due to de Boor, DeVore, and Hollig [3].

6. n-Widths of Analytic Functions

Let $\Delta_R = \{z : |z| < R\}$. Let f be analytic in Δ_R and for $r<R$, set

$$M_p(r,f) = \begin{cases} \left(1/2\pi \int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p} & , 1\le p<\infty \\ \max\{|f(re^{i\theta})| : 0\le\theta<2\pi\} & , p=\infty . \end{cases}$$

We define the Hardy space

$$H_R^p = \{f : f \text{ analytic in } \Delta_R,\ \lim_{r\uparrow R} M_p(r,f) < \infty\}.$$

H_R^∞ is simply the set of bounded analytic functions on Δ_R, while H_R^2 is the class of power series $\sum_{n=0}^{\infty} a_n z^n$ for which $\sum_{n=0}^{\infty} R^{2n}|a_n|^2 < \infty$.

For $p \in [1,\infty]$, set $\|f\|_{H_R^p} = \lim_{r \uparrow R} M_p(r,f)$, and for fixed $m = 0,1,\ldots$, $R \geq 1$, set

$$A_R(m,p) = \{f : f^{(m)} \in H_R^p , \; \|f^{(m)}\|_{H_R^p} \leq 1\}.$$

In Section 1 we showed that $d_n\left(A_R(m,\infty);H_1^\infty\right) = R^{m-n}(n-m)!/n!$ for $n \geq m$. In fact the following result holds.

THEOREM 6.1 (Taikov [46]): Let $R \geq 1$, $m = 0,1,\ldots$, $p \in [1,\infty]$. Then

$$d_n\left(A_R(m,p);H_1^p\right) = d^n\left(A_R(m,p);H_1^p\right) = \delta_n\left(A_R(m,p);H_1^p\right) =$$

$$b_n\left(A_R(m,p);H_1^p\right) = \begin{cases} \infty & , \; n < m \\ R^{m-n}(n-m)!/n! & , \; n \geq m . \end{cases}$$

When considering the n-widths of $A_R(m,p)$ in H_1^q for $p \neq q$, nothing is known. When attempting to carry over the above result to the area norm, rather than the boundary norm, then again nothing is known except when $m = 0$ (in which case all the n-widths equal $R^{-n-2/p}$).

In a Hilbert space setting, rather more can be said. Let $H_1^2 = H^2$ and

$$A = A_1(0,2) = \{f : f \in H^2 , \; \|f\|_{H^2} \leq 1\} .$$

Let μ be a positive measure on a compact set K, $K \subset \Delta_1$. The calculation of the n-widths $d_n\left(A;L^2(K,d\mu)\right)$ $(= d^n = \delta_n = b_n)$ is equivalent, by Theorem 2.1, to the evaluation of the eigenvalues of

$$\lambda f(z) = \int_K \frac{f(w)}{1 - \bar{w}z} \, d\mu(w) \quad , \; z \in \Delta_1 .$$

Fisher, Micchelli [6] consider this problem and prove that

1) the eigenvalues are simple and distinct
2) if f_k is the $k^{\underline{th}}$ eigenfunction of the above eigenvalue problem, then f_k has exactly k-1 zeros in Δ_1 (counting multiplicities).

From Theorem 2.1, it may be seen that span$\{f_1,\ldots,f_n\}$ is an optimal subspace for d_n, with the analogous statement for

d^n. Using the above results, they obtain:

THEOREM 6.2 (Fisher, Micchelli [6]): If $z_1^*,\ldots,z_n^* \in \Delta_1$ are the n zeros of f_{n+1}, then

$$L^n = \{g : g \in H^2 , g(z_j^*) = 0, j = 1,\ldots,n\}$$

is optimal for $d^n(A;L^2(K,d\mu))$ (with the understanding that the appropriate derivatives are taken in case of equal z_j^*'s).

Let X_n denote the orthogonal complement to L^n in H^2. Then X_n is also optimal for $d_n(A;L^2(K,d\mu))$.

If the z_j^*'s are distinct, set $B(z) = \prod_{j=1}^{n} \frac{z-z_j^*}{1-\bar{z}_j^* z}$, and $M_j(z) = \frac{B(z)}{(z-z_j^*)} \frac{1}{B'(z_j^*)}$, $j = 1,\ldots,n$. Then $M_j(z_k^*) = \delta_{jk}$, and the M_j span X_n.

In certain cases, e.g. if $f_k(z) = z^{k-1}$, $k = 1,2,\ldots$, then this method does not give us any additional optimal subspace.

