Reprinted from:
STUDIES IN SPLINE FUNCTIONS AND
APPROXIMATION THEORY

ACADEMIC PRESS, INC.
New York San Francisco London

INTERPOLATION BY SPLINES WITH MIXED BOUNDARY CONDITIONS

Samuel Karlin and Allan Pinkus

1. INTRODUCTION

A polynomial spline of degree n with knots $\{\xi_i\}_1^r$, $0 < \xi_1 < \ldots < \xi_r < 1$, has the representation

$$(1) \qquad S(x) = \sum_{i=0}^{n} c_i x^i + \sum_{i=1}^{r} d_i (x-\xi_i)_+^n ,$$

where c_i, d_i are real constants, and as usual, $x_+ = \max\{x,0\}$.

We treat in this work the following problem. When is it possible to uniquely interpolate arbitrary real data $\{y_\nu\}_{\nu=1}^{\ell}$ at the points $X = \{x_\nu ; \ \nu = 1,\ldots,\ell , \ 0 < x_\nu < 1\}$ by a spline $S(x)$, having the prescribed knots, that also satisfies the mixed boundary conditions

$$(2) \qquad \sum_{j=0}^{n} a_{ij} S^{(j)}(0) + \sum_{j=0}^{n} b_{ij} S^{(j)}(1) = e_i , \quad i = 1,2,\ldots,k ,$$

where e_i are fixed given real numbers. In other words we wish to ascertain criteria that assure the existence of unique real coefficients $\{c_i^*\}_0^n$ and $\{d_i^*\}_1^r$ such that $S^*(x) = \sum_{i=0}^{n} c_i^* x^i + \sum_{i=1}^{r} d_i^*(x-\xi_i)_+^n$ obeys the boundary conditions (2) and fulfills the interpolation requirements $S^*(x_\nu) = y_\nu$, $\nu = 1,2,\ldots,\ell$. The problem is tantamount to the analysis of an appropriate system of linear equations. To guarantee the feasibility of unique interpolation for arbitrary data, it is essential that $\ell = n+r+1-k$.

The above interpolation problem was resolved (see [3]) in considerable generality for the situation where the boundary conditions involving the endpoints 0 and 1 are separated, viz.,

$$(3)\qquad \begin{aligned} \mathcal{B}_0 &: \sum_{j=0}^{n} a_{ij}\, s^{(j)}(0) = e_i \quad , \quad i = 1,\dots,p \\ \mathcal{B}_1 &: \sum_{j=0}^{n} b_{ij}\, s^{(j)}(1) = f_i \quad , \quad i = 1,\dots,q \end{aligned}$$

(e_i and f_i are given real constants, $p+q = k$).

The solution relies on the precise total positivity character of the kernel $\phi(x,\xi) = (x-\xi)_+^n$ and refinements thereof. The results of [lot cit] are extended in this paper to handle the case of mixed boundary conditions as those displayed in (2). The formulation of the interpolation Theorem 1 below was first discovered by Melkman. His analysis depends on various elaborate ramifications of the Budan-Fourier Theorem on zeros of functions, (Melkman [7], [8]).

We lay out in this note, following the analysis in [3], a proof of Theorem 1. The corresponding interpolation problem with periodic boundary conditions considered in [4] is subsumed in the present formulation. With the aid of Theorem 1 we also describe some results on Green's functions for certain differential operators with mixed boundary conditions.

The requirements on the coefficients a_{ij} and b_{ij} occurring in the mixed boundary conditions (2) essential to ensure unique interpolation are embedded in Postulate J now stated.

<u>Postulate J</u> :

i) $k \leq \min\{2n+2, n+r+1\}$. <u>Let</u> $D = ||d_{ij}||$ <u>be the</u>

$k \times 2n+2$ matrix defined explicitly to be

$$(4)\quad d_{ij} = \begin{cases} a_{ij}(-1)^{j+n+r}, & i = 1,\dots,k\ ;\ j = 0,1,\dots,n \\ b_{i,2n+1-j}, & i = 1,\dots,k\ ;\ j = n+1,\dots,2n+1\ . \end{cases}$$

(Notice that the column indices of b_{ij} have been reversed in the construction of D .)

(ii) D is SC_k (sign consistent of order k , i.e., all $k \times k$ non-zero sub-determinants maintain the same sign) and has rank k .

