Some Remarks on Nonnegative Polynomials on Polyhedra

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1. INTRODUCTION

A classical theorem of Lukács, see Szegö (1939), p. 4, states that an algebraic polynomial $p(t)$ of degree at most n, nonnegative on the interval $[a, b]$, has a representation of the form

$$(1.1) \qquad p(t) = \begin{cases} (t-a)(b-t)R^2(t) + Q^2(t), & n = 2m \\ (t-a)S^2(t) + (b-t)T^2(t), & n = 2m-1 \end{cases}$$

where R, S and T are real polynomials of degree $\leq m-1$, and Q is a polynomial of degree $\leq m$. This result and its corresponding versions on $[0, \infty)$ and $(-\infty, \infty)$ have spawned numerous generalizations and applications. They have been used, for example, in the theory of orthogonal polynomials, in the solution of moment problems, and in the study of certain extremal problems for polynomials. One recent use was given in Edelman, Micchelli (1987), where they studied the problem of interpolation by monotone piecewise polynomials.

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A refinement of Lukács' Theorem was given in Karlin, Shapley (1953), where it was proven that an algebraic polynomial $p(t)$ of degree at most n, nonnegative on $[a,b]$, with at most $n-1$ zeros on $[a,b]$ counting multiplicities, admits a unique representation of the form

$$p(t) = \alpha(t-a)(b-t)\prod_{j=1}^{m-1}(t-t_{2j})^2 + \beta\prod_{j=1}^{m}(t-t_{2j-1})^2, \tag{1.2a}$$

for $n = 2m$, and

$$p(t) = \alpha(t-a)\prod_{j=1}^{m-1}(t-t_{2j})^2 + \beta(b-t)\prod_{j=1}^{m-1}(t-t_{2j-1})^2, \tag{1.2b}$$

for $n = 2m-1$, where $\alpha, \beta > 0$ and $a \leq t_1 \leq \cdots \leq t_{n-1} \leq b$. In Karlin (1963), see also Karlin, Studden (1966), this result is generalized to Chebyshev systems. The role of this result in the geometry of moment spaces is well documented in Karlin, Studden (1966). In this regard (1.2) not only identifies the extreme rays of the cone of nonnegative polynomials of degree n on $[a,b]$, but also states that any other element of this cone is a positive combination of at most two such rays.

Attempts to generalize Lukács' Theorem in its various forms to polynomials of more than one variable quickly lead to negative results. For instance, in one dimension, if $p(t)$ is nonnegative for all real t, it is a sum of two squares. However, Hilbert (1888) showed that there exists a real polynomial of degree six in two variables which is nonnegative on $\mathbb{R}^2$, but which is not a finite sum of squares. His argument did not lead to an explicit polynomial. Motzkin (1967) was the first to provide such an example. Subsequently, several authors have provided other examples cf. Robinson (1973), Choi (1975), Schmüdgen (1979) and Berg, Christensen, Jensen (1979). It is interesting to note that Hilbert (1893) later showed that nonnegative polynomials are a sum of squares of rational functions.

Our initial starting point was the modest problem of a representation theorem, similar to (1.2), for *quadratic* polynomials of two variables on a *triangle*. Keeping (1.2) in mind for $n=2$, we believed that any quadratic polynomial $p(x)$ nonnegative on the triangle $\sigma = [v^1, v^2, v^3]$, with vertices $v^1, v^2, v^3 \in \mathbb{R}^2$ could be represented in the form

$$p(x) = \alpha\lambda_1(x)\lambda_2(x) + \beta\lambda_1(x)\lambda_3(x) + \gamma\lambda_2(x)\lambda_3(x) + q(x) \tag{1.3}$$

where the $(\lambda_1(x), \lambda_2(x), \lambda_3(x))$ are the barycentric coordinates of x relative to σ, the numbers α, β, and γ are nonnegative, and q is a bivariate quadratic polynomial nonnegative on all of $\mathbb{R}^2$, and vanishing at some point in σ. To somewhat explain this analogue to (1.2), each of the first three terms vanish on two sides of the triangle and are nonnegative thereon, and therefore correspond in a sense to the factor $(t-a)(b-t)$ of (1.2a). Our intuition was correct and we will prove this result in the course of our discussion.

Nevertheless, even representations of quadratic polynomials in more variables on a simplex becomes a murky problem indeed. We were only able to prove an analogue of (1.1), and not (1.2), for nonnegative quadratic polynomials on simplices in $\mathbb{R}^3$, while an analogue of both (1.1) and (1.2) on simplices in $\mathbb{R}^4$ is in general false. The reason, as we shall later see, is that this question relates to the notion of *copositive matrices* introduced by Motzkin (1952), (1965). Unfortunately the extreme rays of $n \times n$ copositive matrices have yet to be classified and they are not, for $n \geq 5$, what is demanded by a representation of the form (1.3) for simplices in $\mathbb{R}^{n-1}$. More explanation will be provided shortly.

Returning for the moment to univariate polynomials to obtain some guidance and motivation, we remark that there is yet another characterization theorem for *strictly* positive polynomials on $[0,1]$. It is given in terms of a nonnegative sum of *Bernstein* polynomials of some degree, generally higher than the degree of the polynomial they represent, cf. Pólya, Szegö (1976), p. 78, and Karlin, Studden (1966), p. 126. We will generalize this theorem to any number of variables by using the concept of *degree raising*. This idea provides us with a simple iterative test for determining when a polynomial is strictly positive on a simplex, and a rather simple iterative test for strict copositivity of a matrix. Tests for copositivity and strict copositivity have been the subject of numerous papers, cf. Motzkin (1952), (1965), Garsia (1964), Cottle, Habetler, Lemke (1970), Martin (1981), Hadeler (1983), and Väliaho (1986), (1988). The above test is linear in the matrix elements as opposed to various determinantal criteria of other authors.

We will now discuss these ideas and related ones in detail. Our first task is the precise formulation of the problem which we will address.

