MOMENT THEORY FOR WEAK CHEBYSHEV SYSTEMS WITH APPLICATIONS TO MONOSPLINES, QUADRATURE FORMULAE AND BEST ONE-SIDED L^1-APPROXIMATION BY SPLINE FUNCTIONS WITH FIXED KNOTS*

C. A. MICCHELLI† AND ALLAN PINKUS‡

Abstract. The chief purpose of this paper is to present an alternative approach to results concerning the existence and uniqueness of monosplines which have a maximum number of zeros (the fundamental theorem of algebra for monosplines). In addition, we discuss the related problems of "double precision" quadrature formulae and one-sided L^1-approximation by spline functions with fixed knots.

1. Introduction. The chief purpose of this paper is to present an alternative approach to the results of S. Karlin and L. Schumaker [6] and S. Karlin and C. A. Micchelli [3] concerning the existence and uniqueness of monosplines which have a maximum number of zeros (the fundamental theorem of algebra for monosplines). In addition, we discuss the related problems of "double precision" quadrature formulae and one-sided L^1-approximation by spline functions with fixed knots.

Our approach to these problems is based on moment theory. The relationship of the above problems to moment theory is not surprising. In fact, I. J. Schoenberg originally suggested this relationship in [13] and S. Karlin and W. J. Studden [7] discuss a special case of the fundamental theorem of algebra for monosplines by means of moment theory.

However, the method used in [6] (and later in [3], [4] and [10]) to prove Schoenberg's conjecture [13] does not use moment theory and is needlessly complicated. Our proof uses Theorem 2.1; see Theorem 5.1 and Corollary 5.1. Nevertheless, the methods of [6] are indeed valuable when the simplicity of moment theory is not applicable, as in [4] and [10].

A thorough treatment, with improvements, of M. G. Krein's work [8] on moment theory for Chebyshev systems is contained in [7]. The basic difficulty that we face here is to provide a suitable version of these results for weak Chebyshev systems. In his thesis [1], H. Burchard studied the problem of interpolation of data by generalized convex functions and was also led to the problem to extending moment theory to weak Chebyshev systems. His extension, however, is too restrictive for the application we have in mind. For the related problem of determining the "envelope" of smooth functions "pinned down" on some partition, see Micchelli and Miranker [14].

In § 2 we present an extension of moment theory to weak Chebyshev systems, which improves on Burchard's result. We also discuss the related problem of one-sided approximation for weak Chebyshev systems. In § 3 we apply the general theory to certain classes of spline functions to obtain "double precision" quadrature formulae. Section 4 contains our version of the fundamental theorem of algebra for monosplines satisfying mixed boundary conditions, which subsumes [3] and [6].

2. Moment theory for weak Chebyshev systems. We begin by recalling the main results of moment theory for Chebyshev systems. Let M be a linear subspace of $C[0, 1]$ of dimension n spanned by the functions $u_1(t), \cdots, u_n(t)$ and let $d\alpha(t)$ be a nonnegative finite measure on $[0, 1]$. If $d\alpha(t)$ is a discrete measure

$$\int_0^1 f(r) \, d\alpha(t) = \sum_{j=1}^N \lambda_j f(t_j), \qquad\qquad f \in C[0, 1],$$

$\lambda_1 > 0, \cdots, \lambda_N > 0, 0 \leq t_1 < \cdots < t_N \leq 1$, then the index of $d\alpha$, denoted by $I(\alpha)$, is defined to be $\sum_{j=1}^N \omega(t_j)$ where $\omega(t) = \frac{1}{2}$ when $t \in \{0, 1\}$, and $\omega(t) = 1$ when $t \in (0, 1)$. Following [8], we call $d\alpha(t)$ a positive measure (relative to M) provided that $\int_0^1 u(t) \, d\alpha(t) > 0$ whenever u is a nontrivial nonnegative function in M. A positive measure corresponds to an interior point of the moment space determined by the set of functions $\{u_1(t), \cdots, u_n(t)\}$ [7]. The measure $d\alpha_0$ is said to be a principal representation of $d\alpha$ provided that $\int_0^1 u(t) \, d\alpha_0(t) = \int_0^1 u(t) \, d\alpha(t)$ for all $u \in M$ and $I(\alpha_0) = n/2$. If $d\alpha_0$ has mass at one, it is referred to as an upper principal representation; otherwise, it is called a lower principal representation. The set of functions $\{u_1(t), \cdots, u_n(t)\}$ is called a Chebyshev system on $[0, 1]$ (M a Chebyshev subspace) provided that

$$\begin{vmatrix} u_1(t_1) & \cdots & u_1(t_n) \\ \vdots & & \vdots \\ u_n(t_1) & \cdots & u_n(t_n) \end{vmatrix} > 0$$

for all $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$.

The following result of Krein is proven in [7]. If M is a Chebyshev subspace then every positive measure $d\alpha(t)$ has exactly two principal representations,

$$\int_0^1 u(t) \, d\alpha(t) = \int_0^1 u(t) \, d\underline{\alpha}(t) = \int_0^1 u(t) \, d\bar{\alpha}(t), \qquad u \in M,$$

where $d\underline{\alpha}$ is a lower principal representation and $d\bar{\alpha}$ is an upper principal representation.

Let us denote by K the convexity cone generated by M. Thus every $f \in K$ is a function defined on $(0, 1)$ which satisfies the inequality

$$\begin{vmatrix} u_1(t_1) & \cdots & u_1(t_{n+1}) \\ u_2(t_1) & \cdots & u_2(t_{n+1}) \\ \vdots & & \\ u_n(t_1) & \cdots & u_n(t_{n+1}) \\ f(t_1) & \cdots & f(t_{n+1}) \end{vmatrix} \geq 0,$$

for all $0 < t_1 < \cdots < t_{n+1} < 1$. The principal representations corresponding to a positive measure have the additional property that for $f \in K_0 = C[0, 1] \cap K$,

$$(2.1) \qquad \int_0^1 f(t) \, d\underline{\alpha}(t) \leq \int_0^1 f(t) \, d\alpha(t) \leq \int_0^1 f(t) \, d\bar{\alpha}(t).$$

This inequality is called the Markoff–Krein inequality.

We shall now discuss the extension of the above results to a weak Chebyshev system which contains a positive function.

A set of linearly independent continuous functions $\{u_i(t)\}_{i=1}^n$ form a weak Chebyshev system (M a weak Chebyshev subspace) on $[0, 1]$ provided that for all points $0 \le t_1 < \cdots < t_n \le 1$,

$$
\begin{vmatrix}
u_1(t_1) & \cdots & u_1(t_n) \\
u_2(t_1) & \cdots & u_2(t_n) \\
\vdots & & \vdots \\
u_n(t_1) & \cdots & u_n(t_n)
\end{vmatrix} \ge 0.
$$

THEOREM 2.1. *Let M be a weak Chebyshev subspace of dimension $2l$ on $[0, 1]$, which contains a strictly positive function on $(0, 1)$. Then every positive measure relative to M has a lower principal representation.*

Proof. The basic idea of the proof is to "smooth" the weak Chebyshev system $\{u_i(t)\}_{i=1}^{2l}$ into a Chebyshev system and then apply the previous results for Chebyshev systems. Specifically, let $\delta > 0$ and define

$$
u_i(t; \delta) = \frac{1}{\sqrt{2\pi\delta}} \int_0^1 e^{-(x-t)^2/2\delta} u_i(x) \, dx.
$$

Then $\{u_i(t; \delta)\}_{i=1}^{2l}$ is a Chebyshev system [7], and $\lim_{\delta \to 0^+} u_i(t; \delta) = u_i(t)$ uniformly in any closed subinterval of $(0, 1)$ and $\lim_{\delta \to 0^+} u_i(t; \delta) = u_i(t)/2$, for $t \in \{0, 1\}$, $i = 1, \cdots, 2l$. Thus for every positive measure $d\alpha$, there exists a lower principal representation $d\alpha_\delta$ such that for $i = 1, \cdots, 2l$,

$$
(2.2) \qquad \int_0^1 u_i(t; \delta) \, d\alpha(t) = \int_0^1 u_i(t; \delta) \, d\alpha_\delta(t) = \sum_{j=1}^l \lambda_j(\delta) u_i(t_j(\delta); \delta),
$$

where $\lambda_j(\delta) > 0, j = 1, \cdots, l$, and $0 < t_1(\delta) < \cdots < t_l(\delta) < 1$. (We may assume that

$$
(2.3) \qquad \lim_{\delta \to 0^+} \int_0^1 u_i(t; \delta) \, d\alpha_\delta(t) = \int_0^1 u_i(t) \, d\alpha(t), \qquad i = 1, \cdots, 2l.
$$

Otherwise, we construct another measure $d\hat{\alpha}(t)$, positive relative to M, such that

$$
\lim_{\delta \to 0^+} \int_0^1 u_i(t; \delta) \, d\hat{\alpha}(t) = \int_0^1 u_i(t) \, d\alpha(t), \qquad i = 1, \cdots, 2l,
$$

and let $d\alpha_\delta(t)$ be the lower principal representation of $d\hat{\alpha}(t)$. Then clearly (2.3) is valid.) Construct the "polynomial" $u(t; \delta) = \sum_{i=1}^{2l} a_i(\delta) u_i(t; \delta)$ which satisfies $u(t_i(\delta); \delta) = 0$, $i = 1, \cdots, l$, $u(t; \delta) \ge 0$ for $t \ge t_1(\delta)$, and $\sum_{i=1}^{2l} [a_i(\delta)]^2 = 1$. Since $\{u_i(t; \delta)\}_{i=1}^{2l}$ is a Chebyshev system on $[0, 1]$, the above conditions uniquely determine $u(t; \delta)$. Hence from (2.2), we conclude that $\int_0^1 u(t; \delta) \, d\alpha(t) = 0$ for all $\delta > 0$. Since $0 < t_1(\delta) < \cdots < t_l(\delta) < 1$, there exists a subsequence $\{\delta_k\}_{k=1}^\infty$, $\delta_k \downarrow 0$, such that $t_i(\delta_k) \to t_i$, $i = 1, \cdots, l$, where $0 \le t_1 \le \cdots \le t_l \le 1$. Choosing a subsequence of $\{\delta_k\}_{k=1}^\infty$, if necessary, it follows that $u(t; \delta_k) \to u(t)$ as $k \uparrow \infty$, uniformly in any closed subinterval of $(0, 1)$. Thus $\int_0^1 u(t) \, d\alpha(t) = 0$, and $u(t_i) = 0$, $i = 1, \cdots, l$,

while $u(t) \geqq 0$ for $t \geqq t_1$, and $u(t) \not\equiv 0$. Since $d\alpha$ is a positive measure relative to M, $t_1 > 0$. In a similar manner, one proves that $t_l < 1$. Furthermore, using the fact that M contains a function which positive on $(0, 1)$, it follows that $\lambda_1(\delta), \cdots, \lambda_l(\delta)$ are uniformly bounded in δ. Now we may easily pass to the limit in (2.2), perhaps through a subsequence, and obtain a limit $d\underline{\alpha}(t)$ for $d\underline{\alpha}_\delta(t)$, where

$$\int_0^1 u_i(t) \, d\alpha(t) = \int_0^1 u_i(t) \, d\underline{\alpha}(t), \qquad i = 1, \cdots, 2l.$$

The proof of Theorem 2.1 will be complete if we can show that $I(d\underline{\alpha}) = l$. From the above analysis, $I(d\underline{\alpha}) \leqq l$. If strict inequality holds, then we construct by smoothing, a nonnegative nontrivial polynomial in $\{u_i(t)\}_{i=1}^{2l}$ which vanishes at the points of increase of $d\underline{\alpha}$. This contradicts the positivity of the measure $d\underline{\alpha}$. Thus $I(d\underline{\alpha}) = l$, and Theorem 2.1 is proven.