Where the boundary conditions are specialized as in (3) involving linear constraints separately on the endpoints 0 and 1 with $k = p+q$, we establish in Lemma 1, Section 3 that (ii) of Postulate J is equivalent to separate sign consistency demands on the matrices A and B , viz.

(5) (ii') The $p \times n$ matrix $\tilde{A} = ||a_{ij}(-1)^j||$, $(i = 1,\dots,p\ ;\ j = 0,1,\dots,n)$ is SC_p having actual rank p and $B = ||b_{ij}||$ $(i = 1,\dots,q\ ;\ j = 0,1,\dots,n)$ is SC_q of rank q .

(Notice that now the column indices of B appear in natural order unlike the set-up in (4).) Postulate I in [3] is precisely statement (ii').

A number of other examples of boundary conditions fulfilling Postulate J will be highlighted in Section 3. We are now prepared to state the principal theorem of this paper.

Theorem 1. (cf. Melkman [7]). Let the knots $\{\xi_i\}_1^r$, $0 < \xi_1 < \dots < \xi_r < 1$, be prescribed and fixed. Given interpolation points $\{x_i\}_1^{n+r+1-k}$, $0 < x_1 < \dots < x_{n+r+1-k} < 1$, and associated $n+r+1$ real data, $\{e_i\}_1^k$ and $\{y_i\}_1^{n+r+1-k}$, then there exists a unique spline $S(x)$ of degree n with knots $\{\xi_i\}_1^r$ satisfying the boundary conditions

$$(6) \quad \sum_{j=0}^{n} a_{ij}\, S^{(j)}(0) + \sum_{j=0}^{n} b_{ij} S^{(j)}(1) = e_i \quad , \quad i = 1,\ldots,k$$

and interpolation stipulations

$$S(x_i) = y_i \quad , \quad i = 1,\ldots,n+r+1-k \ ,$$

provided Postulate J holds and then if and only if there exists at least one collection of indices $i_1 < i_2 < \ldots < i_s$, $(0 \leq i_\nu \leq n)$ and $j_1 < j_2 < \ldots < j_{k-s}$ $(n+1 \leq j_\mu \leq 2n+1)$ for some s, $(0 \leq s \leq k)$ for which

$$D\begin{pmatrix} 1,2, & \ldots\ldots\ldots\ldots & ,k \\ i_1,\ldots,i_s, & j_1,\ldots,j_{k-s} \end{pmatrix} \neq 0$$

takes place*, while the sets $\{x_\mu\}$, $\{\xi_\nu\}$, $\{i_\alpha\}$, $\{j_\beta\}$ fulfill the conditions

$$(7) \quad \begin{array}{ll} x_{\nu-s} < \xi_\nu < x_{n+1-s+\nu} \quad , & \nu = 1,\ldots,r \\ 2n+1-j_{k-s+1-\mu} < i'_{\mu+n+1+r-k} \quad , & \mu = 1,\ldots,k-r-s \end{array}$$

whenever the above inequalities are meaningful and $\{i'_\ell\}_1^{n-s+1}$ comprise the complementary ordered set of indices to $\{i_\ell\}_1^s$ among the collection $\{0,1,\ldots,n\}$.

Remark 1. The strictness impositions $x_i < x_{i+1}$, $i = 1,\ldots,n+r-k$ and $\xi_i < \xi_{i+1}$, $i = 1,\ldots,r-1$ may be much

---

*The notation $D\begin{pmatrix} \mu_1,\mu_2,\ldots,\mu_\ell \\ \lambda_1,\lambda_2,\ldots,\lambda_\ell \end{pmatrix}$ stands for the subdeterminant of the matrix D composed of row and column indices $\mu_1 < \mu_2 < \ldots < \mu_\ell$ and $\lambda_1 < \lambda_2 < \ldots < \lambda_\ell$, respectively.

relaxed (see [3, Theorem 1'] and [7]), such that effectively

a) no more than $n+1$ consecutive x's or ξ's coincide, and

b) at most $n+2$ of the x's and ξ's together assume a given value.

In Section 2 we present the proof of the theorem while in Section 3 we record a number of important examples of boundary conditions subsumed by Postulate J. The implications of Theorem 1 pertaining to the construction and properties of Green's functions associated with mixed boundary value problems of certain n-th order linear differential operators is discussed in Section 4.