2. NONNEGATIVE POLYNOMIALS: PROJECTIVE COORDINATES

We let $\Pi_n(\mathbb{R}^d)$ denote the class of real algebraic polynomials of degree at most n on $\mathbb{R}^d$, i.e., $p \in \Pi_n(\mathbb{R}^d)$ if

$$p(x) = \sum_{|\alpha|\le n} a_\alpha x^\alpha \Big(= \sum_{|\alpha|\le n} a_\alpha x_1^{\alpha_1}\cdots x_d^{\alpha_d}\Big)$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_+^d$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and $a_\alpha \in \mathbb{R}$. We consider polynomials $p \in \Pi_n(\mathbb{R}^d)$ nonnegative over a convex polyhedron Ω in $\mathbb{R}^d$. The polyhedron is described by a finite set of linear inequalities. Namely,

$$\Omega = \{x : \Lambda(x) \ge 0,\ x \in \mathbb{R}^d\}$$

where the mapping $\Lambda : \mathbb{R}^d \to \mathbb{R}^m$, $\Lambda(x) = (\lambda_1(x), \ldots, \lambda_m(x))$ has affine coordinates given by

$$\lambda_i(x) := \delta^i \cdot x + \mu_i, \quad i = 1, \ldots, m,$$

with $\{\delta^1, \ldots, \delta^m\} \subseteq \mathbb{R}^d$, $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m$, and $\delta \cdot x = \sum_{j=1}^d \delta_j x_j$. Since we are concerned with nonnegative polynomials over Ω, we can and will always assume, without loss of generality, that $0 \in \Omega$. This simply means that $\mu \ge 0$. We also suppose that $\mu \ne 0$. This can always be assumed by adding redundant inequalities in the description of Ω.

We first list several concrete examples of polyhedra Ω to motivate and illustrate our subsequent remarks.

EXAMPLE 2.1 (Simplex). Let $\sigma := [v^1, \ldots, v^{d+1}]$ denote the convex hull of $v^1, \ldots, v^{d+1}$ in $\mathbb{R}^d$, where we assume that the $v^1, \ldots, v^{d+1}$ do not all lie on a hyperplane in $\mathbb{R}^d$. In other words, this is an arbitrary simplex in $\mathbb{R}^d$. We define $\Lambda(x) = (\lambda_1(x), \ldots, \lambda_{d+1}(x))$ as the *barycentric coordinates* of x relative to σ. The $\{\lambda_i(x)\}_{i=1}^{d+1}$ are determined by the equations

$$x = \sum_{j=1}^{d+1} \lambda_j(x) v^j$$

$$1 = \sum_{j=1}^{d+1} \lambda_j(x) .$$

Then $\Omega = \sigma$ and the map $c_\sigma(\lambda) := \sum_{j=1}^{d+1} \lambda_j v^j$ takes the standard simplex $\{\lambda : \lambda \geq 0, \sum_{j=1}^{d+1} \lambda_j = 1\}$ onto σ.

EXAMPLE 2.2 (First Quadrant in $\mathbb{R}^2$). Let $x = (x_1, x_2)$ and $\lambda_1(x) = x_1$, $\lambda_2(x) = x_2$, $\lambda_3(x) = 1$. (Note the redundant inequality $\lambda_3(x) \geq 0$.)

EXAMPLE 2.3 (Vertical Slab in $\mathbb{R}^2$). $\lambda_1(x) = x_1$, $\lambda_2(x) = x_2$ and $\lambda_3(x) = 1 - x_1$.

EXAMPLE 2.4 (Oblique Slab in $\mathbb{R}^2$). $\lambda_1(x) = x_1$, $\lambda_2(x) = x_2$ and $\lambda_3(x) = 1 - x_1 + x_2$.

EXAMPLE 2.5 (Square in $\mathbb{R}^2$). $\lambda_1(x) = x_1$, $\lambda_2(x) = x_2$, $\lambda_3(x) = 1 - x_1$ and $\lambda_4(x) = 1 - x_2$.

EXAMPLE 2.6 (Right-Half Plane in $\mathbb{R}^2$). $\lambda_1(x) = x_1$ and $\lambda_2(x) = 1$.

Each of these examples offer certain peculiarities in delineating a representation for quadratic polynomials on their respective domains. We will deal with each in detail in Section 3.

We now consider the problem of when a polynomial p is nonnegative on Ω. We will first introduce projective coordinates so that later, in the case when p is a quadratic polynomial, we can relate the nonnegativity of p on Ω to the notion of conditionally positive semi-definite matrices. To this end we introduce $y = (x, \tau) \in \mathbb{R}^{d+1}$ and define $L(y) = \Delta x + \tau\mu$, where Δ is the $m \times d$ matrix whose columns are $\delta^1, \ldots, \delta^m$. We assume that the linear map $L : \mathbb{R}^{d+1} \to \mathbb{R}^m$ has a trivial null space, i.e., the $m \times (d+1)$ matrix $[\Delta, \mu]$ is of rank $d+1$. Examples 2.1–2.5 have this property. Example 2.6 does not. Under this condition, there exist integers $1 \leq i_1 < \cdots < i_{d+1} \leq m$ such that the map $\widetilde{L} : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ given by

$$\widetilde{L}(y) = (\delta^{i_1} \cdot x + \tau\mu_{i_1}, \ldots, \delta^{i_{d+1}} \cdot x + \tau\mu_{i_{d+1}})$$

is invertible on $\mathbb{R}^{d+1}$. By reordering, we may assume that $i_j = j$, $j = 1, \ldots, d+1$. Thus there exist vectors $v^1, \ldots, v^{d+1} \in \mathbb{R}^{d+1}$ and constants $w_1, \ldots, w_{d+1}$ such that

$$x = \sum_{i=1}^{d+1} v^i \lambda_i(x)$$
$$1 = \sum_{i=1}^{d+1} w_i \lambda_i(x). \tag{2.1}$$

We use the tilde notation $\widetilde{\Lambda}(x) = (\lambda_1(x), \ldots, \lambda_{d+1}(x)) = \widetilde{\Delta}x + \widetilde{\mu}$, where $\widetilde{\Delta}$ is the matrix obtained from the first $d+1$ rows of Δ, and $\widetilde{\mu}$ is the vector of the first $d+1$ elements of μ. Note the following correspondance with projective coordinates.

$$L(y) = \begin{cases} \tau\Lambda(\tau^{-1}x), & y = (x,\tau),\ \tau \neq 0 \\ \Delta x, & \tau = 0 \end{cases}$$

and similarly for the tilde maps.

From the above we can associate with every polynomial $p \in \Pi_n(\mathbb{R}^d)$ a unique polynomial $\widetilde{p}$ of degree n on $\mathbb{R}^{d+1}$ which is homogeneous of degree n in its variables $(\lambda_1, \ldots, \lambda_{d+1})$, such that

$$\widetilde{p}(\widetilde{\Lambda}(x)) = p(x), \quad x \in \mathbb{R}^d.$$

The existence of $\widetilde{p}$ is easily established. For a given monomial $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq n$, we can use (2.1) to rewrite it as

$$x_1^{\alpha_1} \cdots x_d^{\alpha_d} = \prod_{j=1}^{d} \Big(\sum_{i=1}^{d+1} v_j^i \lambda_i(x) \Big)^{\alpha_j} \Big(\sum_{i=1}^{d+1} w_i \lambda_i(x) \Big)^{n-|\alpha|}$$

which is homogeneous of degree n in the λ_i. The uniqueness of $\widetilde{p}$ is obvious.