Under the stronger assumption that M contains a strictly positive function on the *closed* interval $[0, 1]$, we may prove the following result.

THEOREM 2.2. *Let M be a weak Chebyshev subspace on $[0, 1]$ which contains a function which is strictly positive in $[0, 1]$. Then every measure which is positive relative to M has an upper and lower representation.*

Remark 2.1. Theorem 2.2 was proven in [1] under the stronger hypothesis that M is a weak Chebyshev subspace on some closed interval strictly containing $[0, 1]$.

Remark 2.2. Theorems 2.1 and 2.2 are not valid if there does not exist a positive function within the weak Chebyshev subspace. For example, consider the two-dimensional weak Chebyshev subspace composed of cubic polynomials which have a double zero at $\frac{1}{2}$. On $[0, 1]$ the positive measure dt has no lower principal representation relative to this subspace.

LEMMA 2.1. *Let M be a weak Chebyshev subspace of dimension n on $[0, 1]$ and let $f \in K_0 - M$. If $d\underline{\alpha}$ is a lower principal representation of some measure which is positive relative to M, then there exists a nontrivial nonnegative function $g(t) = c_0 f(t) + \sum_{i=1}^n c_i u_i(t)$ such that $\int_0^1 g(t) \, d\underline{\alpha}(t) = 0$, and for any $g(t)$ of the above form, $c_0 > 0$. If we replace $d\underline{\alpha}(t)$ by an upper principal representation $d\bar{\alpha}(t)$, then the above holds with $c_0 < 0$.*

Proof. If M is a Chebyshev subspace, then the proof of Lemma 2.1 may be found in [7]. For M as above, the existence of a $g(t)$ as indicated in the statement of the lemma follows by smoothing as in the proof of Theorem 2.1. Thus for $d\underline{\alpha}(t)$, $c_0 \geqq 0$, and for $d\bar{\alpha}(t)$, $c_0 \leqq 0$. However, if in either case $c_0 = 0$, then we contradict the positivity of $d\underline{\alpha}$, or $d\bar{\alpha}$ relative to M.

COROLLARY 2.1. *Let M be a weak Chebyshev subspace of dimension n on $[0, 1]$. If $d\underline{\alpha}(t)$ and $d\bar{\alpha}(t)$ are lower and upper principal representations of a positive measure $d\alpha(t)$ relative to M, then*

$$\int_0^1 f(t) \, d\underline{\alpha}(t) \leqq \int_0^1 f(t) \, d\alpha(t) \leqq \int_0^1 f(t) \, d\bar{\alpha}(t),$$

for any $f \in K_0$.

Proof. We may assume without loss of generality that $f \in K_0 - M$. Then from Lemma 2.1 we conclude that there exists a nontrivial nonnegative function

$g(t) = c_0 f(t) + \sum_{j=1}^{n} c_j u_j(t)$ with $c_0 > 0$ such that $\int_0^1 g(t) \, d\underline{\alpha}(t) = 0$. Hence

$$0 \le \int_0^1 g(t) \, d\alpha(t) = \int_0^1 g(t) \, d\alpha(t) - \int_0^1 g(t) \, d\underline{\alpha}(t)$$

$$= c_0 \left(\int_0^1 f(t) \, d\alpha(t) - \int_0^1 f(t) \, d\underline{\alpha}(t) \right).$$

This proves the lower inequality. The upper inequality is similarly proven.

Remark 2.3. When $d\alpha(t)$ satisfies the hypothesis of Corollary 2.1 and is also a positive measure relative to the subspace M_f which is spanned by the functions $\{f, u_1, \cdots, u_n\}$, then strict inequality holds in the above inequality whenever $f \in K_0 - M$.

We will denote the smallest linear subspace containing K_0 by $[K_0]$.

The following useful corollary appears in [1] in a weaker form.

COROLLARY 2.2. *Let M be a weak Chebyshev subspace of dimension n. If $[K_0]$ contains an n-dimensional Chebyshev system on $[0, 1]$, then every positive measure relative to M has at most one upper and one lower principal representation.*

Proof. Suppose $d\bar{\alpha}_1$ and $d\bar{\alpha}_2$ are two upper principal representations for $d\alpha(t)$. Then according to Corollary 2.1, $\int_0^1 f(t) \, d\bar{\alpha}_1(t) = \int_0^1 f(t) \, d\bar{\alpha}_2(t)$ for all $f \in K_0$. Since $[K_0]$ contains a Chebyshev system of dimension n, and $I(d\bar{\alpha}_1) = I(d\bar{\alpha}_2) = n/2$, we conclude that $d\bar{\alpha}_1 = d\bar{\alpha}_2$.

Chebyshev systems have the property that for any points $0 \le x_1 < \cdots < x_n \le 1$ and any data $y_1, y_2, \cdots, y_n$ there exists a unique $u \in M$ satisfying

(2.4) $u(x_i) = y_i,$ $i = 1, 2, \cdots, n.$

For a weak Chebyshev system, the determinant of the linear system (2.4) may be zero. However there does exist at least one set of points in $[0, 1]$ for which (2.4) has a unique solution. We will show that the support of the principal representations of positive measures has this property under the assumptions of Corollary 2.2. To explain this further let us suppose for the moment that $n = 2l$, $M \subseteq C^1[0, 1]$, and $d\alpha$ is a positive measure with lower principal representation $d\underline{\alpha}$. Then

$$\int_0^1 f(t) \, d\alpha(t) = \sum_{j=1}^{l} \lambda_j f(t_j), \qquad f \in M,$$

where $\lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_l > 0, 0 < t_1 < \cdots < t_l < 1$.

We associate with $d\alpha$ the interpolation problem

$$u(t_i) = y_i^0, \qquad\qquad i = 1, 2, \cdots, l,$$

$$u'(t_i) = y_i^1, \qquad\qquad i = 1, 2, \cdots, l.$$

This set of equations has a unique solution for all real data $\{y_i^j\}$, $i = 1, \cdots, l$, $j = 0, 1$, provided that the homogeneous set of equations

(2.5)
$$u(t_i) = 0, \qquad\qquad i = 1, \cdots, l,$$
$$u'(t_i) = 0, \qquad\qquad i = 1, \cdots, l,$$

has only the trivial solution in M. We will denote (2.5) simply by $u(d\alpha) = 0$. In general, for any discrete measure, we interpret $u(d\beta) = 0$ as the interpolation

problem which sets $u(t)$ and its first derivative equal to zero at an interior point of the support of $d\beta$ while at an endpoint of $[0, 1]$ only the value of $u(t)$ is set equal to zero.

COROLLARY 2.3. *Suppose M is a weak Chebyshev system contained in $C^1[0, 1]$, and $[K_0] \cap C^1[0, 1]$ contains an n-dimensional Chebyshev system on $[0, 1]$. If for all $f \in K_0$, $d\alpha$ is a positive measure for the subspace M_f, then $u(d\beta) = 0$ has only the zero solution in M, if $d\beta$ is a principal representation for $d\alpha$.*

Proof. Let us again restrict ourselves to the case when $n = 2l$ and to the interpolation problem corresponding to the lower principal representation. Thus we are required to show that the only solution to the system of equations

$$(2.6) \qquad \sum_{i=1}^{2l} a_i u_i(t_j) = 0, \qquad j = 1, 2, \cdots, l,$$

$$\sum_{i=1}^{2l} a_i u_i'(t_j) = 0, \qquad j = 1, 2, \cdots, l,$$

is the zero solution. Suppose to the contrary that (2.6) has a nontrivial solution. Then we conclude that there exist constants c_i^0, c_i^1, $i = 1, 2, \cdots, l$, not all zero, such that

$$F(u) \equiv \sum_{i=1}^{l} c_i^0 u(t_i) + \sum_{i=1}^{l} c_i^1 u'(t_i) = 0,$$

for all $u \in M$. From our hypothesis, there exists an $f_0 \in K \cap C^1[0, 1]$ such that $F(f_0) \neq 0$. Choose a constant c such that

$$\int_0^1 v \, d\alpha = \int_0^1 v \, d\underline{\alpha} + cF(v), \qquad v \in M_{f_0}.$$

We arrive at a contradiction, as before, by constructing a nontrivial nonnegative $v_0 \in M_{f_0}$ which vanishes on the support of $d\underline{\alpha}$ and necessarily has the property that $F(v_0) = 0$. Thus we have contradicted the fact that $d\alpha$ is a positive measure for M_{f_0}.

Remark 2.4. Let $w(t) > 0$, $t \in [0, 1]$. Then the measure $d\alpha(t) = w(t) \, dt$ is a positive measure for all subspaces M_f, $f \in K_0$.

Corollary 2.3 enables us to treat the question of one-sided approximation by weak Chebyshev systems. Let us consider the minimum problem

$$(2.7) \qquad \min_{\substack{u \le f \\ u \in M}} \int_0^1 (f(t) - u(t)) \, d\alpha(t).$$

COROLLARY 2.4. *Let the hypothesis of Corollary 2.3 hold and suppose $d\underline{\alpha}$ is a lower principal representation for $d\alpha$. Then every $f \in K \cap C^1[0, 1]$ has a unique best one-sided approximation from below. The best approximation u_0 to f is determined uniquely by the interpolation conditions $(u_0 - f)(d\underline{\alpha}) = 0$.*

Proof. Let u_0 be uniquely determined by the conditions $(u_0-f)(d\underline{\alpha})=0$. From Lemma 2.1 and Corollary 2.3, we conclude that $u_0 \leq f$. Now let u be any element in M such that $u \leq f$. Then

(2.8)
$$\int_0^1 u(t)\, d\alpha(t) = \int_0^1 u(t)\, d\underline{\alpha}(t) \leq \int_0^1 f(t)\, d\underline{\alpha}(t)$$

$$= \int_0^1 u_0(t)\, d\underline{\alpha}(t) = \int_0^1 u_0(t)\, d\alpha(t).$$

Thus u_0 is a best one-sided approximation to f from below. Furthermore, if u^0 is any other best one-sided approximation to f from below, then according to (2.8) we have $\int_0^1 (f(t)-u^0(t))\, d\underline{\alpha}(t) = 0$. Thus $(f-u^0)(d\underline{\alpha}) = 0$, and from Corollary 2.3 we conclude that $u^0 \equiv u_0$.

Remark 2.5. The unique one-sided approximation from above for $f \in K \cap C^1[0,1]$ is determined by the interpolation conditions $(f-u_0)(d\bar{\alpha}) = 0$, if $d\bar{\alpha}$ exists.

We end this section with some remarks concerning weak Chebyshev systems which satisfy linear constraints. This will enable us to conveniently apply the above results to certain classes of spline functions.