Now, let

$$A = A_1(0,\infty) = \{f : f \in H^\infty , \|f\|_{H^\infty} \le 1\}$$

and let K be again a compact subset of Δ. Set $X = C(K)$ or $X = L^q(\mu)$ for some $q \in [1,\infty)$ and some positive measure μ with support on K.

DEFINITION 6.1: A Blaschke product of degree m is any function of the form

$$B(z) = \sigma \prod_{j=1}^{m} \frac{z-\alpha_j}{1-\bar{\alpha}_j z} ,$$

where $|\alpha_j| < 1$, $j = 1,\ldots, m$ and $|\sigma| = 1$.

Let $\mathcal{B}_m$ denote the class of Blaschke products of degree $\le m$. The following striking result holds.

THEOREM 6.3 (Fisher, Micchelli [5]): For A and X as above,

$$d_n(A;X) = d^n(A;X) = \delta_n(A;X) = \min\{\|B\|_X : B \in \mathcal{B}_n\}.$$

It is an interesting open question as to whether, for $X = C(K)$ in particular, the minimum norm Blaschke product is unique.

In Example 2.4 we considered the class of time- and band-limited functions in an L^2 setting. The following L^∞ version of the result was recently proved by Melkman [30].

Let $B(\sigma,T)$ denote the class of entire functions f of exponential type σ for which $|f(t)| \leq 1$ for all $t \in (-\infty, -T) \cup (T,\infty)$.

THEOREM 6.4 (Melkman [30]): There is a function $F_n \in B(\sigma,T)$, which is real on $\mathbb{R}$ and such that:

1) F_n equioscillates on $[-T,T]$ between the values $\pm \|F_n\|_{C[-T,T]}$ exactly n+1 times at the points $\rho_1 < \ldots < \rho_{n+1}$
2) F_n equioscillates outside $(-T,T)$ between ± 1.
3) F_n has only the real, simple zeros implied by (1) and (2).
4) If $\|F_n\|_{C[-T,T]} < 1$, then $|F_n(\pm T)| = \|F_n\|_{C[-T,T]}$. Otherwise $|F_n(\pm T)| = 1$.

Let $\{t_i\}_{i=1}^n$ denote the n unique zeroes of F_n in $(-T,T)$. Then

THEOREM 6.5 (Melkman [30]):

$$d_n\big(B(\sigma,T)\big); C[-T,T] = d^n\big(B(\sigma,T) ; C[-T,T]\big)$$

$$= \delta_n\big(B(\sigma,T) ; C[-T,T]\big) = b_n\big(B(\sigma,T) ; C[-T,T]\big) = \|F_n\|_{C[-T,T]}.$$

Furthermore

1) $X_n = \mathrm{span}\{h(t)t^j\}_{j=0}^{n-1}$ is optimal for d_n, where
$$h(t) = \frac{F_n'(t)(t^2-T^2)}{\prod_{i=1}^{n+1}(t-\rho_i)}$$
2) $L^n = \{g : g \in B(\sigma,T), g(t_i) = 0, i=1,\ldots n\}$ is optimal for d^n.
3) Interpolation at the $\{t_i\}_{i=1}^n$ from X_n is an optimal operator for δ_n.
4) $Y_{n+1} = \left\{F_n(t)p(t)/w(t) : p \in \pi_n, w(t) = \prod_{i=1}^{n}(t-t_i)\right\}$ is optimal for b_n.

Set $S_\beta = \{z : z \in \mathbb{C}, |\text{Im } z| < \beta\}$. Let $\tilde{H}^p_\beta$ denote the class of functions f which are analytic in S_β, real and 2π-periodic on the x-axis, and satisfy $1/2\pi \int_0^{2\pi} |\text{Re } f(t+i\beta)|^p \, dt \le 1$. We wish to calculate $d_n(\tilde{H}^p_\beta; L^q[0,2\pi])$. This class was considered by Tichomirov [48] for $p = q = \infty$. A complete proof in that case was given by Forst [7]. A function f is in $\tilde{H}^p_\beta$ if and only if it is of the form

$$f(z) = 1/2\pi \int_0^{2\pi} K_\beta(z-t) \; \text{Re}[f(t+i\beta]dt,$$

for $z \in S_\beta$, where $K_\beta(z) = 1+2\sum_{k=1}^{\infty} \frac{\cos kz}{\cosh k\beta}$. As it turns out the kernel K_β is cyclic variation diminishing. It therefore follows that the analogues of Theorems 3.3 and 3.4 hold for this class.

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