Any given p has, of course, different representations from different choices of the tilde map. We formalize this observation in the following:

PROPOSITION 2.1. *Given any invertible linear map $\widetilde{L} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d+1}$, there is a one-to-one correspondance between $p \in \Pi_n(\mathbb{R}^d)$ and homogeneous $\widetilde{p} \in \Pi_n(\mathbb{R}^{d+1})$ given by the equation*

$$p(x) = \widetilde{p}(\widetilde{L}(x,1)), \quad x \in \mathbb{R}^d.$$

Going back to the polyhedron Ω, we see that fixing a choice of the tilde map leads to a cone in $\mathbb{R}^{d+1}$, namely

$$\widetilde{\Omega}_+ := \{\nu : \nu = \widetilde{L}(x,\tau),\ \tau \geq 0,\ x \in \mathbb{R}^d,\ L(x,\tau) \geq 0\}.$$

The nonnegativity of p on Ω transforms into the nonnegativity of $\widetilde{p}$ on $\widetilde{\Omega}_+$. For clearly if $\widetilde{p}(\nu) \geq 0$ for all $\nu \in \widetilde{\Omega}_+$ then $p(x) = \widetilde{p}(\widetilde{L}(x,1)) \geq 0$ for all

$x \in \Omega$ since $\nu = \widetilde{L}(x,1) \in \widetilde{\Omega}_+$. Conversely, assume $p(x) \geq 0$ for all $x \in \Omega$. Let $\nu_0 \in \widetilde{\Omega}_+$ be such that $\nu_0 = \widetilde{L}(x_0, \tau_0)$, $\tau_0 \geq 0$, $L(x_0, \tau_0) \geq 0$. If $\tau_0 > 0$, then $\tau_0^{-1}\nu_0 = \widetilde{L}(\tau_0^{-1}x_0, 1)$ and $\tau_0^{-1}x_0 \in \Omega$. Therefore

$$0 \leq p(\tau_0^{-1}x_0) = \widetilde{p}(\tau_0^{-1}\nu_0).$$

By the homogeneity of $\widetilde{p}$, we get

$$0 \leq \widetilde{p}(\tau_0^{-1}\nu_0) = \tau_0^{-n}\widetilde{p}(\nu_0)$$

and thus $\widetilde{p}(\nu_0) \geq 0$. If $\tau_0 = 0$, i.e., $\nu_0 = \widetilde{L}(x_0, 0)$, we let $\nu_\varepsilon := \widetilde{L}(\varepsilon x_0, 1)$. Since by assumption $0 \in \Omega$, we have $\nu_\varepsilon \in \widetilde{\Omega}_+$. Therefore

$$0 \leq \lim_{\varepsilon \to 0+} \varepsilon^{-n} p(\varepsilon x_0) = \lim_{\varepsilon \to 0+} \varepsilon^{-n} \widetilde{p}(\nu_\varepsilon) = \widetilde{p}(\nu_0).$$

We can improve the above observation by noting that under an additional hypothesis on Ω, the polynomial $\widetilde{p}$ is actually nonnegative on the larger cone

$$\widetilde{\Omega} = \{\nu : \nu = \widetilde{L}(x,\tau),\, L(x,\tau) \geq 0,\, (x,\tau) \in \mathbb{R}^{d+1}\}$$

which is determined by one less inequality (i.e., $\tau \geq 0$ is deleted), and as a further consequence *strict positivity* of p on Ω is then equivalent to strict positivity of $\widetilde{p}$ on $\widetilde{\Omega}$.

PROPOSITION 2.2. *With the above assumptions, set*

$$C = \{x : \Delta x \geq 0,\, x \in \mathbb{R}^d\}.$$

Assume $C = \{0\}$. Then $\widetilde{\Omega}_+ = \widetilde{\Omega}$ and futhermore, whenever p is a polynomial, then $p(x) > 0$, $x \in \Omega$ if and only if $\widetilde{p}(\nu) > 0$, $\nu \in \widetilde{\Omega}\backslash\{0\}$.

REMARK 2.1. Since $0 \in \Omega$ we have $\mu \geq 0$ and therefore $C \subseteq \Omega$. Furthermore if Ω is bounded it is easily seen that $C = \{0\}$. Therefore this condition holds for Examples 2.1 and 2.5, but does not hold for the other examples. However, as may be seen from Example 2.3, it is not necessary that $C = \{0\}$ in order that $\widetilde{\Omega}_+ = \widetilde{\Omega}$.

PROOF. Let $(x,\tau) \in \mathbb{R}^{d+1}$ satisfy $L(x,\tau) \geq 0$. Then $\Delta x \geq -\tau\mu$ and so, if $\tau \leq 0$ we conclude (since $\mu \geq 0$) that $x \in C$. Therefore $x = 0$, and since

$\mu \neq 0$, we also get $\tau = 0$. We have established not only that $\widetilde{\Omega}_+ = \widetilde{\Omega}$, but also that the only $\nu = \widetilde{L}(x, 0) \in \widetilde{\Omega}$ is $\nu = 0$. Hence the remaining claim of the proposition is also immediate. ∎

REMARK 2.2. It is easy to see that the condition $C = \{0\}$ is equivalent to the fact that the cone spanned by the $\delta^1, \ldots, \delta^m$ is all of $\mathbb{R}^d$. We also remark that since we have shown that the inequalities $L(x, \tau) \geq 0$ imply $\tau \geq 0$ whenever $C = \{0\}$, it follows that there exist, in this case, nonnegative γ_i, $i = 1, \ldots, m$, for which

$$1 = \sum_{i=1}^{m} \gamma_i \lambda_i(x), \quad x \in \mathbb{R}^d .$$

We now investigate in some detail the special cases alluded to earlier. Most of our remarks pertain to Example 2.1. Here Ω is a given simplex σ, $m = d+1$ so that the tilde and un-tilde maps are the same, and $\widetilde{\Omega} = \mathbb{R}^{d+1}_+$. Furthermore, in this case Proposition 2.1 merely gives us what is commonly referred to as the Bernstein-Bézier representation of a polynomial $p \in \Pi_n(\mathbb{R}^d)$ in barycentric coordinates relative to σ. We adopt the usual notation and write

$$p(x) = \sum_{|\alpha|=n} b_\alpha B^n_\alpha(\lambda) \tag{2.2}$$

where $\lambda^\alpha = \lambda_1^{\alpha_1} \cdots \lambda_{d+1}^{\alpha_{d+1}}$, and $B^n_\alpha(\lambda) = \binom{n}{\alpha} \lambda^\alpha$.