Given linear functionals $L_1(u), \cdots, L_k(u)$ defined on the linear subspace M spanned by the functions $\{u_i\}_{i=1}^{n+r}$, we denote by $M(L)$ the subspace of functions in M which satisfy the linear constraints $L_i(u) = 0$, $i = 1, 2, \cdots, k$. We may construct a basis for $M(L)$ in the following way. We define the $(k+s)$th order determinants

$$U\begin{pmatrix} i_1, \cdots, i_k, i_{k+1}, \cdots, i_{k+s} \\ L_1, \cdots, L_k, x_1, \cdots, x_s \end{pmatrix}$$

$$= \begin{vmatrix} L_1(u_{i_1}) & \cdots L_k(u_{i_1}) & u_{i_1}(x_1) & \cdots & u_{i_1}(x_s) \\ \vdots & \vdots & \vdots & & \vdots \\ L_1(u_{i_{k+s}}) & \cdots L_k(u_{i_{k+s}}) & u_{i_{k+s}}(x_1) & \cdots & u_{i_{k+s}}(x_s) \end{vmatrix}$$

for $1 \leq i_1 < \cdots < i_{k+s} \leq n+r$ and $0 \leq x_1 < \cdots < x_s \leq 1$. If the set of linear functionals $\{L_i\}_{i=1}^k$ is independent over M, that is, rank $\|L_i(u_j)\|_{i=1}^k {}_{j=1}^{n+r} = k$, then there exists exist $i_1 < \cdots < i_k$ such that

$$d = U\begin{pmatrix} i_1, \cdots, i_k \\ L_1, \cdots, L_k \end{pmatrix} \neq 0,$$

and the functions

$$v_{i'_l}(t) = U\begin{pmatrix} i_1, \cdots, i_k i'_l \\ L_1, \cdots, L_k, t \end{pmatrix}, \qquad l = 1, 2, \cdots, n+r-k,$$

where $\{i'_1, \cdots, i'_{n+r-k}\}$ are the set of complementary ordered indices to $\{i_1, \cdots, i_k\}$ in $\{1, 2, \cdots, n+r\}$, form a basis for $M(L)$. Furthermore, employing

Sylvester's determinant identity, see Karlin [2], we have for some $\sigma = \pm 1$

$$\begin{vmatrix} v_{i_1'}(x_1) & \cdots & v_{i_1'}(x_{n+r-k}) \\ v_{i_2'}(x_1) & \cdots & v_{i_2'}(x_{n+r-k}) \\ \vdots & & \vdots \\ v_{i_{n+r-k}'}(x_1) & \cdots & v_{i_{n+r-k}'}(x_{n+r-k}) \end{vmatrix} = \sigma d^{n+r-k-1} U\begin{pmatrix} 1, 2, \cdots, k, k+1, \cdots, n+r \\ L_1, \cdots, L_k, x_1, \cdots, x_{n+r-k} \end{pmatrix}.$$

Thus if $M(L)$ has dimension $n+r-k$, rank $\|L_i(u_j)\| = k$ and

$$(2.9) \qquad \sigma U\begin{pmatrix} 1, \cdots, k, k+1, \cdots, n+r \\ L_1, \cdots, L_k, x_1, \cdots, x_{n+r-k} \end{pmatrix} \geqq 0,$$

for all $0 \leqq x_1 < \cdots < x_{n+r-k} \leqq 1$, then the set of functions $\{v_l\}_{l=1}^{n+r-k}$ form a weak Chebyshev system on $[0, 1]$.

Furthermore, let us note that if there exists a set of points for which strict inequality holds in (2.9), then it follows that the rank $\|(L_i(u_j)\| = k$ and the dimension of $M(L)$ is $n+r-k$.

Let f be a function defined on $[0, 1]$ such that

$$(2.10) \qquad \sigma \begin{vmatrix} L_1(u_1) & \cdots & L_1(u_{n+r}) & L_1(f) \\ \vdots & & \vdots & \vdots \\ L_k(u_1) & \cdots & L_k(u_{n+r}) & L_k(f) \\ u_1(x_1) & \cdots & u_{n+r}(x_1) & f(x_1) \\ \vdots & & \vdots & \vdots \\ u_1(x_{n+r-k+1}) & \cdots & u_{n+r}(x_{n+r-k+1}) & f(x_{n+r-k+1}) \end{vmatrix} \geqq 0,$$

for $0 \leqq x_1 < \cdots < x_{n+r-k+1} \leqq 1$. Then according to Sylvester's determinant identity, we may express the determinant in (2.10) as

$$d^{n+r-k} \begin{vmatrix} v_{i_1'}(x_1) & \cdots & v_{i_1'}(x_{n+r-k-1}) \\ v_{i_2'}(x_1) & \cdots & v_{i_2'}(x_{n+r-k+1}) \\ \vdots & & \vdots \\ v_{i_{n+r-k}'}(x_1) & \cdots & v_{i_{n+r-k}'}(x_{n+r-k+1}) \\ \bar{f}(x_1) & \cdots & \bar{f}(x_{n+r-k+1}) \end{vmatrix}$$

where

$$(2.11) \qquad \bar{f}(t) = \begin{vmatrix} L_1(u_{i_1}) & \cdots & L_1(u_{i_k}) & L_1(f) \\ \vdots & & \vdots & \vdots \\ L_k(u_{i_1}) & \cdots & L_k(u_{i_k}) & L_k(f) \\ u_{i_1}(t) & \cdots & u_{i_k}(t) & f(t) \end{vmatrix}.$$

Thus $\bar{f}$ is in the convexity cone of $M(L)$ and $L_i(\bar{f}) = 0$, $i = 1, \cdots, k$. Among all functions which satisfy these relations, (2.11) gives us a correspondence between

the elements in the cone of $M(L)$ and functions for which (2.10) is valid.

Finally, observe that we may expand the determinant in (2.11) by the last column and express $\bar{f}(t)$ in the form

$$\bar{f}(t) = df(t) + \sum_{j=1}^{k} a_j(t)L_j(f),$$

where $a_1(t), \cdots, a_k(t)$ are elements of M.

We now consider some application of our previous results.

3. Quadrature formulae for spline functions with boundary conditions.

Let Δ_r denote the partition $0 = \xi_0 < \xi_1 < \cdots < \xi_r < \xi_{r+1} = 1$ of the unit interval $[0, 1]$. The class of spline functions on $[0, 1]$ of degree $n-1$ with simple knots at Δ_r is defined by

$$\mathscr{S} = \mathscr{S}_{n-1}(\Delta_r) = \{S : S \in C^{n-2}[0, 1], S|_{(\xi_\nu, \xi_{\nu+1})} \in \prod_{n-1}, \nu = 0, 1, \cdots, r\},$$

where $\prod_{n-1}$ denotes all polynomials of degree $\leqq n-1$. Every element $S \in \mathscr{S}$ has a representation of the form

$$S(t) = \sum_{i=0}^{n-1} a_i t^i + \sum_{i=1}^{r} c_i (t - \xi_i)_+^{n-1},$$

where $t_+ = \max\{0, t\}$ (we shall always assume $n \geqq 2$).

We are interested in the subclass of $\mathscr{S}$ which satisfies boundary conditions of the following form.

Let $n + r = k + m$, and define

$$(3.1) \qquad C_i(f) = \sum_{j=0}^{n-1} A_{ij} f^{(j)}(0) + \sum_{j=0}^{n-1} B_{ij} f^{(j)}(1), \qquad i = 1, \cdots, k.$$

Denote by $\mathscr{S}(\mathscr{C}_k)$ the subset of $\mathscr{S}$ satisfying $C_i(S) = 0$, $i = 1, \cdots, k$, and let $C = \|C_{ij}\|_{i=1}^{k} {}_{j=0}^{2n-1}$, where

$$(3.2) \qquad C_{ij} = \begin{cases} A_{ij}(-1)^{j+n+r+1}, & i = 1, \cdots, k, \quad j = 0, 1, \cdots, n-1, \\ B_{i,2n-1-j}, & i = 1, \cdots, k, \quad j = n, \cdots, 2n-1. \end{cases}$$

The following conditions on the matrix C are assumed to prevail throughout this paper.

(i) $0 \leqq k \leqq \min\{2n, n+r\}$.

(ii) *There exist* $\{i_1, \cdots, i_s, j_1, \cdots, j_{k-s}\} \subseteq \{0, 1, \cdots, 2n-1\}$, $0 \leqq i_1 < \cdots < i_s \leqq n-1 < j_1 < \cdots < j_{k-s} \leqq 2n-1$, *satisfying* $M_{\nu-1} + m \geqq \nu$, $\nu = m+1, \cdots, n$, *where M_ν counts the number of terms in* $\{i_1, \cdots, i_s, 2n-1-j_1, \cdots, 2n-1-j_{k-s}\}$ *less than or equal to ν, and*

$$(3.3) \qquad C\begin{pmatrix} 1, & \cdots, & k \\ i_1, \cdots, i_s, j_1, \cdots, j_{k-s} \end{pmatrix} \neq 0.$$

(iii) *For all* $\{i_1, \cdots, i_s, j_1, \cdots, j_{k-s}\}$ *satisfying* (ii),

$$C\begin{pmatrix} 1, & \cdots, & k \\ i_1, \cdots, i_s, j_1, \cdots, j_{k-s} \end{pmatrix},$$

is of one fixed sign.

Remark 3.1. Note that we make no assumptions on the $k \times k$ minors of C for which $M_{\nu-1} + m \geqq \nu$, $\nu = m + 1, \cdots, n$ does not hold.

The main theorem of this section is the following result.

THEOREM 3.1. *Given a positive weight function $w(t) > 0$ and nonnegative integers n, r, k, l with $n \geqq 2$, and $n + r = k + 2l$, then there exists a quadrature formula of the form*

$$(3.4) \qquad \int_0^1 f(t) w(t) \, dt = \sum_{j=1}^k c_j C_j(f) + \sum_{j=1}^l \lambda_j f(t_j),$$

which is exact for all $s \in \mathcal{S}$, where $\lambda_j > 0$, $j = 1, \cdots, l$, and $0 < t_1 < \cdots < t_l < 1$.

We remark that the formula appearing above is of "double precision" since the dimension of $\mathcal{S}$ is $n + r$ while the number of "free" parameters appearing on the right hand side of (3.4) is $k + 2l$.

In general, the above quadrature formula is not unique as the following example demonstrates.

Example 3.1. Let $n = 2$ and $r = 3$ with the knots chosen at $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{1}{2}$, $\xi_3 = \frac{2}{3}$, and $l = 2$ and $k = 1$ where the boundary condition is $S(0) + S(1) = 0$. This boundary condition satisfies (3.3) and the following two quadrature formulae hold for $f \in \{1, t, (t - \frac{1}{3})_+^1, (t - \frac{1}{2})_+^1, (t - \frac{2}{3})_+^1\}$:

$$\int_0^1 f(t) \, dt = \frac{1}{6}(f(0) + f(1)) + \frac{1}{4} f\left(\frac{1}{3}\right) + \frac{5}{12} f\left(\frac{3}{5}\right),$$

$$\int_0^1 f(t) \, dt = \frac{1}{6}(f(0) + f(1)) + \frac{5}{12} f\left(\frac{2}{5}\right) + \frac{1}{4} f\left(\frac{2}{3}\right).$$

Whether uniqueness persists for all $n \geqq 3$ remains unresolved. However, we will later give some partial results on the uniqueness of (3.4).

The main idea in the proof of Theorem 3.1 is simply to show that the subspace $\mathcal{S}(\mathcal{C}_k)$ has a lower principal representation. The remainder of the section is devoted to the details of the proof of this fact.

Let us write

$$u_i(t) = t^{i-1}, \qquad\qquad\qquad i = 1, \cdots, n,$$

and

$$u_{n+i}(t) = (t - \xi_i)_+^{n-1}, \qquad\qquad\qquad i = 1, \cdots, r.$$

Then in the notation of § 2, Melkman [9] (see also [5]) proved.

THEOREM 3.2. *If $n + r = k + m$ and (3.3) is valid, then*

$$(3.5) \qquad U\begin{pmatrix} 1, & \cdots, & n+r \\ C_1, \cdots, & C_k, x_1, \cdots, & x_m \end{pmatrix} \sigma \geqq 0,$$

where $\sigma = +1$ or -1 fixed, for all choices of $0 < x_1 \leqq \cdots \leqq x_m < 1$ (where at most n of the x_i's coincide), and (3.5) is strictly positive iff there exists an $\{s, k - s\}$ for which (ii) of (3.3) is satisfied and

$$(3.6) \qquad \xi_{\mu+s-n} < x_\mu < \xi_{\mu+s}, \qquad\qquad \mu = 1, \cdots, m$$

(whenever the inequalities are meaningful).

Thus according to the discussion at the end of § 2, we conclude that $\mathscr{S}(\mathscr{C}_k)$ is a weak Chebyshev subspace of dimension m on $[0, 1]$.

We list below some examples of boundary conditions which satisfy (3.3).