2. PROOF OF THEOREM 1

For convenience we introduce the notation $u_i(x) = x^i$ and $\phi(x,\xi) = (x-\xi)_+^n$, $D^j = \frac{d}{dx^j}$. The boundary and interpolation conditions for homogeneous (zero) data lead to the following set of linear equations in the $n+r+1$ variables $\{c_i\}_0^n$ and $\{d_i\}_1^r$, viz.,

$$\sum_{i=0}^{n} c_i \Big(\sum_{j=0}^{n} a_{\mu j} D^j u_i(0) + b_{\mu j} D^j u_i(1) \Big) +$$

$$+ \sum_{i=1}^{r} d_i \Big[\sum_{j=0}^{n} b_{\mu j} D_x^j \, \phi(1,\xi_i) \Big] = 0 \quad , \quad \mu = 1,2,\ldots,k \ ,$$

(8)

$$\sum_{i=0}^{n} c_i u_i(x_\nu) + \sum_{j=1}^{r} d_j \, \phi(x_\nu,\xi_j) = 0 \ , \quad \nu = 1,2,\ldots,n+r+1-k.$$

Since we require to have the problem well-posed, necessarily $k \leq \min(2n+2,n+r+1)$. The matrix M of the system (8) can be represented as a product of two matrices $M = \tilde{D} \cdot K$, where $\tilde{D}$ is the $n+r+1 \times 3n+3+r-k$ matrix of the form

$$\tilde{D} = \left[\begin{array}{c|c|c} a_{ij}(-1)^j & 0 & b_{ij} \\ i=1,\ldots,k;\ j=0,1,\ldots,n & & i=1,\ldots,k;\ j=0,\ldots,n \\ \hline 0 & I_{n+r+1-k} & 0 \end{array}\right]$$

(I stands for the appropriate identity matrix) and K is the $3n+3+r-k \times n+r+1$ matrix of the form

$$K = \left[\begin{array}{cccccc} u_0(0) & 0 \cdots & 0 & & & \\ 0 & -Du_1(0) & \vdots & & 0 & \\ \vdots & \ddots & 0 & & & \\ 0 & \cdots\ 0 & (-1)^n D^n u_n(0) & & & \\ u_0(x_1) & & u_n(x_1) & \phi(x_1,\xi_1) & \cdots & \phi(x_1,\xi_r) \\ \vdots & & \vdots & \vdots & & \vdots \\ u_0(x_{n+r+1-k}) & \cdots & u_n(x_{n+r+1-k}) & \phi(x_{n+r+1-k},\xi_1) & \cdots & \phi(x_{n+r+1-k},\xi_r) \\ u_0(1) & \cdots & u_n(1) & \phi(1,\xi_1) & \cdots & \phi(1,\xi_r) \\ \vdots & & \vdots & \vdots & & \vdots \\ D^n u_0(1) & \cdots & D^n u_n(1) & D^n\phi(1,\xi_1) & \cdots & D^n\phi(1,\xi_r) \end{array}\right]$$

We now proceed to calculate the determinant of M .

Invoking the Cauchy-Binet formula ([1, p.1]),

$$(9)\quad \det \tilde{D}K = \sum_{1\le \ell_1<\cdots<\ell_{n+r+1}\le 3n+3+r-k} \tilde{D}\begin{pmatrix} 1,\ldots,n+r+1 \\ \ell_1,\ldots,\ell_{n+r+1}\end{pmatrix} \times$$

$$\times\ K\begin{pmatrix} \ell_1,\ldots,\ell_{n+r+1} \\ 1,\ldots,n+r+1\end{pmatrix} .$$

Observe that

$$(10)\quad \tilde{D}\begin{pmatrix} 1,\ldots,n+r+1 \\ \ell_1,\ldots,\ell_{n+r+1}\end{pmatrix} = 0$$ unless columns $n+2,\ldots,2n+2+r-k$ are all included among the indices $\{\ell_i\}_{i=1}^{n+r+1}$,

and the expansion by minors based on the last rows yields

$$(11)\quad \tilde{D}\begin{pmatrix} 1, \ldots, n+r+1 \\ \ell_1, \ldots, \ell_{n+r+1} \end{pmatrix} = (-1)^{(k-s)(n+r+1-k)} \times D^*\begin{pmatrix} 1, & \ldots & , k \\ i_1, \ldots, i_s, j_1, \ldots, j_{k-s} \end{pmatrix}$$

where D^* is the contracted $k \times 2n+2$ matrix of the form

$$(12)\quad D^* = \left[\begin{array}{c|c} a_{ij}(-1)^j & b_{ij} \\ i=1,\ldots,k; j=0,1,\ldots,n & i=1,\ldots,k; j=0,1,\ldots,n \end{array}\right]$$