From Proposition 2.2 we have that $p(x)$ is nonnegative (positive) on σ if and only if the Bernstein-Bézier polynomial is nonnegative (positive) on $\mathbb{R}^{d+1}$ ($\mathbb{R}^{d+1} \backslash \{0\}$). Recall that an $m \times m$ symmetric matrix A is *copositive* if $(Ax, x) \geq 0$ for all $x \in \mathbb{R}^m_+$. It is *strictly copositive* if $(Ax, x) > 0$ for all $x \in \mathbb{R}^m_+ \backslash \{0\}$. As a corollary we have:

COROLLARY 2.3. *Let p be a quadratic polynomial on $\mathbb{R}^d$. Then p is nonnegative (positive) on a simplex σ if and only if in the Bernstein-Bézier form (2.2), the $(d+1) \times (d+1)$ matrix $A = (a_{ij})$ defined by $a_{ij} = b_{e^i + e^j}$ where $e^i \in \mathbb{R}^{d+1}$, $(e^i)_k = \delta_{ik}$, is copositive (strictly copositive).*

Using a result of Hadeler (1983) characterizing 3×3 copositive matrices, we immediately obtain a condition for a bivariate quadratic polynomial to be nonnegative over a triangle σ, in terms of the coefficients of its Bernstein-Bézier representation of degree 2.

COROLLARY 2.4. *Let p be a bivariate quadratic polynomial. Then $p(x) \geq 0$, $x \in \sigma$, σ a given triangle if and only if when*

$$p(x) = \sum_{i,j=1}^{3} b_{ij} \lambda_i(x) \lambda_j(x) \tag{2.3}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ are the barycentric coordinates of x, we have
(i) $b_{ii} \geq 0, \quad i = 1, 2, 3$
(ii) $b_{ij} \geq -\sqrt{b_{ii} b_{jj}}$, *all* $i \neq j$
and at least one of the following
(iii) $b_{12}\sqrt{b_{33}} + b_{23}\sqrt{b_{11}} + b_{13}\sqrt{b_{22}} + \sqrt{b_{11} b_{22} b_{33}} \geq 0$
(iv) $\det B \geq 0$
holds. Furthermore, p is positive on σ if and only if (i) and (ii) hold with strict inequality, and either (iii) holds or (iv) holds with strict inequality.

REMARK 2.3. In (2.3) it is to be assumed that $b_{ij} = b_{ji}$ for all i and j.

REMARK 2.4. This result was recently discussed by Nadler (1988) from another point of view.

Another characterization of 3×3 copositive matrices leads us to an analogue of (1.1). To explain, we first observe that any symmetric matrix A which is a sum of a positive semi-definite matrix plus a matrix with non-negative elements is necessarily copositive. More importantly, it is known that all copositive matrices of order ≤ 4 can be so represented, see Diananda (1962). However there are 5×5 copositive matrices which cannot be decomposed in this form. This leads us to the following representation of bivariate and trivariate quadratic polynomials, nonnegative on a simplex.

COROLLARY 2.5. *Let p be a quadratic polynomial of either two or three variables. Then $p(x)$ is nonnegative on the simplex σ if and only if $p(x)$ has the form*

$$p(x) = \sum_{1 \leq i < j \leq d+1} \alpha_{ij} \lambda_i(x) \lambda_j(x) + q(x), \quad x \in \mathbb{R}^d, \tag{2.4}$$

$d = 2, 3$, where $\lambda = (\lambda_1, \dots, \lambda_{d+1})$ are barycentric coordinates of x relative to σ, $\alpha_{ij} \geq 0$, $1 \leq i < j \leq d+1$, and q is nonnegative for all $x \in \mathbb{R}^d$.

REMARK 2.5. It is known that the homogeneous quadratic

$$(A\lambda, \lambda) := (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)^2 - 4\lambda_1\lambda_2 - 4\lambda_2\lambda_3 - 4\lambda_3\lambda_4 - 4\lambda_4\lambda_5 - 4\lambda_5\lambda_1$$

is nonnnegative on $\mathbb{R}^5_+$, but the matrix A is not the sum of a positive semi-definite and nonnegative matrix, cf. Hall (1986). The corresponding quadratic polynomial $p(x) := (A\lambda(x), \lambda(x))$, $x \in \sigma$, σ a simplex, cannot be written in the form (2.4).

The representation (2.4) is an analogue of (1.1) in that we only know that q is nonnegative on $\mathbb{R}^d$. In the next section we show that in the case $d = 2$, we can choose q so that it vanishes at some point in σ. Firstly, however, we give an alternative characterization of *positive* polynomials on σ, of any degree in any number of variables.

To this end, we need to review the process of degree raising, an idea well-known in the methodology associated with modelling curves and surfaces by the Bernstein-Bézier technique. Any $p \in \Pi_n(\mathbb{R}^d)$ has a representation in Bernstein-Bézier form

$$p(x) = \sum_{|\alpha|=m} b^m_\alpha B^m_\alpha(\lambda), \tag{2.5}$$

$b^m_\alpha = b^m_\alpha(p)$, for all $m \geq n$. Furthermore, there is a simple iterative relationship between successive representations as witnessed by the formula

$$b^{m+1}_\alpha = \frac{1}{m+1} \sum_{j=1}^{d+1} \alpha_j b^m_{\alpha - e^j}, \quad |\alpha| = m+1\,.$$

At each stage, the new representation (2.5) is formed by a simple convex combination of the previous coefficients. Hence

$$\max_{|\alpha|=m+1} |b^{m+1}_\alpha| \leq \max_{|\alpha|=m} |b^m_\alpha|, \quad m \geq n\,.$$

We combine this with the fact that there exists a positive constant c, depending on σ and n, such that for all $p \in \Pi_n(\mathbb{R}^d)$

$$c \max_{|\alpha|=n} |b^n_\alpha(p)| \leq \max_{x \in \sigma} |p(x)|$$

to obtain

$$c \max_{|\alpha|=m} |b_\alpha^m(p)| \le \max_{x\in\sigma} |p(x)|, \quad m \ge n\,.$$