Example 3.2.

$$S^{(i_\mu)}(0) = 0, \qquad\qquad\qquad \mu = 1, \cdots, p,$$

$$S^{(j_\mu)}(1) = 0, \qquad\qquad\qquad \mu = 1, \cdots, q,$$

where $p + q = k, 0 \leq i_1 < \cdots < i_p \leq n-1, 0 \leq j_1 < \cdots < j_q \leq n-1$, and $M'_{\nu-1} + m \geq \nu, \nu = m+1, \cdots, n$, where M'_ν counts the number of terms in $\{i_1, \cdots, i_p, j_1, \cdots, j_q\}$ less than or equal to ν.

Example 3.3. $S^{(i)}(0) = S^{(i)}(1), i = 0, 1, \cdots, k-1$, if $k + n + r$ is odd.

Example 3.4. $S^{(i)}(0) = -S^{(i)}(1), i = 0, 1, \cdots, k-1$, if $k + n + r$ is even.

Example 3.5. Separated boundary conditions. Let

$$A_i(f) = \sum_{j=0}^{n-1} A_{ij} f^{(j)}(0) = 0, \qquad\qquad i = 1, \cdots, p,$$

$$B_i(f) = \sum_{j=0}^{n-1} B_{ij} f^{(j)}(1) = 0, \qquad\qquad i = 1, \cdots, q,$$

where $p + q = k$. It may be easily seen (see [5]) that these boundary conditions satisfy (3.3) provided that

   (i) $0 \leq p, q \leq n$;

   (ii) *there exist* $\{i_1, \cdots, i_p\}$, $\{j_1, \cdots, j_q\} \subseteq \{0, 1, \cdots, n-1\}$ *satisfying* $M'_{\nu-1} + m \geq \nu, \nu = m+1, \cdots, n$, *where* M'_ν *counts the number of terms in* $\{i_1, \cdots, i_p, j_1, \cdots, j_q\}$ *less than or equal to ν and*

(3.7)
$$\tilde{A}\begin{pmatrix} 1, \cdots, p \\ i_1, \cdots, i_p \end{pmatrix} B\begin{pmatrix} 1, \cdots, q \\ j_1, \cdots, j_q \end{pmatrix} \neq 0,$$

where $\tilde{A} = \|A_{ij}(-1)^j\|_{i=1, j=0}^{p, n-1}$, $B = \|B_{ij}\|_{i=1, j=0}^{q, n-1}$;

   (iii) *for all* $\{i_1, \cdots, i_p\}, \{j_1, \cdots, j_q\}$ *satisfying* (ii),

$$\tilde{A}\begin{pmatrix} 1, \cdots, p \\ i_1, \cdots, i_p \end{pmatrix} B\begin{pmatrix} 1, \cdots, q \\ j_1, \cdots, j_q \end{pmatrix},$$

is of one fixed sign.

Note that Example 3.5 includes Example 3.2.

Returning to the general case (3.1), let T denote the set of integers s for which (ii) of (3.3) is satisfied. Then we have the following interesting corollary of Theorem 3.2.

COROLLARY 3.1. *If there exists an* $s \in T$ *for which* $\min\{s, k-s\} \geq r$, *then* $\mathscr{S}(\mathscr{C}_k)$ *has a basis of m functions which form a Chebyshev system on* $(0, 1)$.

It may also be shown that if $s \in T$ is such that $\min\{s, k-s\} \leq r$, then $M_{\nu-1} + m \geq \nu, \nu = m+1, \cdots, n$ for all $\{i_1, \cdots, i_s, j_1, \cdots, j_{k-s}\}$ satisfying $0 \leq i_1 < \cdots < i_s \leq n-1 < j_1 < \cdots < j_{k-s} \leq 2n-1$. Note that when the boundary conditions are separated (Example 3.5), then $T = \{p\}$.

In the discussion which follows, we set $m = 2l$. Since we wish to prove the existence of a lower principal representation for any positive measure $d\alpha(t)$ relative to $\mathscr{S}(\mathscr{C}_k)$, we shall assume $l > 0$ (i.e., $k < n + r$). If $l = 0$, Theorem 3.1 is

easily proven. To apply Theorem 2.1, we must show that there exists a function in $\mathscr{S}(\mathscr{C}_k)$ which is positive on $(0, 1)$. However, this is not always possible. To circumvent this difficulty, we introduce the following notion of a zero of degree α for the subspace $\mathscr{S}(\mathscr{C}_k)$.

DEFINITION 3.1. If $S \in \mathscr{S}(\mathscr{C}_k)$ implies $S(0) = S'(0) = \cdots = S^{(\alpha-1)}(0) = 0$, while there exists an $S \in \mathscr{S}(\mathscr{C}_k)$ for which $S^{(\alpha)}(0) \neq 0$, then we say that $\mathscr{S}(\mathscr{C}_k)$ has a zero of degree α at 0. If there exists no such α, i.e., $S^{(i)}(0) = 0$, $i = 0, 1, \cdots, n-1$, then we say that $\mathscr{S}(\mathscr{C}_k)$ has a zero of degree n at 0. Similarly we define the degree of the zero of $\mathscr{S}(\mathscr{C}_k)$ at 1.

The following result in the case of separated boundary conditions is to be found in [12]. The proof of the general case below is essentially the same as the proof in [12]. We include it here for completeness.

PROPOSITION 3.1. *For $k < n + r$, $\mathscr{S}(\mathscr{C}_k)$ has a zero of degree α at 0 iff for all $\{i_1, \cdots, i_s, j_1, \cdots, j_{k-s}\}$ satisfying* (ii) *of* (3.3), *$i_1 = 0$, $i_2 = 1, \cdots, i_\alpha = \alpha - 1$. A similar result holds at* 1.

Proof. Assume $\mathscr{S}(\mathscr{C}_k)$ has a zero of degree α at zero, $\alpha > 0$. Assume, as well, that for all $\{i_1, \cdots, i_s, j_1, \cdots, j_{k-s}\}$ satisfying (ii) of (3.3), $i_1 = 0, \cdots, i_\gamma = \gamma - 1$, but that there exists an $\{i_1, \cdots, i_s, j_1, \cdots, j_{k-s}\}$ satisfying (ii) of (3.3) for which $i_{\gamma+1} \neq \gamma$, $0 \leq \gamma < \alpha$. Consider the matrix $\bar{C} = \|\bar{C}_{ij}\|_{i=1 \ j=0}^{k+1 \ 2n-1}$, where

$$\bar{C}_{ij} = \begin{cases} C_{ij}, & i = 1, \cdots, k; \quad j = 0, 1, \cdots, 2n-1, \\ \delta_{\gamma j}, & i = k+1; \quad j = 0, 1, \cdots, 2n-1. \end{cases}$$

It is easily shown, since $i_1 = 0, \cdots, i_\gamma = \gamma - 1$ for all $\{i_1, \cdots, i_s, j_1, \cdots, j_{k-s}\}$ satisfying (ii) of (3.3), that $\bar{C}$ satisfies (3.3) unless $k = 2n$. However, if $k = 2n$, then the proposition is immediate. Let $\mathscr{S}(\bar{\mathscr{C}}_{k+1})$ denote the subset of $\mathscr{S}$ satisfying the boundary conditions associated with the matrix $\bar{C}$. Since $\mathscr{S}(\bar{\mathscr{C}}_{k+1})$ satisfies (3.3), $\mathscr{S}(\bar{\mathscr{C}}_{k+1})$ is a weak Chebyshev subspace of dimension $2l - 1$ on $(0, 1)$ (recall that $n + r = k + 2l$). However, every $S \in \mathscr{S}(\mathscr{C}_k)$ satisfies $S^{(\gamma)}(0) = 0$ since $\gamma < \alpha$, and thus $\mathscr{S}(\mathscr{C}_k) \subseteq \mathscr{S}(\bar{\mathscr{C}}_{k+1})$. $\mathscr{S}(\mathscr{C}_k)$ is a subspace of dimension $2l$, and a contradiction follows.

Now let us assume that $\mathscr{S}(\mathscr{C}_k)$ has a zero of degree α at 0, and for all $\{i_1, \cdots, i_s, j_1, \cdots, j_{k-s}\}$ satisfying (ii) of (3.3), we have $i_1 = 0, \cdots, i_\alpha = \alpha - 1$, and $i_{\alpha+1} = \alpha$. Construct $\bar{C}$ as above with $\gamma = \alpha$. Then (ii) of (3.3) is not satisfied, and from the analysis of Theorem 3.2 (see [5]) the determinant associated with the conditions $S \in \mathscr{S}(\bar{\mathscr{C}}_{k+1})$, $S(x_i) = 0$, $i = 1, \cdots, 2l - 1$, is singular for every choice of $\{x_i\}_{i=1}^{2l-1}$ in $(0, 1)$.

Now $\mathscr{S}(\bar{\mathscr{C}}_{k+1}) \subseteq \mathscr{S}(\mathscr{C}_k)$ and there exists an $S \in \mathscr{S}(\mathscr{C}_k)$ for which $S^{(\alpha)}(0) \neq 0$. Thus $\mathscr{S}(\bar{\mathscr{C}}_{k+1})$ has dimension at most $2l - 1$. Since we may choose a basis $\{S_j(x)\}_{j=1}^{2l}$ for $\mathscr{S}(\mathscr{C}_k)$ such that $S_j^{(\alpha)}(0) = 0$, $j = 1, \cdots, 2l - 1$, $\mathscr{S}(\bar{\mathscr{C}}_{k+1})$ has dimension $2l - 1$. The $\{S_j(x)\}_{j=1}^{2l-1}$ are linearly independent functions and thus there exist points $\{y_i\}_{i=1}^{2l-1}$, $0 < y_1 < \cdots < y_{2l-1} < 1$, such that any $S \in \mathscr{S}(\bar{\mathscr{C}}_{k+1})$, i.e., $S(x) = \sum_{j=1}^{2l-1} a_j S_j(x)$, satisfying $S(y_i) = 0$, $i = 1, \cdots, 2l - 1$, implies $S(x) \equiv 0$. This contradicts the fact that the determinant associated with the conditions $S \in \mathscr{S}(\bar{\mathscr{C}}_{k+1})$, $S(x_i) = 0$, $i = 1, \cdots, 2l - 1$, is singular for every choice of $\{x_i\}_{i=1}^{2l-1}$. The proposition is proven by applying the same analysis at 1.

From Proposition 3.1, we have

COROLLARY 3.2. *For all $S \in \mathcal{S}(\mathcal{C}_k)$, $S(t) \equiv 0$ in $[0, \xi_1]$ if and only if $s = n$ for all $s \in T$, and $S(t) \equiv 0$ on $[\xi_r, 1]$ for all $S \in \mathcal{S}(\mathcal{C}_k)$ if and only if $k - s = n$ for all $s \in T$.*

Now, let $d\alpha(t)$ be any positive measure relative to $\mathcal{S}(\mathcal{C}_k)$, and construct, as in the proof of Theorem 2.1, the points $\{t_i\}_{i=1}^{l}$, $0 < t_1 \le \cdots \le t_l < 1$ for the subspace $\mathcal{S}(\mathcal{C}_k)$. Corollary 3.2 shows that we cannot, in general, expect $\mathcal{S}(\mathcal{C}_k)$ to contain a positive function. However, in the proof of Theorem 2.1, we see that we only require the existence of a function which is positive on the set $\{t_i\}_{i=1}^{l}$ to conclude that $d\alpha$ has a lower principal representation. In our next proposition we will explore the relationship between the $\{t_i\}_{i=1}^{l}$ and the $\{\xi_i\}_{i=1}^{r}$.