To evaluate the determinants $K\begin{pmatrix} \ell_1, \ldots, \ell_{n+r+1} \\ 1, \ldots, n+r+1 \end{pmatrix}$ we use the fact that rows $n+2,\ldots,2n+2+r-k$ are included (otherwise the contribution to (9) is zero owing to the comment of (10)) and then apply the Laplace expansion by minors to obtain

$$(13)\quad K\begin{pmatrix} \ell_1, \ldots, \ell_{n+r+1} \\ 1, \ldots, n+r+1 \end{pmatrix}$$

$$= K\begin{pmatrix} i_1, i_2, \ldots, i_s, n+2, \ldots, 2n+2+r-k, j_1, \ldots, j_{k-s} \\ 1, & \ldots\ldots\ldots & , n+r+1 \end{pmatrix}$$

$$= \left(\prod_{\ell=1}^{s} (i_\ell)!\right)(-1)^{\frac{s(s-1)}{2}} K^*\begin{pmatrix} x_1, \ldots, x_{n+1+r-k}, j_1, \ldots, j_{k-s} \\ i_1', \ldots, i_{n-s+1}', \xi_1, \ldots, \xi_r \end{pmatrix}$$

where $\{i_\ell'\}_{\ell=1}^{n-s+1}$ appearing in (13) is the complementary set of indices to $\{i_\ell\}_{\ell=1}^{s}$ in $\{0,1,\ldots,n\}$, and $K^*(\vdots\vdots\vdots)$ is the Fredholm determinant based on the kernel $K(z,w)$ defined explicitly as

$$\begin{aligned} K^*(x_\nu,\ell_\mu) &= u_{\ell_\mu}(x_\nu) \\ K^*(x_\nu,\xi_m) &= \phi(x_\nu,\xi_m) \\ K^*(j_\alpha,i_m) &= u_{i_m}^{(j_\alpha)}(1) \\ K^*(j_\alpha,\xi_m) &= \phi^{(j_\alpha)}(1,\xi_m) \end{aligned} \tag{14}$$

, ($\phi^{(j)}(x,\xi)$ refers to differentiation in the variable x).

(The kernel K^* of (14) possesses a plethora of total positivity properties elaborated in [1] and [3] and features in several contexts connected to boundary value problems of ordinary differential operators of n-th order.) The above reductions lead to the formula

$$\det M = \sum_{s=0}^{k} \sum_{\substack{0\le i_1<\dots<i_s\le n \\ 0\le j_1<\dots<j_{k-s}\le n}} \left[\prod_{\ell=1}^{s} (i_\ell)!\right] \times (-1)^{s(s-1)/2+(k-s)(n+r+1-k)} \times D^*\begin{pmatrix} 1, & \dots & ,k \\ i_1,\dots,i_s,j_1,\dots,j_{k-s} \end{pmatrix} \times K^*\begin{pmatrix} x_1,\dots,x_{n+r+1-k},j_1,\dots,j_{k-s} \\ i_1',\dots,i_{n-s+1}',\xi_1,\dots,\xi_r \end{pmatrix} . \tag{15}$$

From [3, Theorem 1],

$$K^*\begin{pmatrix} x_1,\dots,x_{n+r+1-k},j_1,\dots,j_{k-s} \\ i_1',\dots,i_{n-s+1}',\xi_1,\dots,\xi_r \end{pmatrix} \ge 0 . \tag{16}$$

From the hypothesis of Postulate J we know that all the determinants of $D\begin{pmatrix} 1, & \dots & ,k \\ i_1,\dots,i_s,j_1,\dots,j_{k-s} \end{pmatrix}$ maintain a single sign. Moreover, by reversing the last $k-s$ columns, we have

$$(17)\quad D^*\begin{pmatrix}1, & \ldots & , k\\ i_1,\ldots,i_s,j_1,\ldots,j_{k-s}\end{pmatrix}$$

$$= (-1)^{s(n+r)+\frac{(k-s)(k-s-1)}{2}}$$

$$\times\ D\begin{pmatrix}1, & \ldots & , k\\ i_1,\ldots,i_s,2n+1-j_{k-s},\ldots,2n+1-j_1\end{pmatrix}$$