Using the Bernstein operator

$$B_m(f(\sigma))(x) := \sum_{|\beta|=m} f\Big(c_\sigma\Big(\frac{\beta}{m}\Big)\Big) B_\beta^m(\lambda)$$

where, as before, $x = c_\sigma(\lambda) = \sum_{j=1}^{d+1} \lambda_j v^j$, we get

$$c \max_{|\alpha|=m} \Big|b_\alpha^m - p\Big(c_\sigma\Big(\frac{\alpha}{m}\Big)\Big)\Big| \le \max_{x\in\sigma} |p(x) - B_m(p(\sigma))(x)|$$

since for every m, $B_m(p(\sigma))$ is in fact a polynomial of degree n, cf. Dahmen, Micchelli (1988). It is well-known that there exists a constant K depending on p and σ such that the error in approximating p by its Bernstein operator value is bounded by $\frac{K}{m}$. We therefore finally obtain

$$\max_{|\alpha|=m} \Big|b_\alpha^m - p\Big(c_\sigma\Big(\frac{\alpha}{m}\Big)\Big)\Big| \le \frac{K}{cm}\,. \tag{2.6}$$

This demonstrates the well-known fact that in degree raising the coefficients converge uniformly to values of the polynomial. As a consequence we obtain:

THEOREM 2.6. *Let $p \in \Pi_n(\mathbb{R}^d)$. Then $p(x) > 0$, all $x \in \sigma$, σ some simplex, if and only if there exists an $m \ge n$ such that*

$$p(x) = \sum_{|\alpha|=m} b_\alpha^m B_\alpha^m(\lambda)$$

with $b_\alpha^m > 0$, all α, where $\lambda = (\lambda_1, \ldots, \lambda_{d+1})$ are the barycentric coordinates of x relative to σ.

REMARK 2.6. For polynomials of one variable, the above result says that $p(x) = \sum_{i=0}^n a_i x^i$ is positive on $[0,1]$ if and only if there exists an $m \ge n$ such that in the representation

$$p(x) = \sum_{j=0}^{m} b_j \binom{m}{j} x^j (1-x)^{m-j} \tag{2.7}$$

we have $b_j > 0$ for $j = 0, 1, \ldots, m$, see Pólya, Szegö (1976), p. 78, Karlin, Studden (1966), p. 126. This result continues to fascinate. See the recent paper Erdélyi, Szabados (1988) where a study is made of the least m needed in (2.7).

Somewhat more can be said in the case where $d = 1$. Namely, if $p(x) > 0$ on the open interval $(0, 1)$, then we can always divide $p(x)$ by $x^r(1-x)^t$, where r and t are the multiplicities of the zero of p at zero and one, respectively, to obtain a polynomial strictly positive on $[0, 1]$. Thus (2.7) persists (with some zero coefficients) when p is positive on $(0, 1)$. In two or more dimensions we cannot relax the hypothesis of strict positivity. For example, if p is nonnegative on all of $\mathbb{R}^2$ and vanishes only at a point $x^0 \in \partial\sigma$, which is not a vertex (say for definiteness on the face $\lambda_1(x) = 0$) then $\lambda_1(x^0) = 0$ and $\lambda_2(x^0), \ldots, \lambda_{d+1}(x^0) > 0$. Thus if

$$p(x) = \sum_{|\alpha|=m} b_\alpha^m B_\alpha^m(\lambda) \tag{2.8}$$

and $b_\alpha^m \geq 0$, it follows that $b_\alpha^m = 0$ if $\alpha_1 = 0$. But then the polynomial on the right of equation (2.8) vanishes identically on the face $\lambda_1(x) = 0$. Thus for such a polynomial we never have $b_\alpha^m \geq 0$ for all $|\alpha| = m$. The above Theorem 2.6 and Corollary 2.3 also give us a test for *strict* copositivity of a matrix.

COROLLARY 2.7. *Let $B = (b_{ij})$ be a $(d+1) \times (d+1)$ symmetric matrix. Set*

$$b^2_{e^i+e^j} := b_{ij}, \quad i, j = 1, \ldots, d+1,$$

and

$$b_\alpha^{m+1} = \frac{1}{m+1} \sum_{i=1}^{d+1} \alpha_i b^m_{\alpha - e^i}, \quad m \geq 2,$$

where e^i is the ith unit vector in $\mathbb{R}^{d+1}$ and $\alpha = (\alpha_1, \ldots, \alpha_{d+1}) \in \mathbb{N}^{d+1}$, $|\alpha| = \alpha_1 + \cdots + \alpha_{d+1} = m+1$. Then B is strictly copositive if and only if there exists an $m \geq 2$ such that $b_\alpha^m > 0$ for all $|\alpha| = m$.

As the previous example shows, this test may fail if the matrix B is only copositive. However from (2.6), we see that the above B is copositive if and only if

$$\liminf_{m \to \infty} \min_{|\alpha|=m} b_\alpha^m \geq 0\,.$$

Our final remark of this section pertains to even degree polynomials strictly positive on bounded polyhedra. For this purpose we require the following result.

PROPOSITION 2.8. *Let* Γ *be an* $m \times (d+1)$ *matrix of rank* $d+1$, *and* p *a homogeneous polynomial of degree* $2k$ *on* $\mathbb{R}^{d+1}$ *satisfying* $p(x) > 0$ *if* $\Gamma x \geq 0$, $x \neq 0$. *There then exists a homogeneous polynomial* q *of degree* $2k$ *on* $\mathbb{R}^m$ *such that* $q(\Gamma x) = p(x)$, $x \in \mathbb{R}^{d+1}$, *and* $q(y) > 0$ *if* $y \in \mathbb{R}^m_+ \backslash \{0\}$.

PROOF. Let $Q : \mathbb{R}^m \to \mathbb{R}^{d+1}$ be the orthogonal projection of $\mathbb{R}^m$ onto the columns of Γ. That is,

$$Qy = \Gamma(\Gamma^T\Gamma)^{-1}\Gamma^T y\,.$$

For any $t \in \mathbb{R}$, set

$$q_t(y) = t[(y,y) - (Qy,y)]^k + p((\Gamma^T\Gamma)^{-1}\Gamma^T y)\,.$$

Then q_t is a homogeneous polynomial of degree $2k$ on $\mathbb{R}^m$. Furthermore, for all $x \in \mathbb{R}^{d+1}$

$$q_t(\Gamma x) = t[(\Gamma x, \Gamma x) - (\Gamma x, \Gamma x)]^k + p(x) = p(x)\,.$$