Let us note that from the proof of Theorem 2.1, there exists a nontrivial $\hat{S}(t) \in \mathcal{S}(\mathcal{C}_k)$ such that

$$\hat{S}(t_i) = 0, \qquad\qquad\qquad i = 1, \cdots, l,$$

$$\hat{S}(t) \ge 0, \qquad t \ge t_1,$$

and

$$\int_0^1 \hat{S}(t) \, d\alpha(t) = 0.$$

Therefore, since $d\alpha(t)$ is a positive measure with respect to $\mathcal{S}(\mathcal{C}_k)$, $\hat{S}(t) < 0$ for some $t < t_1$. On the basis of this observation, we have

PROPOSITION 3.2. *If $\mathcal{S}(\mathcal{C}_k)$ has a zero of degree n at 0, then $t_1 > \xi_2$, while if $\mathcal{S}(\mathcal{C}_k)$ has a zero of degree n at 1, then $t_l < \xi_{r-1}$.*

Proof. If $\mathcal{S}(\mathcal{C}_k)$ has a zero of degree n at zero, then $S \in \mathcal{S}(\mathcal{C}_k)$ implies $S(t) \equiv 0$, $t \in [0, \xi_1]$. Since $S^{(i)}(\xi_1) = 0$, $i = 0, 1, \cdots, n-2$, and $S|_{(\xi_1, \xi_2)} \in \Pi_{n-1}$, if $S(t_1) = 0$ for $t_1 \le \xi_2$, then $S(t) \equiv 0$ for $t \le t_1$. However, $\hat{S}(t) < 0$ for some $t < t_1$, and thus we conclude that $t_1 > \xi_2$. By an analogous argument we obtain the corresponding result at one.

PROPOSITION 3.3. *If $k < n + r$, then there exists an $S \in \mathcal{S}(\mathcal{C}_k)$ which is strictly positive on an open interval containing $[t_1, t_l]$.*

Proof. If $k = 2n$, then $r = n + 2l$, and we may easily construct, by the use of B-splines (see [2] or § 5), a spline $S \in \mathcal{S}(\mathcal{C}_k)$ such that $S(t) > 0$, $t \in (\xi_1, \xi_r)$. The result then emanates from Proposition 3.2.

In what follows, we shall assume $n \ge 3$. For the case $n = 2$, the required spline may be explicitly constructed.

Define, for $\varepsilon, \delta \in (0, 1)$,

$S_1(t) =$

$$\int \cdots \int_{\varepsilon \le y_1 < \cdots \le y_{l-1} \le 1} U\binom{1, \quad\quad\quad \cdots, \quad\quad\quad n+r}{C_1, \cdots, C_k, \varepsilon, y_1, y_1, \cdots, y_{l-1}, y_{l-1}, t} \sigma \, dy_1 \cdots dy_{l-1},$$

$S_2(t) =$

$$\int \cdots \int_{0 \le y_1 \le \cdots \le y_{l-1} \le 1-\delta} U\binom{1, \quad\quad\quad \cdots, \quad\quad\quad n+r}{C_1, \cdots, C_k, y_1, y_1, \cdots, y_{l-1}, y_{l-1}, t, 1-\delta} \sigma \, dy_1 \cdots dy_{l-1},$$

and

$$S(t) = S_1(t) + S_2(t).$$

From (3.5), $S_1(t) \geq 0$ for $t \in (\varepsilon, 1)$, while $S_2(t) \geq 0$ for $t \in (0, 1-\delta)$.

Case I. There exists an $s \in T$ such that $s, k-s < n$. Let $\varepsilon, \delta > 0$ be chosen arbitrarily small. By (3.6), if $s < n-1$, $S_1(t) > 0$ for $t \in (\varepsilon, 1)$, while if $s = n-1$, $S_1(t) > 0$ for $t \in (\xi_1, 1)$. Similarly, if $k-s < n-1$, $S_2(t) > 0$ for $t \in (0, 1-\delta)$, and if $k-s = n-1$, $S_2(t) > 0$ for $t \in (0, \xi_r)$. Thus it follows that $S(t) > 0$ for $t \in (\varepsilon, 1-\delta)$ for all ε, δ positive and small. Since $t_1 > 0$, $t_l < 1$, the result follows.

Case II. $\mathscr{S}(\mathscr{C}_k)$ has a zero of degree n at 0 or 1. Assume $\mathscr{S}(\mathscr{C}_k)$ has a zero of degree n at 1. Thus for $s \in T$, $k-s = n$. Since we have already considered the case $k = 2n$, we assume $k < 2n$, implying $s \leq n-1$. Choose $\varepsilon, \delta > 0$, ε small and $\xi_{r-1} < 1-\delta < \xi_r$. If $s < n-1$, then $S_1(t) > 0$ for $t \in (\varepsilon, \xi_r)$, while if $s = n-1$, then $S_1(t) > 0$, for $t \in (\xi_1, \xi_r)$. However, $S_2(t) > 0$ for $t \in (0, \xi_{r-1})$, and the result then follows from Proposition 3.2.

Case III. $\mathscr{S}(\mathscr{C}_k)$ has no zero of degree n at 0 or 1, but for all $s \in T$, either $s = n$ or $k-s = n$.

From Corollary 3.2, it follows that since $\mathscr{S}(\mathscr{C}_k)$ has no zero of degree n at 0 or 1, there exist $s_1, s_2 \in T$ such that $s_1 = n$, $k-s_1 < n$, and $s_2 < n$, $k-s_2 = n$. Obviously, $k-s_1 = s_2$.

If $k-s_1 = s_2 < n-1$, let $\varepsilon, \delta > 0$ be chosen small. Since $s_2 < n-1$, $S_1(t) > 0$ for $t \in (\varepsilon, \xi_r)$, and since $k-s_1 < n-1$, $S_2(t) > 0$ for $t \in (\xi_1, 1-\delta)$. Thus $S(t) > 0$ for $t \in (\varepsilon, 1-\delta)$.

Assume $k-s_1 = s_2 = n-1$. By the above construction, $S(t) > 0$ for $t \in (\xi_1, \xi_r)$. We shall show that $t_1 > \xi_1$ and $t_l < \xi_r$, proving the proposition.

LEMMA 3.1. *Assume, as above, that $k = 2n-1$ and $n, n-1 \in T$. Then $t_1 > \xi_1$ and $t_l < \xi_r$.*

Proof. Let $\mathscr{S}'$ denote the subset of $\mathscr{S}$ satisfying $S^{(i)}(0) = S^{(i)}(1) = 0$, $i = 0, 1, \cdots, n-1$. Since $k = 2n-1$, $\mathscr{S}'$ is a subset of $\mathscr{S}(\mathscr{C}_k)$ of dimension $2l-1$ and every $S \in \mathscr{S}'$ vanishes identically on $[0, \xi_1] \cup [\xi_r, 1]$. For $\xi_1 < \varepsilon < \xi_2$, $S_1(t) < 0$ for $t < \varepsilon$ and $S_1(t) > 0$ for $t > \varepsilon$ where $S_1(t)$ is defined above. Thus $S_1 \in \mathscr{S}(\mathscr{C}_k)$, and $S_1(t) \neq 0$ for all $t \in (0, \xi_1] \cup [\xi_r, 1)$. Therefore the subset of $\mathscr{S}(\mathscr{C}_k)$ which vanishes at some point $x \in (0, \xi_1] \cup [\xi_r, 1)$ has dimension $2l-1$. However, since this subset still contains $\mathscr{S}'$, it must equal $\mathscr{S}'$. Let $\hat{S}(t)$ be as constructed in Theorem 2.1. If $t_1 \leq \xi_1$, then by the above analysis, $\hat{S} \in \mathscr{S}'$ and thus $\hat{S}(t) \equiv 0$ for $t \leq t_1$. This contradicts the properties of $\hat{S}$ and therefore $t_1 > \xi_1$. Similarly, $t_l < \xi_r$. The lemma is proven.

We are now ready to prove

THEOREM 3.3. *Assume $n + r = k + 2l$ and $d\alpha(t)$ is a positive measure with respect to the weak Chebyshev subspace $\mathscr{S}(\mathscr{C}_k)$ of dimension $2l$. Then $d\alpha(t)$ has a lower principal representation.*

Proof. The proof of Theorem 3.3 follows Theorem 2.1 and Proposition 3.3.

Remark 3.2. If there exists an $s \in T$ such that $\min\{s, k-s\} \geq r$, then from Corollary 3.1, $\mathscr{S}(\mathscr{C}_k)$ is a Chebyshev subspace on $(0, 1)$. The existence and uniqueness of the lower principal representation for $\mathscr{S}(\mathscr{C}_k)$ is immediate in this case.

We may now prove Theorem 3.1.

Proof of Theorem 3.1. From Theorem 3.3, there exists a quadrature formula of the form

$$(3.8) \qquad \int_0^1 f(t) \, dt = \sum_{i=1}^{l} \lambda_i f(t_i),$$

which is exact for all $S \in \mathcal{S}(\mathscr{C}_k)$, where $\lambda_i > 0$, $i = 1, \cdots, l$ and $0 < t_1 < \cdots < t_l < 1$.

From (2.10) and the subsequent analysis, any $S \in \mathcal{S}$ may be expressed in the form

$$S(t) = d^{-1} \left[\bar{S}(t) - \sum_{j=1}^{k} c_j(t) C_j(S) \right],$$

where $d \neq 0$ and $\bar{S} \in \mathcal{S}(\mathscr{C}_k)$. Substituting this relation with $f = \bar{S}$ into (3.8), we obtain (3.4). The theorem is proven.

If the boundary conditions under consideration are separated (see Example 3.5), then the quadrature formula (3.8) and (3.4) are unique. The proof of this fact is based upon Corollary 2.2 and the following proposition.

PROPOSITION 3.4. *Assume* $n + r = p + q + 2l$, *and that the boundary conditions* (3.1) *are separated and satisfy* (3.7). *If* $f \in C^n[0, 1]$ *and* $C_i(f) = 0$, $i = 1, \cdots, p + q = k$, *then* $f \in [K]$, *where* $[K]$ *is the smallest linear subspace containing* K, *the convexity cone generated by* $\mathcal{S}(\mathscr{C}_k)$.

Proof. Any $f \in C^n[0, 1]$ may be written as $f = f_1 - f_2$, where $f_j^{(n)}(t)(-1)^i \geq 0$ for $t \in (\xi_{i-1}, \xi_i)$, $i = 1, \cdots, r + 1$; $j = 1, 2$, and $f_j \in C^{n-1}[0, 1]$, $f_j \in C^n(\xi_{i-1}, \xi_i)$, $i = 1, \cdots, r + 1$. Let $g_1(t) = f_1(t) + S(t)$, where $S \in \mathcal{S}$, such that $C_i(g_1) = 0$, $i = 1, \cdots, k$. Since $C_i(f) = 0$, $i = 1, \cdots, p + q$, and $f = (f_1 + S) - (f_2 + S)$, we conclude that $C_i(f_2 + S) = 0$, $i = 1, \cdots, k$. Let $g_2(t) = f_2(t) + S(t)$.