It is useful to substitute into (15) using (17) yielding

$$(18)\quad \det M = \sum_{s=0}^{k}\ \sum_{\substack{0\le i_1<\ldots<i_s\le n\\ 0\le j_1<\ldots<j_{k-s}\le n}}$$

$$\times\ D\begin{pmatrix}1, & \ldots & , k\\ i_1,\ldots,i_s,2n+1-j_{k-s},\ldots,2n+1-j_1\end{pmatrix}$$

$$\times\ K^*\begin{pmatrix}x_1,\ldots,x_{n+1+r-k},j_1,\ldots,j_{k-s}\\ i_1',\ldots,i_{n-s+1}',\xi_1,\ldots,\xi_r\end{pmatrix}\left[\prod_{\ell=1}^{s}(i_\ell!)\right]$$

$$\times\ (-1)^{s(n+r)+\frac{(k-s)(k-s-1)}{2}+\frac{s(s-1)}{2}+(k-s)(n+r+1-k)}.$$

Observe that

$$(19)\quad \begin{aligned}&(-1)^{s(n+r)+\frac{(k-s)(k-s-1)}{2}+\frac{s(s-1)}{2}+(k-s)(n+r+1-k)}\\ &= (-1)^{s(s-1)+\frac{k(k-1)}{2}+k(n+r+1-k)} = (-1)^{\frac{k(k-1)}{2}+k(n+r)}\end{aligned}$$

is of one sign independent of s, when k, n and r are held fixed. Thus in view of (19), the sign consistency postulate imposed on D and the fact of (16), we see that all terms in the sum (18) are of the same sign and consequently the determinant of M is non-zero provided some term is non-zero. The exact specifications on the variables and indices where the determinants of type (16) are positive are recorded in [3, Theorem 1]. It is therefore essential that D have a non-zero subdeterminant that meshes properly with these criteria. The conditions of our theorem are made to order to satisfy this requirement. The proof of Theorem 1 is complete.

3. EXAMPLES OF BOUNDARY CONDITIONS SATISFYING POSTULATE J

Example (a): Consider the case of the separated boundary conditions considered in [3], of the form

$$(20)\qquad \begin{aligned} &\sum_{j=0}^{n} a_{ij}\, S^{(j)}(0) = \alpha_i \quad , \quad i = 1,\ldots,p \\ &\sum_{j=0}^{n} b_{ij}\, S^{(j)}(1) = \beta_i \quad , \quad i = 1,\ldots,q \ , \end{aligned}$$

where

$$(21)\qquad \tilde{A} = ||a_{ij}(-1)^j||_{1\ 0}^{p\ n} \quad \text{and} \quad B = ||b_{ij}||_{1\ 0}^{q\ n}$$

are SC_p (sign consistent of order p) and SC_q , respectively, each with full rank.

Lemma 1. Assume $p+q \leq \min\{2n+2, n+r+1\}$ and $\tilde{A}$, B are SC_p, SC_q , respectively, with full rank. Then Postulate J is satisfied.

Proof. Let $k = p+q$. Comparison with (4) reveals that $D = ||d_{ij}||$ is the $k \times 2n+2$ matrix of the form

$$d_{ij} = \begin{cases} a_{ij}(-1)^{j+n+r} & , \quad i = 1,\ldots,p \ ; \quad j = 0,1,\ldots,n \\ b_{i-p,2n+1-j} & , \quad i = p+1,\ldots,q \ ; \ j = n+1,\ldots,2n+1 \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

We claim that $D\begin{pmatrix} 1, & \dots & , k \\ i_1,\dots,i_s,j_1,\dots,j_{k-s} \end{pmatrix} = 0$ unless $s = p$ (and accordingly $k-s = q$). Suppose $s > p$, then the first s columns of this specific determinant involve vectors of length $p < s$ with possibly non-zero elements, while the remaining entries in these rows are zero. Hence we have linear dependence of these columns. The analogous result holds for $p > s$ by examining the first p rows of the sub-matrix at hand. Thus only $p = s$ is to be considered. Then

$$D\begin{pmatrix} 1, & \dots & , k \\ i_1,\dots,i_p,j_1,\dots,j_q \end{pmatrix} = A^*\begin{pmatrix} 1,\dots,p \\ i_1,\dots,i_p \end{pmatrix} B^*\begin{pmatrix} 1,\dots,q \\ j_1,\dots,j_q \end{pmatrix},$$