We claim that there exists a $t_0 > 0$ for which $q_{t_0}(y) > 0$ if $y \in \mathbb{R}^m_+\backslash\{0\}$. (Note that if q_{t_0} is nonnegative for t_0, then it is nonnegative for all $t \geq t_0$.) Assume this is false. There then exists a sequence $y_n \in \mathbb{R}^m_+\backslash\{0\}$, $(y_n, y_n) = 1$, such that $q_n(y_n) \leq 0$. We choose a limit point y_0 of the sequence $\{y_n\}$. Then $(y_0, y_0) = 1$ and $y_0 \in \mathbb{R}^m_+\backslash\{0\}$. Futhermore we have

$$[(y_n, y_n) - (Qy_n, y_n)]^k \leq -\frac{1}{n}p((\Gamma^T\Gamma)^{-1}\Gamma^T y_n)$$

for each n. Let $n \uparrow \infty$. Then

$$[(y_0, y_0) - (Qy_0, y_0)]^k \leq 0\,.$$

Since Q is an orthogonal projection, $(y,y) - (Qy,y) \geq 0$ for all $y \in \mathbb{R}^m$. Thus

$$(y_0, y_0) = (Qy_0, y_0)\,.$$

But this implies, since Q is an orthogonal projection, that there exists an $x_0 \in \mathbb{R}^{d+1}\backslash\{0\}$ such that $y_0 = \Gamma x_0$. Thus $\Gamma x_0 \geq 0$, $x_0 \neq 0$, implying by assumption that $p(x_0) > 0$. However

$$p((\Gamma^T\Gamma)^{-1}\Gamma^T y_n) \leq -n[(y_n, y_n) - (Qy_n, y_n)]^k \leq 0$$

for all n, and therefore $p((\Gamma^T\Gamma)^{-1}\Gamma^T y_0) \leq 0$. But $p((\Gamma^T\Gamma)^{-1}\Gamma^T y_0) = p(x_0)$. This contradiction proves the proposition. ∎

REMARK 2.7. This proposition extends a result of Martin, Jacobson (1981) (Theorem 4.2, p. 239) which corresponds to the case $k = 1$. The proof is patterned after that special case.

From this above result we obtain using Propositions 2.1, 2.2 and 2.8:

COROLLARY 2.9. *Let p be a polynomial of even degree on $\mathbb{R}^d$ which is strictly positive on a bounded polyhedron $\Omega = \{x : \Lambda(x) \geq 0,\ x \in \mathbb{R}^d\}$, where $\Lambda(x) = \Delta x + \mu$, and $[\Delta, \mu]$ is a matrix of rank $d+1$. There then exists a homogeneous polynomial q on $\mathbb{R}^m$, of the same degree as p, satisfying $q(\Lambda(x)) = p(x)$, all $x \in \mathbb{R}^d$, and $q(y) > 0$ for all $y \in \mathbb{R}^m_+\backslash\{0\}$.*

3. NONNEGATIVE BIVARIATE QUADRATIC POLYNOMIALS

We begin this section with a proof of the theorem mentioned in the introduction.

THEOREM 3.1. *Let $p(x)$, $x \in \mathbb{R}^2$ be any quadratic polynomial nonnegative on a triangle σ with vertices v^1, v^2, v^3. Let $(\lambda_1(x), \lambda_2(x), \lambda_3(x))$ denote the barycentric coordinates of x relative to σ. Then there are nonnegative constants α, β, γ, and a quadratic polynomial q which is nonnegative on all of $\mathbb{R}^2$ and vanishes at some point $y \in \sigma\backslash\{v^1, v^2, v^3\}$ such that*

$$p(x) = \alpha\lambda_1(x)\lambda_2(x) + \beta\lambda_1(x)\lambda_3(x) + \gamma\lambda_2(x)\lambda_3(x) + q(x). \tag{3.1}$$

We base the proof of this result on the following proposition.

PROPOSITION 3.2. *Let C be a positive semi-definite 3×3 (real) matrix. There then exist 3×3 matrices A and N such that $C = A + N$, where*
(i) $N = (n_{ij})_{i,j=1}^{3}$ satisfies $n_{ij} \geq 0$, $n_{ii} = 0$, $i, j = 1, 2, 3$.
(ii) A is a positive semi-definite matrix with a vector λ^0 in its null space, where $\lambda^0 \in \mathbb{R}_+^3$ and at least two components of λ^0 are strictly positive.

PROOF. We divide the proof into various cases, depending on the sign patterns of the elements of C. We first observe that we may assume that the diagonal elements of C are all positive. For if not, then after permutation of rows and columns, if necessary, we may assume that C has the form

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & f \\ 0 & f & c \end{pmatrix}$$

with $b, c \geq 0$ and $bc \geq f^2$. We then set

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & -\sqrt{bc} \\ 0 & -\sqrt{bc} & c \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f + \sqrt{bc} \\ 0 & f + \sqrt{bc} & 0 \end{pmatrix}$$

and note that $\lambda^0 = (0, \sqrt{c}, \sqrt{b})$ is an eigenvector of A. If $b = 0$, then $\lambda^0 = (1, 1, 0)$ is also an eigenvector of A. The similar result holds if $c = 0$.

We therefore assume that

$$C = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

with $a, b, c > 0$.

We consider the various possible sign patterns of the off-diagonal elements of C. Since we can always permute rows and columns, we may assume that d and e have the same sign (weakly). Thus we need only consider four cases.

CASE 1. $d, e, f \geq 0$. In this case we set

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & -\sqrt{bc} \\ 0 & -\sqrt{bc} & c \end{pmatrix}, \quad N = \begin{pmatrix} 0 & d & e \\ d & 0 & f + \sqrt{bc} \\ e & f + \sqrt{bc} & 0 \end{pmatrix}.$$

$\lambda^0 = (0, \sqrt{c}, \sqrt{b})$ is an eigenvector of A.

CASE 2. $d, e \geq 0$ and $f \leq 0$. We decompose C exactly as in Case 1.

CASE 3. $d, e, f \leq 0$. We assume from Case 2 that $d, e < 0$. We claim that there exists an $\alpha_0 \geq 1$ such that the matrix

$$A = \begin{pmatrix} a & \alpha_0 d & \alpha_0 e \\ \alpha_0 d & b & \alpha_0 f \\ \alpha_0 e & \alpha_0 f & c \end{pmatrix}$$

is positive semi-definite and singular. To see this note that $\det A$ is a cubic or quadratic polynomial in α_0 with leading coefficient negative, and is nonnegative for $\alpha_0 = 1$. Thus there exists an $\alpha_0 \geq 1$ for which

$$\det A = abc + 2\alpha_0^3 def - \alpha_0^2(af^2 + be^2 + cd^2) = 0. \tag{3.2}$$

Furthermore, it necessarily follows from (3.2) that $ab \geq \alpha_0^2 d^2$, $ac \geq \alpha_0^2 e^2$ and $bc \geq \alpha_0^2 f^2$. Thus A is positive semi-definite. In fact if $ab = \alpha_0^2 d^2$, then from (3.2),

$$2\alpha_0^3 def - \alpha_0^2(af^2 + be^2) = 0\,.$$

Since all terms are nonpositive, this implies that $be^2 = 0$ contradicting our previous hypothesis. Thus $ab > \alpha_0^2 d^2$, and similarly $ac > \alpha_0^2 e^2$.