We shall prove that for any function $g(t)$ which has the form

$$(3.9) \qquad g(t) = S(t) + \frac{1}{(n-1)!} \int_0^1 (t-x)_+^{n+1} g^{(n)}(x) \, dx,$$

where $S \in \mathcal{S}$, and which satisfies $g^{(n)}(t)(-1)^i \geq 0$, $t \in (\xi_{i-1}, \xi_i)$, $i = 1, \cdots, r + 1$, and $C_i(g) = 0$, $i = 1, \cdots, k$, then either g or $-g$ lie in K. By Taylor's theorem, both $g_1(t)$ and $g_2(t)$ are of the requisite form. From the properties of $g(t)$, (3.9) may be rewritten in the form

$$g(t) = S(t) + \sum_{i=1}^{r+1} \frac{(-1)^i}{(n-1)!} \int_{\xi_{i-1}}^{\xi_i} (t-x)_+^{n+1} |g^{(n)}(x)| \, dx.$$

Let $u_{n+r+1}(t) = (t - \xi)_+^{n-1}$. An application of Theorem 3.2 and (3.7) yields

$$(3.10) \qquad U \binom{1, \quad \cdots, \quad n+r+1}{C_1, \cdots, C_k, x_1, \cdots, x_{2l+1}} (-1)^i \sigma \geq 0,$$

for $\xi \in (\xi_{i-1}, \xi_i)$, $i = 1, \cdots, r+1$, where $\sigma = +1$ or -1, fixed. Now from (3.9),

$$
\begin{vmatrix}
C_1(u_1) & \cdots & C_1(u_{n+r}) & C_1(g) \\
\vdots & & \vdots & \vdots \\
C_k(u_1) & \cdots & C_k(u_{n+r}) & C_k(g) \\
u_1(x_1) & \cdots & u_{n+r}(x_1) & g(x_1) \\
\vdots & & \vdots & \vdots \\
u_1(x_{2l+1}) & \cdots & u_{n+r}(x_{2l+1}) & g(x_{2l+1})
\end{vmatrix}
$$

$$
= \sum_{i=1}^{r+1} \frac{(-1)^i}{(n-1)!} \int_{\xi_{i-1}}^{\xi_i} U\!\left(\begin{matrix} 1, \cdots, n+r+1 \\ C_1, \cdots, C_k, x_1, \cdots, x_{2l+1} \end{matrix} \right) |g^{(n)}(\xi)| \, d\xi.
$$

From (3.10) it follows that the above determinant is nonnegative (or nonpositive) for all $0 < x_1 \leq \cdots \leq x_{2l+1} < 1$. Thus by (2.11), $\bar{g}$ (or $-\bar{g}$) is in K. Since, by assumption $C_i(g) = 0$, $i = 1, \cdots, k$, it follows that $g = \bar{g}$, and the proposition is proven.

Thus we have also proven

THEOREM 3.4. *For separated boundary conditions which satisfy* (3.7), *the quadrature formula* (3.4) *is unique.*

Remark 3.3. In the case of separated boundary conditions, it follows from Corollary 2.3 and Theorem 3.2 that

$$
\xi_{2i+p-n} < t_i < \xi_{2i-1+p}, \qquad\qquad i = 1, \cdots, l,
$$

where the $\{t_i\}_{i=1}^l$ are the nodes of the unique quadrature formula (3.4).

Remark 3.4. The analysis of this section also holds for spline functions with knots of multiplicity at most $n - 2$ and for Chebyshevian spline functions (see [3] and [6]).

4. Monosplines satisfying boundary conditions. In this section, we shall study the Peano kernel of the quadrature formula of Theorem 3.1 and state our version of the fundamental theorem of algebra for monosplines.

A monospline of degree n with l knots $\{x_i\}_{i=1}^l$, $0 < x_1 < \cdots < x_l < 1$, is a function of the form

(4.1) $$ M(x) = \frac{x^n}{n!} + \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^l b_i (x - x_i)_+^{n-1}. $$

Let $C_i(f)$, $i = 1, \cdots, k$, be boundary conditions of the form (3.1) such that the $k \times 2n$ matrix C satisfies (3.3). Let

(4.2) $$ Q(f) = \sum_{i=1}^k c_i C_i(f) + \sum_{i=1}^l \lambda_i f(t_i), $$

be the quadrature formula constructed in Theorem 3.1, i.e., $\int_0^1 f(x) \, dx = Q(f)$ for all $f \in \mathcal{S}$. Recall that $n + r = k + 2l$. Every $f \in C^n[0, 1]$ has the representation

$$
f(t) = \sum_{i=0}^{n-1} f^{(i)}(0) \frac{t^i}{i!} + \frac{1}{(n-1)!} \int_0^1 (t-x)_+^{n-1} f^{(n)}(x) \, dx.
$$

Define $R(f) = \int_0^1 f(x)\, dx - Q(f)$. Then for $f \in C^n[0, 1]$,

$$R(f) = \int_0^1 R_t((t-x)_+^{n-1}) f^{(n)}(x)\, dx.$$

This, of course, is the Peano representation of the remainder $R(f)$. We define

$$\frac{(-1)^n}{(n-1)!} R_t((t-x)_+^{n-1}) = M^*(x),$$

and note that $M^*(x)$ is a monospline of degree n with the l knots $\{t_i\}_{i=1}^l$. Thus for all $f \in C^n[0, 1]$,

$$(4.3) \qquad \int_0^1 f(x)\, dx = \sum_{i=1}^k c_i C_i(f) + \sum_{i=1}^l \lambda_i f(t_i) + (-1)^n \int_0^1 M^*(x) f^{(n)}(x)\, dx.$$

Since $R_t((t-\xi_i)_+^{n-1}) = 0$, we conclude that $M^*(\xi_i) = 0$, $i = 1, \cdots, r$.

Let $M(x)$ be any monospline of the form (4.1), and $f \in C^n[0, 1]$. Then integration by parts yields

$$
\begin{aligned}
(4.4) \qquad \int_0^1 f(x)\, dx &= \sum_{i=0}^{n-1} (-1)^{i+1} f^{(i)}(0) M^{(n-1-i)}(0) \\
&\quad + \sum_{i=0}^{n-1} (-1)^i f^{(i)}(1) M^{(n-1-i)}(1) \\
&\quad - \sum_{i=1}^l (n-1)! b_i f(x_i) + (-1)^n \int_0^1 M(x) f^{(n)}(x)\, dx.
\end{aligned}
$$

Thus from (4.3) and (4.4) we obtain

LEMMA 4.1. *The monospline $M^*(x)$ defined above satisfies*

$$
\begin{aligned}
(4.5) \qquad \sum_{i=1}^k c_i C_i(f) &= \sum_{i=0}^{n-1} (-1)^{i+1} f^{(i)}(0) (M^*)^{(n-1-i)}(0) \\
&\quad + \sum_{i=0}^{n-1} (-1)^i f^{(i)}(1) (M^*)^{(n-1-i)}(1),
\end{aligned}
$$

for all $f \in C^n[0, 1]$.

Since the $k \times 2n$ matrix C has rank k, we may construct a $2n \times 2n$ nonsingular matrix whose first k rows agree with C. We shall also denote this enlarged matrix by C. Define $D = (C^T)^{-1}$, and let

$$\bar{\mathbf{g}} = ((-1)^{n+r+1} g(0), (-1)^{n+r+2} g'(0), \cdots, (-1)^r g^{(n-1)}(0), g^{(n-1)}(1), \cdots, g(1))$$

and

$$\hat{\mathbf{g}} = ((-1)^{n+r} g^{(n-1)}(0), \cdots, (-1)^{n+r} g(0), (-1)^{n-1} g(1), (-1)^{n-2} g'(1), \cdots, g^{(n-1)}(1)).$$

Thus $(\bar{\mathbf{f}}, \hat{\mathbf{M}})$ (the inner product of the vectors $\bar{\mathbf{f}}$ and $\hat{\mathbf{M}}$) represents the right-hand side of (4.5), and $(C\bar{\mathbf{f}})_i = C_i(f)$, $i = 1, \cdots, k$.

LEMMA 4.2. *For $M^*(x)$ as above,*

$$(D\hat{\mathbf{M}}^*)_i = 0, \qquad\qquad i = k+1, \cdots, 2n.$$

Proof. The proof follows from Lemma 4.1 and the equation

$$(\bar{\mathbf{f}}, \hat{\mathbf{M}}^*) = (C\bar{\mathbf{f}}, D\hat{\mathbf{M}}^*) = \sum_{i=1}^{2n} (C\bar{\mathbf{f}})_i (D\hat{\mathbf{M}})_i.$$

THEOREM 4.1. *If* $n + r = k + 2l$ *and* $D = (C^T)^{-1}$, *where the matrix composed of the first k rows of C satisfies* (3.3), *then given* $\{\xi_i\}_{i=1}^r$, $0 < \xi_1 < \cdots < \xi_r < 1$, *there exists a monospline $M(x)$ of degree n with l knots such that*

(4.6)
$$M(\xi_i) = 0, \qquad\qquad i = 1, \cdots, r,$$
$$(D\hat{\mathbf{M}})_i = 0, \qquad\qquad i = k + 1, \cdots, 2n.$$

Furthermore, the knots of the monospline $M(x)$ are the nodes of the quadrature formula (3.4), *and $M(x)$ is unique if and only if the corresponding quadrature formula is unique.*

Proof. From the above analysis, every quadrature formula of the form (3.4) gives rise to a monospline $M(x)$ satisfying (4.6).

If $M(x)$ satisfies (4.6), then (4.5) holds for $f \in \mathcal{S}$. Let

$$Q^0(f) = \sum_{i=1}^k c_i C_i(f) - \sum_{i=1}^l (n-1)! b_i f(x_i)$$

and

$$R^0(f) = \int_0^1 f(x)\, dx - Q^0(f).$$

From (4.4), $R^0(f) = 0$ for $f \in \{1, t, \cdots, t^{n-1}\}$, and since $R_t^0((t-x)_+^{n-1}) = M(x)$, the theorem is proven.

The following two theorems represent a partial converse to Theorem 4.1. To prove these theorems, we demand an additional assumption on the $k \times 2n$ matrix C (see Remark 3.1).

(4.7)   Assume the $k \times 2n$ matrix C satisfies (3.3) and *all* nonzero $k \times k$ minors of C are of one sign.

THEOREM 4.2. *If D is as above, where C satisfies* (4.7), *and if $M(x)$ is a monospline of degree n with l knots for which* $(D\hat{\mathbf{M}})_i = 0$, $i = k + 1, \cdots, 2n$, *then $M(x)$ has at most $r + 1$ distinct zeros in* $(0, 1)$.

Proof. Assume $M(x)$ has $r + 2$ distinct zeros $\{\xi_i\}_{i=1}^{r+2}$ in $(0, 1)$. Then there exists a quadrature formula

$$\int_0^1 f(t)\, dt = \sum_{i=1}^k c_i C_i(f) + \sum_{i=1}^l \lambda_i f(t_i),$$

which is exact for $f \in \mathcal{S}^* = \{1, t, \cdots, t^{n-1}, (t-\xi_1)_+^{n-1}, \cdots, (t-\xi_{r+2})_+^{n-1}\}$. The $k \times 2n$ matrix C satisfies (3.3) with respect to $n, k, 2l$ and r. Since r and $r + 2$ are of the same parity and C satisfies (4.7), it follows that $\mathcal{S}^*(\mathcal{C}_k)$ is a weak Chebyshev subspace of dimension $2l + 2$, and

$$\int_0^1 f(t)\, dt = \sum_{i=1}^l \lambda_i f(t_i),$$

for all $f \in \mathcal{S}^*(\mathcal{C}_k)$. This is impossible since we may construct, by smoothing (see

Theorem 2.1), a nonnegative nontrivial $S \in \mathcal{S}^*(\mathscr{C}_k)$ which vanishes at the nodes $\{t_i\}_{i=1}^{l}$. The theorem is proven.

THEOREM 4.3. *Under the assumptions of Theorem 4.2, if the boundary conditions represented by C are separated, then M(x) has at most r distinct zeros in* (0, 1).

Proof. The proof is the same as that of Theorem 4.2, where we use (3.7) and note that for separated boundary conditions, the parity of r plays no role. Thus the addition of one more knot to $\mathcal{S}(\mathscr{C}_k)$ gives rise to a weak Chebyshev subspace of dimension $2l+1$.