where $A^* = ||a_{ij}^*||_{1\ 0}^{p\ n}$, $B^* = ||b_{ij}^*||_{1\ 0}^{p\ n}$, and

$$a_{ij}^* = a_{ij}(-1)^{j+n+r}, \quad i = 1,\dots,p\ ;\ j = 0,1,\dots,n$$

$$b_{ij}^* = b_{i,n-j}, \quad i = 1,\dots,q\ ;\ j = 0,1,\dots,n\ .$$

But,

$$A^*\begin{pmatrix} 1,\dots,p \\ i_1,\dots,i_p \end{pmatrix} = (-1)^{p(n+r)}\,\tilde{A}\begin{pmatrix} 1,\dots,p \\ i_1,\dots,i_p \end{pmatrix} \quad \text{and}$$

$$B^*\begin{pmatrix} 1,\dots,q \\ j_1,\dots,j_q \end{pmatrix} = (-1)^{\frac{q(q-1)}{2}}\, B\begin{pmatrix} 1, & \dots & , q \\ n-j_q,\dots,n-j_1 \end{pmatrix}.$$

By assumption $\tilde{A}$ and B are SC_p and SC_q, respectively, with full rank. The desired conclusion is now evident and the lemma is proven.

The lemma shows that in the case of separated boundary conditions, we may rid ourselves of the summation on s occurring in (15). In this case, Postulate J can be stated without involving n and r.

Example (b): For periodic boundary conditions $s^{(i)}(0) = s^{(i)}(1)$, $i = 0,1,\ldots,n$, we have $k = n+1$, and D reduces to

$$\begin{bmatrix} (-1)^{n+r} & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \\ 0 & (-1)^{n+r+1} & \cdots & 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 0 & \cdots & (-1)^{r} & -1 & \cdots & 0 & 0 \end{bmatrix}$$

This matrix is SC_{n+1} if and only if r is odd (see [2, p.76]) for all n .

Example (c): Anti-periodic boundary conditions are delimited as

$$s^{(i)}(0) = -s^{(i)}(1) \quad , \quad i = 0,1,\ldots,n \quad .$$

It is immediately checked that the corresponding D is SC_{n+1} if and only if r is even.

Example (d): Partial periodicity. Consider boundary conditions $s^{(i)}(0) = s^{(i+\ell)}(1)$, $i = 0,1,\ldots,n-\ell$, for some fixed ℓ . The associated D is the $n-\ell+1 \times 2n+2$ matrix

$$\underbrace{\begin{bmatrix} (-1)^{n+r} & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & (-1)^{r-\ell} & \underbrace{0 \quad \cdots \quad 0}_{\ell} \end{bmatrix}}_{n+1}\begin{matrix} & & -1 \\ & ⋰ & \\ \underbrace{-1 \quad\quad}_{n-\ell+1} & & \end{matrix}\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ \underbrace{0 \quad \cdots \quad 0}_{\ell} \end{bmatrix}$$

This matrix is $SC_{n-\ell+1}$ if and only if $r-\ell$ is odd.

The restrictions (7) of our main theorem reduce in the case of examples (b) and (c) to the condition

$x_{\nu-s} < \xi_\nu < x_{n+1-s+\nu}$, $\nu = 1,\ldots,r$, for some $s = 0,1,\ldots,n+1$. This is equivalent to the assumption given in [4]. In example (c), these same conditions must hold for some $s = 0,1,\ldots,n-\ell+1$.

Example (e): Let the mixed boundary conditions have the form

$$\sum_{j=0}^{\ell} a_{ij} s^{(j)}(0) + \sum_{j=0}^{\ell} b_{ij} s^{(j)}(1) = e_i \ , \quad i = 1,\ldots,p$$

$$\sum_{j=\ell+1}^{n} a_{ij} s^{(j)}(0) + \sum_{j=\ell+1}^{n} b_{ij} s^{(j)}(1) = f_i \ , \quad i = p+1,\ldots,k \ .$$

$$D = \left[\begin{array}{c|c|c|c} \begin{matrix} a_{ij}(-1)^{j+n+r} \\ i=1,\ldots,p\,; \\ j=0,1,\ldots,\ell \end{matrix} & 0 & 0 & \begin{matrix} b_{i,2n+1-j} \\ i=1,\ldots,p\,; \\ j=2n-\ell+1,\ldots,2n+1 \end{matrix} \\ \hline 0 & \begin{matrix} a_{ij}(-1)^{j+n+r} \\ i=p+1,\ldots,k\,; \\ j=\ell+1,\ldots,n \end{matrix} & \begin{matrix} b_{i,2n+1-j} \\ i=p+1,\ldots,k\,; \\ j=n+1,\ldots,2n-\ell \end{matrix} & 0 \end{array}\right]$$