We set $N = C - A$. Since $\alpha_0 \geq 1$ and $d, e, f \leq 0$, N has the desired form. The vector

$$\lambda^0 = (\alpha_0^2 df - \alpha_0 eb, -\alpha_0 af + \alpha_0^2 ed, ab - \alpha_0^2 d^2)$$

is an eigenvector of A. Since $a, b, c > 0$, $d, e, < 0$, $f \leq 0$ and $ab > \alpha_0^2 d^2$, it follows that all three components of λ^0 are strictly positive.

CASE 4. $d, e \leq 0$ and $f \geq 0$. If $def = 0$, then we are in one of the previous cases. As such we assume that $d, e < 0$ and $f > 0$.

If $abc - be^2 - cd^2 \geq 0$, set

$$\widetilde{C} = \begin{pmatrix} a & d & e \\ d & b & 0 \\ e & 0 & c \end{pmatrix}, \quad \widetilde{N} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f \\ 0 & f & 0 \end{pmatrix}.$$

Then $\widetilde{C}$ is positive semi-definite and we have reduced our problem to case 3.

As such we assume that

$$abc - be^2 - cd^2 < 0\,. \tag{3.3}$$

Since $\det C = abc + 2def - (af^2 + be^2 + cd^2)$, and (3.3) holds, there exists an $\alpha \in (0,1]$ such that

$$C_\alpha = \begin{pmatrix} a & d & e \\ d & b & \alpha f \\ e & \alpha f & c \end{pmatrix}$$

is singular and positive semi-definite. Since $f \geq 0$, $C - C_\alpha$ has nonnegative entries. As such it suffices to work with C_α.

If $a(\alpha f)^2 \geq cd^2$, set

$$A = \begin{pmatrix} a & d & -\sqrt{ac} \\ d & b & -\sqrt{\frac{c}{a}}d \\ -\sqrt{ac} & -\sqrt{\frac{c}{a}}d & c \end{pmatrix},$$

and

$$N = \begin{pmatrix} 0 & 0 & e + \sqrt{ac} \\ 0 & 0 & \alpha f + \sqrt{\frac{c}{a}}d \\ e + \sqrt{ac} & \alpha f + \sqrt{\frac{c}{a}}d & 0 \end{pmatrix}.$$

By assumption $\alpha f + \sqrt{\frac{c}{a}}d \geq 0$, and since C_α is positive semi-definite $e + \sqrt{ac} \geq 0$. The vector $\lambda^0 = (\sqrt{c}, 0, \sqrt{a})$ is an eigenvector of A. If $a(\alpha f)^2 \geq be^2$, a similar analysis holds.

The final possibility we must consider is that $a(\alpha f)^2 < cd^2$ and $a(\alpha f)^2 < be^2$, while C_α is positive semi-definite and singular. In this case we set $A = C_\alpha$. The vector

$$\lambda^0 = (bc - (\alpha f)^2, e(\alpha f) - cd, d(\alpha f) - be)$$

is an eigenvector of A. The first component is nonnegative since C_α is positive semi-definite. Now

$$c(-d) = \sqrt{c}\sqrt{cd^2} > \sqrt{c}\sqrt{a(\alpha f)^2} = \sqrt{ac}(\alpha f) \geq -e(\alpha f)$$

and similarly

$$b(-e) = \sqrt{b}\sqrt{be^2} > \sqrt{b}\sqrt{a(\alpha f)^2} = \sqrt{ab}(\alpha f) \geq -d(\alpha f).$$

Thus the second and third components of λ^0 are strictly positive. This proves the proposition. ∎

PROOF OF THEOREM 3.1. Assume p is a quadratic polynomial nonnegative over a triangle σ with vertices v^1, v^2, v^3. From Corollary 2.3 (see also Corollary 2.5), p may be written in the form

$$p(x) = (B\lambda(x), \lambda(x))$$

where $\lambda(x) = (\lambda_1(x), \lambda_2(x), \lambda_3(x))$ are the barycentric coordinates of x relative to σ, and B is a 3×3 copositive matrix. From Diananda (1962), B can be decomposed as $C + \widetilde{N}$ where C is positive semi-definite and $\widetilde{N}$ is a matrix with nonnegative entries. Since adding positive terms to the diagonal entries of a positive semi-definite matrix maintains the positive semi-definiteness of the matrix, we may assume that the diagonal entries of $\widetilde{N}$ are all zero. We now decompose C as in Proposition 3.2 to obtain

$$B = A + N$$

where A and N are as in the statement of Proposition 3.2. Thus

$$p(x) = (A\lambda(x), \lambda(x)) + (N\lambda(x), \lambda(x)) .$$

¿From the definition of N,

$$(N\lambda(x), \lambda(x)) = \alpha\lambda_1(x)\lambda_2(x) + \beta\lambda_1(x)\lambda_3(x) + \gamma\lambda_2(x)\lambda_3(x)$$

for some $\alpha, \beta, \gamma \geq 0$. In addition,

$$q(x) = (A\lambda(x), \lambda(x))$$

is a quadratic polynomial nonnegative on all of $\mathbb{R}^2$, and vanishes at some point $y \in \sigma\backslash\{v^1, v^2, v^3\}$ corresponding to $\lambda(y) = \lambda^0$, where λ^0 has been normalized so that the sum of its coefficients is one. ∎

We now present representation theorems for each of Examples 2.2–2.6.