Remark 4.1. The bound given in Theorem 4.2 is sharp as the following example indicates. Consider the case $n = 2$ and $r = 1$ with the knot $\xi = \frac{1}{3}$. Let $l = 1$ and $k = 1$, with the boundary condition $S(0) + S(1) = 0$ which satisfies (4.7). The quadrature formula

$$\int_0^1 f(t)\,dt = \frac{1}{6}(f(0)+f(1)) + \frac{2}{3}f\left(\frac{1}{2}\right),$$

holds for $f \in \{1, t, (t-\frac{1}{3})_+^1\}$ and $f(t) = (t-\frac{2}{3})_+^1$. The associated monospline

$$M(x) = \frac{x^2}{2} - \frac{x}{6} - \frac{2}{3}\left(x - \frac{1}{2}\right)_+^1,$$

satisfies $M(0) = M(1) = M'(0) + M'(1) = 0$, and $M(\frac{1}{3}) = M(\frac{2}{3}) = 0$.

Remark 4.2. As previously commented upon in Remark 3.4, Theorem 3.1 extends to spline functions with knots of multiplicity $\leq n - 2$. Thus Theorems 4.1–4.3 extend to the case of multiple zeros of order at most $n - 2$ for $M(x)$.

Let $\mathscr{D}_{2n-k}$ denote the set of boundary conditions $(D\hat{M})_i = 0$, $i = k+1, \cdots, 2n$, where $D = (C^T)^{-1}$, and the first k rows of C satisfy (3.3). Our present goal is to present a more workable definition of $\mathscr{D}_{2n-k}$. To this end, define

$$(4.8) \qquad G_i(f) = \sum_{j=0}^{n-1} E_{ij} f^{(j)}(0) + \sum_{j=0}^{n-1} F_{ij} f^{(j)}(1), \qquad i = 1, \cdots, 2n-k,$$

and let $G = \|G_{ij}\|_{i=1 \ j=0}^{2n-k \ 2n-1}$, where

$$(4.9) \qquad G_{ij} = \begin{cases} E_{ij}(-1)^{j+n+1}, & i = 1, \cdots, 2n-k; j = 0, 1, \cdots, n-1, \\ F_{i,2n-j-1}, & i = 1, \cdots, 2n-k; j = n, \cdots, 2n-1. \end{cases}$$

Let $\mathscr{G}_{2n-k}$ denote the set of boundary conditions $G_i(M) = 0$, $i = 1, \cdots, 2n-k$, where G satisfies (3.3) with $m = r$; i.e.,

 (i) max $\{0, n-r\} \leq 2n-k \leq 2n$,

 (ii) *there exist* $\{i_1, \cdots, i_s, j_1, \cdots, j_{2n-k-s}\} \subseteq \{0, 1, \cdots, 2n-1\}$, *satisfying* $M_{\nu-1} + r \geq \nu$, $\nu = r+1, \cdots, n$, *and*

$$(4.10) \qquad G\begin{pmatrix} 1, & \cdots, & 2n-k \\ i_1, \cdots, i_s, j_1, \cdots, j_{2n-k-s} \end{pmatrix} \neq 0,$$

 (iii) *for all* $\{i_1, \cdots, i_s, j_1, \cdots, j_{2n-k-s}\}$ *satisfying* (ii),

$$G\begin{pmatrix} 1, & \cdots, & 2n-k \\ i_1, \cdots, i_s, j_1, \cdots, j_{2n-k-s} \end{pmatrix},$$

is of one sign.

Then we have

PROPOSITION 4.1. $\mathscr{D}_{2n-k} = \mathscr{G}_{2n-k}$.

Proof.

$$(D\hat{\mathbf{M}})_i = \sum_{j=0}^{n-1} D_{ij} M^{(n-j-1)}(0)(-1)^{n+r} + \sum_{j=0}^{n-1} D_{i,j+n}(-1)^{n-j-1} M^{(j)}(1)$$

$$= \sum_{j=0}^{n-1} D_{i,n-j-1}(-1)^{r+j+1}[M^{(j)}(0)(-1)^{n+j+1}]$$

$$+ \sum_{j=0}^{n-1} D_{i,2n-j-1}(-1)^j [M^{(n-j-1)}(1)], \qquad i = k+1, \cdots, 2n.$$

Let

$$H_{ij} = \begin{cases} D_{i+k,n-j-1}(-1)^{r+j+1}, & i = 1, \cdots, 2n-k; j = 0, 1, \cdots, n-1, \\ D_{i+k,3n-j-1}(-1)^{n+j}, & i = 1, \cdots, 2n-k; j = n, \cdots, 2n-1. \end{cases}$$

Then Proposition 4.1 is valid if we can prove that $H = \|H_{ij}\|_{i=1}^{2n-k} {}_{j=0}^{2n-1}$ satisfies (4.10) if and only if C satisfies (3.3).

Let $\{i'_m\}_{m=1}^{n-s}$ denote the complementary set of indices to $\{n-i_m-1\}_{m=1}^s$ in $\{0, 1, \cdots, n-1\}$, and $\{j'_m\}_{m=1}^{k+s-n}$ denote the complementary set of indices to $\{3n-j_m-1\}_{m=1}^{2n-k-s}$ in $\{n, \cdots, 2n-1\}$.

The following two lemmas prove Proposition 4.1.

LEMMA 4.3. *With the above definitions,*

$$H\begin{pmatrix} 1, & \cdots, & 2n-k \\ i_1, \cdots, i_s, j_1, \cdots, j_{2n-k-s} \end{pmatrix} = C\begin{pmatrix} 1, & \cdots, & k \\ i'_1, \cdots, i'_{n-s}, j'_1, \cdots, j'_{k+s-n} \end{pmatrix}.$$

Proof.

$$H\begin{pmatrix} 1, & \cdots, & 2n-k \\ i_1, \cdots, i_s, j_1, \cdots, j_{2n-k-s} \end{pmatrix}$$

$$= D\begin{pmatrix} k+1, & \cdots, & 2n \\ n-i_1-1, \cdots, n-i_s-1, 3n-j_1-1, \cdots, 3n-j_{2n-k-s}-1 \end{pmatrix}(-1)^{\varepsilon_1}$$

$$= D\begin{pmatrix} k+1, & \cdots, & 2n \\ n-i_s-1, \cdots, n-i_1-1, 3n-j_{2n-k-s}-1, \cdots, 3n-j_1-1 \end{pmatrix}(-1)^{\varepsilon_1+\varepsilon_2}$$

$$= C\begin{pmatrix} 1, & \cdots, & k \\ i'_1, \cdots, i'_{n-s}, j'_1, \cdots, j'_{k+s-n} \end{pmatrix}(-1)^{\varepsilon_1+\varepsilon_2+\varepsilon_3},$$

where

$$\varepsilon_1 = (r+1)s + n(2n-k-s) + \sum_{m=1}^{s} i_m + \sum_{m=1}^{2n-k-s} j_m,$$

$$\varepsilon_2 = \frac{s(s-1)}{2} + \frac{(2n-k-s)(2n-k-s-1)}{2},$$

and

$$\varepsilon_3 = \frac{2n(2n+1)}{2} - \frac{k(k+1)}{2} + s(n-1) + (3n-1)(2n-k-s) - \sum_{m=1}^{s} i_m - \sum_{m=1}^{2n-k-s} j_m.$$

Utilizing the fact that $n + r - k = 2l$, it follows that $(-1)^{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} \equiv 1$. The lemma is proven.

LEMMA 4.4. $\{i_1, \cdots, i_s, j_1, \cdots, j_{2n-k-s}\}$ *satisfies* $M_{\nu-1} + r \geqq \nu$, $\nu = r + 1, \cdots, n$, *if and only if* $\{i'_1, \cdots, i'_{n-s}, j'_1, \cdots, j'_{k+s-n}\}$ *satisfies* $M_{\nu-1} + 2l \geqq \nu$, $\nu = 2l + 1, \cdots, n$.

Proof. Due to the symmetry of the analysis we prove only one direction. Assume $\{i_1, \cdots, i_s, j_1, \cdots, j_{2n-k-s}\}$ is such that $M_\mu + r \leqq \mu$ for some $\mu = r, \cdots, n - 1$. Note that since $M_{n-1} = 2n - k$, and $2n - k + r = n + 2l \geqq n$, $\mu < n - 1$. Let i_γ, $2n - j_\beta - 1 \leqq \mu < i_{\gamma+1}$, $2n - j_{\beta-1} - 1$. Thus $M_\mu + r = \gamma + (2n - k - s - \beta + 1) + r \leqq \mu$, i.e., $\gamma - s - \beta + 1 + n + 2l - \mu \leqq 0$. Now $n - i_\gamma - 1$, $j_\beta - n \geqq n - \mu - 1 > n - i_{\gamma+1} - 1$, $j_{\beta-1} - n$, and therefore

$$i'_{n-\mu-s+\gamma}, 2n - 1 - j'_{\beta+k+s-2n+\mu} > n - \mu - 2 \geqq i'_{n-\mu-s+\gamma-1}, 2n - 1 - j'_{\beta+k+s-2n+\mu+1},$$

and for $\{i'_1, \cdots, i'_{n-s}, j'_1, \cdots, j'_{k+s-n}\}$,

$$
\begin{aligned}
M_{n-\mu-2} + 2l &= 2l + (n - \mu - s + \gamma - 1) + (n - \beta - \mu) \\
&= (\gamma - s - \beta + 1 + n + 2l - \mu) + (n - \mu - 2) \leqq n - \mu - 2,
\end{aligned}
$$

since $\gamma - s - \beta + 1 + n + 2l - \mu \leqq 0$. The lemma is proven.

Utilizing Proposition 4.1, Theorems 4.1–4.3 may be restated in terms of the boundary conditions (4.8), where G satisfies (4.10).

Remark 4.3. Note that from the proof of Proposition 4.1, it is easily seen that the boundary forms (4.8) are separated if and only if the corresponding boundary forms (3.1) are also separated. Thus Theorem 4.1 in conjunction with Proposition 4.1 extends the results in [3].

5. An example and a further application of moment theory. In this section we discuss an interesting example of Theorem 4.1, as well as present another application of Theorem 2.1.

We begin by recalling that the nth Bernoulli polynomial, $B_n(x)$, is determined by the relations

$$
\begin{aligned}
&B_0(x) = 1, \qquad B'_n(x) = nB_{n-1}(x), && n = 1, 2, \cdots, \\
(5.1) \quad &B_n(x) = (-1)^n B_n(1 - x) \\
&B_n(0) = 0, && n = 3, 5, \cdots.
\end{aligned}
$$

The periodic extension of period one of the Bernoulli polynomial which we denote by $\bar{B}_n$ is, according to (5.1), a monospline of degree n with knots at the integers. $\bar{B}_n(x)$ is the Peano kernel for the Euler–Maclaurin quadrature formula

$$
\int_0^N f(x)\,dx = \frac{1}{2}f(0) + f(1) + \cdots + f(N-1) + \frac{1}{2}f(N)
$$

$$
(5.2) \qquad + \sum_{0 < 2\nu \leqq n} \frac{B_{2\nu}(0)}{(2\nu)!}\left(f^{(2\nu-1)}(0) - f^{(2\nu-1)}(N)\right)
$$

$$
+ \frac{(-1)^n}{n!}\int_0^N \bar{B}_n(x)f^{(n)}(x)\,dx.
$$

When n is even, $n = 2m$, we may rewrite (5.2) in the form

$$\int_0^N f(x)\, dx = \frac{1}{2}f(0) + f(1) + \cdots + f(N-1) + \frac{1}{2}f(N)$$

(5.3)
$$+ \sum_{\nu=1}^{m-1} \frac{B_{2\nu}(0)}{(2\nu)!} [f^{(2\nu-1)}(0) - f^{(2\nu-1)}(1)]$$

$$+ \frac{1}{(2m)!} \int_0^N M(x) f^{(2m)}(x)\, dx,$$

where $M(x) = \bar{B}_{2m}(x) - B_{2m}(0)$. M has a double zero at every integer. Furthermore, it satisfies the boundary conditions

$$M^{(2m-1-i)}(0) = M^{(2m-1-i)}(N) = 0, \qquad i \notin \{0, 1, 3, \cdots, 2m-3\},$$

which are adjoint to the boundary terms appearing in (5.3). Thus we see that the Euler–Maclaurin quadrature formula (5.3) is exact for all spline functions of degree $2m - 1$ with double knots at $1, 2, \cdots, N-1$. Thus it is of double precision. In the notation of Thorem 4.1, $n = 2m$, $l = N-1$, $k = 2m$ and $r = 2(N-1)$.