THEOREM 3.3 (First Quadrant). *Let p be a nonnegative quadratic polynomial on $\mathbb{R}^2_+$. Then there exist nonnegative constants α, β, γ, and a quadratic polynomial q which is nonnegative on all of $\mathbb{R}^2$ such that*

$$p(x) = \alpha x_1 + \beta x_2 + \gamma x_1 x_2 + q(x), \quad x = (x_1, x_2).$$

PROOF. We write $p(x)$ in the form

$$p(x) = (B\Lambda(x), \Lambda(x))$$

where $\Lambda(x) = (x_1, x_2, 1)$ and B is a 3×3 copositive matrix. Here we have made use of Proposition 2.1 and the remarks thereafter. We now decompose B as a positive semi-definite matrix A plus a matrix N with nonnegative entries and zero diagonal entries. ∎

THEOREM 3.4 (Vertical Slab). *Let p be a nonnegative quadratic polynomial on the vertical slab of Example 2.3. Then there exist nonnegative constants α, β, γ, and a quadratic polynomial q which is nonnegative on all of $\mathbb{R}^2$ and which vanishes at some point y in the slab, but not at $(0,0)$ or $(1,0)$, such that*

$$p(x) = \alpha x_1 x_2 + \beta x_1(1 - x_1) + \gamma x_2(1 - x_1) + q(x).$$

PROOF. First we write p in homogeneous coordinates as

$$p(x) = (B\Lambda(x), \Lambda(x))$$

where $\Lambda(x) = (x_1, x_2, 1 - x_1)$, and $x = (x_1, x_2)$. B is a 3×3 matrix and has the property that $(Bz, z) \geq 0$ whenever $z = (x_1, x_2, \tau - x_1)$ with $x_1 \geq 0$, $x_2 \geq 0$, $\tau - x_1 \geq 0$ and $\tau \geq 0$. Even though the hypothesis of Proposition 2.2 is not satisfied, the inequality $\tau \geq 0$ is still redundant. Hence B is copositive. We now proceed as in the proof of Theorem 3.1. The polynomial q vanishes at some point y as claimed since the eigenvector λ^0 of A can be normalized so that its first and third components sum to one. ∎

THEOREM 3.5 (Oblique Slab). *Let p be a nonnegative quadratic polynomial on the oblique slab of Example 2.4. Then there exist nonnegative constants $\alpha, \beta, \gamma, \delta, \varepsilon, \mu$, and a quadratic polynomial q which is nonnegative on all of $\mathbb{R}^2$ such that*

$$p(x) = \alpha x_1 + \beta x_2 + \gamma(1 - x_1 + x_2) + \delta x_1 x_2 + \varepsilon x_1(1 - x_1 + x_2) + \mu x_2(1 - x_1 + x_2) + q(x).$$

For a proof of this result, we first appeal to the following theorem.

THEOREM (Martin, Jacobson (1981)). *Let Q be an $n \times n$ symmetric matrix and A an $m \times n$ matrix with $m \leq 4$. Suppose $(Qx, x) \geq 0$ whenever $Ax \geq 0$, and there exists an $\widetilde{x} \in \mathbb{R}^n$ with $A\widetilde{x} > 0$. Then there exists a copositive $m \times m$ matrix C, and a positive semi-definite $n \times n$ matrix S such that*

$$Q = A^T C A + S\,. \tag{3.4}$$

REMARK 3.1. Necessary and sufficient conditions for the existence of a decomposition of the type (3.4) is given in Martin, Powell, Jacobson (1981).

PROOF OF THEOREM 3.5. By Proposition 2.1 we write

$$p(x) = (Q\Lambda(x), \Lambda(x))$$

where $\Lambda(x) = (x_1, x_2, 1 - x_1 + x_2)$, $x = (x_1, x_2)$ and Q is a 3×3 matrix. We know that $(Qz, z) \geq 0$ whenever $z = (x_1, x_2, \tau - x_1 + x_2)$ with $x_1 \geq 0$, $x_2 \geq 0$, $\tau - x_1 + x_2 \geq 0$ and $\tau \geq 0$. The last condition is not redundant here. Thus $(Qz, z) \geq 0$ if $Az \geq 0$ where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

We use the above stated theorem to express p in the form

$$p(x) = (CA\Lambda(x), A\Lambda(x)) + (S\Lambda(x), \Lambda(x))$$

where C is a 4×4 copositive matrix and S is a 3×3 positive semi-definite matrix. We use the result of Diananda (1962) to decompose C as $B + N$ where B is positive semi-definite, and N has nonnegative entries with zero diagonals, plus the fact that $A\Lambda(x) = (x_1, x_2, 1 - x_1 + x_2, 1)^T$ to obtain the statement of the theorem. ∎

The representation theorem for the square is proved in a similar fashion. We omit the details.

THEOREM 3.6 (Square). *Let p be a nonnegative quadratic polynomial on the square of Example 2.5. Then there exist nonnegative constants $\alpha, \beta, \gamma, \delta, \varepsilon, \mu$, and a quadratic polynomial q which is nonnegative on all of $\mathbb{R}^2$ such that*

$$p(x) = \alpha x_1 x_2 + \beta x_1(1 - x_1) + \gamma x_1(1 - x_2) + \delta x_2(1 - x_1) + \varepsilon x_2(1 - x_2) + \mu(1 - x_1)(1 - x_2) + q(x).$$

REMARK 3.2. In Theorems 3.3 and 3.5 we cannot choose the q appearing therein to vanish at some point in the respective polyhedra. This may be seen by representing the polynomial $p(x) = 1$ thereon. In these cases we necessarily have $q(x) = 1$. It may be that the q appearing in Theorem 3.6 and the q of three variables in Corollary 2.5 can be chosen to vanish in their respective polyhedra. This fact would seem to require us to prove an analogue of Proposition 3.2 for 4×4 matrices. The existence of such an analogue remains an open question.

Our final characterization theorem requires a special limiting argument as it cannot be directly converted to homogeneous coordinates.

THEOREM 3.7 (Right-Half Plane). *Let p be a nonnegative quadratic polynomial on the right-half plane (Example 2.6). Then there exists a nonnegative constant α and a quadratic polynomial q which is nonnegative on all of $\mathbb{R}^2$ such that*

$$p(x) = \alpha x_1 + q(x).$$

PROOF. For every $t \geq 0$, p is nonnegative on the cone $x_1 \geq 0$, $x_1 + tx_2 \geq 0$. Therefore we can (after a linear change of variables) use Theorem 3.3 to express p in the form

$$p(x) = \alpha_t x_1 + \beta_t(x_1 + tx_2) + \gamma_t x_1(x_1 + tx_2) + q_t(x) \tag{3.5}$$

where $\alpha_t, \beta_t, \gamma_t \geq 0$ and q_t is a quadratic polynomial nonnegative on all of $\mathbb{R}^2$. We claim that

$$\delta_t = \alpha_t + \beta_t + \gamma_t + \max_{(x,x)=1} q_t(x)$$

remains bounded as $t \to 0^+$. If not we would divide both sides of (3.5) by δ_t and pass to the limit through a subsequence to obtain

$$0 = \alpha' x_1 + \beta' x_1 + \gamma' x_1^2 + q'(x)$$

where α', β', γ' and q' are not all zero. Since all these quantities are nonnegative we easily obtain a contradiction. Thus δ_t must remain bounded as $t \to 0^+$. We now pass to a limit in (3.5) through a subsequence, add the β term to the α term, and the γ term to the q term. ∎

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