Similarly, the odd degree Bernoulli monospline $M(x) = \bar{B}_{2m-1}(x)$ is the Peano kernel of the (odd degree) Euler–Maclaurin quadrature formula

$$\int_0^N f(x)\, dx = \frac{1}{2}f(0) + f(1) + \cdots + f(N-1) + \frac{1}{2}f(N)$$

(5.4)
$$+ \sum_{\nu=1}^{m-1} \frac{B_{2\nu}(0)}{(2\nu)!} [f^{(2\nu-1)}(0) - f^{(2\nu-1)}(N)]$$

$$- \frac{1}{(2m-1)!} \int_0^N M(x) f^{(2m-1)}(x)\, dx.$$

In this case, M has a simple zero at each integer and half integer. Also, M satisfies

$$M^{(2m-2-i)}(0) = M^{(2m-2-i)}(N) = 0, \qquad i \notin \{0, 1, 3, \cdots, 2m-3\}.$$

Thus (5.4) is of double precision and corresponds to Theorem 4.1 with $n = 2m - 1$, $l = N-1$, $k = 2m$ and $r = 2N-1$.

The following theorem was suggested to us by A. A. Melkman who indicated a method of proof similar to that used in [6].

THEOREM 5.1. *Let data* $y_1, \cdots, y_{n+2r+1}$ *and points* $x_1 < x_2 < \cdots < x_{n+2r+1}$ *be given. Suppose that the divided differences* $[y_i, \cdots, y_{i+n}]$ *of the data over the points* $x_i, \cdots, x_{i+n}$, $i = 1, \cdots, 2r+1$, *strictly alternates in sign and* $n \geqq 2$. *Then there exists a monospline* $M(x)$ *of degree* n *with* r *knots and a nonzero constant* λ *such that*

$$M(x_i) = \lambda y_i, \qquad\qquad i = 1, \cdots, n+2r+1.$$

Furthermore, if $2r \leqq n$, *then* $M(x)$ *is unique.*

Proof. Assume without loss of generality that $x_1 = 0$ and $x_{n+2r+1} = 1$. Consider the space $\mathcal{S}_0$ of spline functions of the form

$$S(t) = \sum_{i=1}^{2r+1} c_i B(x_i, \cdots, x_{i+n}; t),$$

where $\sum_{i=1}^{2r+1} c_i[y_i, \cdots, y_{i+n}] = 0$, and $B(x_i, \cdots, x_{i+n}; t)$ is (the B-spline) defined to be the nth divided difference of $(x - t)_+^{n-1}$ at $x = x_i, \cdots, x_{i+n}$. It is well known that any subset of B-splines form a weak Chebyshev system (cf. [2]). Since the divided difference of the data strictly alternates, we conclude that $\mathcal{S}_0$ is a weak Chebyshev subspace of dimension $2r$. To prove this fact, let us set $B(x_i, \cdots, x_{i+n}; t) = u_i(t)$, $i = 1, \cdots, 2r+1$, and $[y_i, \cdots, y_{i+n}] = z_i$, $i = 1, \cdots, 2r+1$, and consider the functions $v_i(t) = u_i(t) - (z_i/z_{2r+1})u_{2r+1}(t)$, $i = 1, \cdots, 2r$. Note that

$$z_{2r+1} V\begin{pmatrix} 1, \cdots, 2r \\ t_1, \cdots, t_{2r} \end{pmatrix} = \begin{vmatrix} u_1(t_1) & \cdots & u_1(t_{2r}) & z_1 \\ \vdots & & \vdots & \vdots \\ u_{2r+1}(t_1) & \cdots & u_{2r+1}(t_{2r}) & z_{2r+1} \end{vmatrix}$$

$$= \sum_{i=1}^{2r+1} (-1)^{i+2r+1} z_i U\begin{pmatrix} 1, \cdots, i-1, i+1, \cdots, 2r+1 \\ t_1 & \cdots, & t_{2r} \end{pmatrix}.$$

Now

$$U\begin{pmatrix} 1, \cdots, i-1, i+1, \cdots, 2r+1 \\ t_1, & \cdots, & t_{2r} \end{pmatrix} \geqq 0,$$

and $z_i(-1)^i \sigma > 0$, $\sigma^2 = 1$, fixed. Thus $\{v_i(t)\}_{i=1}^{2r}$ is a weak Chebyshev system of dimension $2r$ on $(0, 1)$ which spans the set $\mathcal{S}_0$.

Since $z_i(-1)^i \sigma > 0$, $i = 1, \cdots, 2r+1$, we may always find positive numbers $c_1^0, \cdots, c_{2r+1}^0$ such that $\sum_{i=1}^{2r+1} c_i^0 z_i = \sum_{i=1}^{2r+1} c_i^0[y_i, \cdots, y_{i+n}] = 0$. Thus the function $S(t) = \sum_{i=1}^{2r+1} c_i^0 B(x_i, \cdots, x_{i+n}; t)$ is strictly positive on $(0, 1)$, and from Theorem 2.1, there exist points $0 < \xi_1 < \cdots < \xi_r < 1$, and $\mu_i > 0$, $i = 1, \cdots, r$, such that

$$(5.5) \qquad\qquad \int_0^1 f(t)\, dt = \sum_{i=1}^r \mu_i f(\xi_i),$$

for all $f \in \mathcal{S}_0$. It easily follows that there is a constant λ for which

$$\int_0^1 B(x_j, \cdots, x_{j+n}; t)\, dt = \sum_{i=1}^r \mu_i B(x_j, \cdots, x_{j+n}; \xi_i) + \lambda[y_j, \cdots, y_{j+n}],$$

for $j = 1, \cdots, 2r+1$.

Let

$$M(x) = \sum_{j=0}^{n-1} a_j x^j + \int_0^1 (x - t)_+^{n-1}\, dt - \sum_{i=1}^r \mu_i (x - \xi_i)_+^{n-1},$$

where $a_0, a_1, \cdots, a_{n-1}$ are chosen so that $M(x_j) = \lambda y_j$, $j = 1, \cdots, n$. Now

$$M(x_j, \cdots, x_{j+n}) = \int_0^1 B(x_j, \cdots, x_{j+n}; t)\, dt - \sum_{i=1}^r \mu_i B(x_j, \cdots, x_{j+n}; \xi_i)$$

$$= \lambda[y_j, \cdots, y_{j+n}], \qquad\qquad\qquad j = 1, \cdots, 2r+1.$$

Since $M(x_j) = \lambda y_j$, $j = 1, \cdots, n$, it then follows that $M(x_j) = \lambda y_j$, $j = 1, \cdots, n + 2r + 1$. Note that if $\lambda = 0$, then $M(x)$ has $n + 2r + 1$ zeros, an impossibility (Theorem 4.3). Thus $\lambda \neq 0$ and $M(x)$ is the desired monospline.

In [11], it is proven that the functions $1, t, \cdots, t^{n-1}$ are each contained in the smallest linear subspace containing the convexity cone generated by $\mathscr{S}_0$. Since uniqueness of the monospline $M(x)$ is equivalent to the uniqueness of the quadrature formula (5.5), we conclude from Corollary 2.2 that M is unique when $2r \leq n$. This completes the proof of the theorem.

Remark 5.1. In the statement of Theorem 5.1, we assumed $z_i(-1)^i \sigma > 0$, $i = 1, \cdots, 2r + 1$. This was done to insure that $\mathscr{S}_0$ is a weak Chebyshev subspace of dimension $2r$ which contains a positive function. In order that $\mathscr{S}_0$ be a weak Chebyshev subspace of dimension $2r$, it is sufficient that $z_i(-1)^i \sigma \geq 0$, $i = 1, \cdots, 2r + 1$, and at least one of the z_i is nonzero. Assuming that this is the case and if the sets $\{i : z_i > 0\}$ and $\{i : z_i < 0\}$ are both nonempty, then we may construct, as in the proof of Theorem 5.1, an element of $\mathscr{S}_0$ which is strictly positive on $(0, 1)$. If one of the above two sets is empty, but the other does not contain either 1 or $2r + 1$, and if $n \geq 3$, then we may still construct a positive function in $\mathscr{S}_0$. These conditions suffice for Theorem 5.1 to hold.

In particular, if we choose $y_i = \delta_{i,n+2r}$, $i = 1, \cdots, n + 2r + 1$, we obtain

COROLLARY 5.1. *Given any points* $s_1 < \cdots < s_{n+2r}$, *there exists a monospline* $M(x)$ *of degree* n *with* r *knots such that*

$$M(s_i) = 0, \qquad\qquad i = 1, \cdots, n + 2r.$$

This is the fundamental theorem of algebra for monosplines as it appears in [6] and [13]. This result is also a special case of Theorem 4.1 with $k = 2n$. The uniqueness as well as the converse, i.e., $M(x)$ has no more than $n + 2r$ zeros, are also results of Theorems 4.1 and 4.3.

REFERENCES

[1] H. BURCHARD, *Interpolation and approximation by generalized convex functions*, Ph.D. dissertation, Purdue Univ., West Lafayette, Ind., 1968.

[2] S. KARLIN, *Total Positivity*, Vol. I, Stanford University Press, Stanford, Calif., 1968.

[3] S. KARLIN AND C. A. MICCHELLI, *The fundamental theorem of algebra for monosplines satisfying boundary conditions*, Israel J. Math., 11 (1972), pp. 405–451.

[4] S. KARLIN AND A. PINKUS, *Gaussian quadrature formulae with multiple nodes*, Studies in Spline functions and Approximation Theory, S. Karlin, C. A. Micchelli, A. Pinkus and I. J. Schoenberg, eds., Academic Press, New York, 1976.

[5] ——, *Interpolation by Splines with Mixed Boundary Conditions*, Academic Press, New York, 1976.

[6] S. KARLIN AND L. SCHUMAKER, *The fundamental theorem of algebra for Tchebycheffian monosplines*, J. Analyse Math., 20 (1967), pp. 233–270.

[7] S. KARLIN AND W. J. STUDDEN, *Tchebycheff Systems: with Applications in Analysis and Statistics*, Interscience, New York, 1966.

[8] M. G. KREIN, *The ideas of P. L. Chebyshev and A. A. Markov in the theory of limiting values of integrals and their further developments*, Amer. Math. Soc. Transl., 12 (1951), pp. 1–122.

[9] A. A. MELKMAN, *Interpolation by splines satisfying mixed boundary conditions*, Israel J. Math., 19 (1974), pp. 369–381.

[10] C. A. MICCHELLI, *The fundamental theorem of algebra for monosplines with multiplicities*, Linear Operators and Approximation, Proc. Conf. in Oberwolfach, P. L. Butzer, J. P. Kahane and B. Sz. Nagy, eds., Birkhaüser Verlag, Basel, Switzerland, 1971, pp. 14–22.

[11] ———, *Best L^1-approximation by weak Chebyshev systems and the uniqueness of interpolating perfect splines*, IBM Res. Rep. #5388, Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1975, to appear J. Approx. Theory.

[12] A. PINKUS, *Representation theorems for Tchebycheffian polynomials with boundary conditions and their applications*, Israel J. Math., 17 (1974), pp. 11–34.

[13] I. J. SCHOENBERG, *Spline functions, convex curves and mechanical quadrature*, Bull. Amer. Math. Soc., 64 (1958), pp. 352–357.

[14] C. A. MICCHELLI AND W. L. MIRANKER, *High order search methods for finding roots*, J. Assoc. Comput. Mach., 22 (1975), pp. 51–60.