Then D is SC_k and has full rank k if and only if the matrices $E = \|e_{ij}\|_{i=1}^{p}\,{}_{j=0}^{2\ell+1}$, and $F = \|f_{ij}\|_{i=p+1}^{k}\,{}_{j=1}^{2n-2\ell}$ are SC_p , SC_{k-p} and of full rank, respectively, where

$$e_{ij} = \begin{cases} (-1)^{j+n+r} a_{ij} & , \quad i = 1,\ldots,p \ ; \ j = 0,1,\ldots,\ell \\ (-1)^{k-p} b_{i,2\ell+1-j} & , \quad i = 1,\ldots,p \ ; \ j = \ell+1,\ldots,2\ell+1 \end{cases}$$

and

$$f_{ij} = \begin{cases} (-1)^{j+\ell+n+r} a_{i,j+\ell} & , \quad i = p+1,\ldots,k \ ; \ j = 1,\ldots,n-\ell \\ b_{i,2n+1-(j+\ell)} & , \quad i = p+1,\ldots,k \ ; \ j = n-\ell+1,\ldots,2n-2\ell. \end{cases}$$

Remark 2. The results of this work also extend to the case of generalized Chebycheffian splines in the usual manner, and to extended complete Chebyshev systems.

4. TOTAL POSITIVITY PROPERTIES OF GREEN'S FUNCTION FOR MIXED BOUNDARY CONDITIONS

Consider the $n+1$ st order differential operator $L_{n+1}f = D_n \cdots D_1 D_0 f$ composed from the first order differential operators

$$(D_i f)(x) = \frac{d}{dx} \frac{f(x)}{w_i(x)} \quad , \quad i = 0,1,\ldots,n \quad ,$$

acting on $f \in C^{n+1}[0,1]$, where $w_i(x) > 0$, $x \in [0,1]$, and $w_i(x) \in C^{n+1-i}[0,1]$, $i = 0,1,\ldots,n$. $\phi(x,\xi)$ is the fundamental solution of $L_{n+1}f = 0$ whose explicit representation is

$$\phi(x,\xi) = \begin{cases} 0 & x < \xi \ , \\ w_0(x)\int_\xi^x w_1(t_1)\int_\xi^{t_1} w_2(t_2)\cdots\int_\xi^{t_{n-1}} w_n(t_n)dt_n\cdots dt_1 \ , & x \geq \xi \ . \end{cases}$$

A base of $n+1$ linearly independent solutions of $L_{n+1}f = 0$ are

$$\phi(x,0) = u_0(x) \ , \ D_{i-1}\cdots D_1D_0[\phi(x,0)] = u_i(x) \ , \ i = 1,\ldots,n .$$

It is well known (e.g., see [1]) that under the above assumptions $\{u_i(x)\}_{i=0}^n$ is an ECT-system on $[0,1]$ satisfying $u_i^{(k)}(0) = 0$, $i = 1,\ldots,n$, $k = 0,1,\ldots,i-1$, and

$$D^j u_i(0) = \delta_{ij} w_i(0) \quad , \quad i,j = 0,1,\ldots,n$$

$$D^j u_i(x) = 0 \quad , \quad j > i \ , \quad i,j = 0,1,\ldots,n$$

where $D^0 = I$, $D^j = D_{j-1}\cdots D_1 D_0$, $j = 1,\ldots,n$.

A Chebycheffian spline, for which the analysis of the previous sections hold, with knots $\{\xi_i\}_1^r$ is a function of the form $S(x) = \sum_{i=0}^{n} a_i u_i(x) + \sum_{i=1}^{r} c_i \phi(x,\xi_i)$. In this setting, the mixed boundary conditions take the form (see [2])

$$\sum_{j=0}^{n} a_{ij} D^j S(0) + \sum_{j=0}^{n} b_{ij} D^j S(1) = e_i \quad , \quad i = 1,\ldots,k \quad .$$

We are concerned with the Green's function associated with the differential operator $L = L_{n+1}$ coupled with the mixed homogeneous boundary conditions

$$(22) \quad \sum_{j=0}^{n} a_{ij} D^j f(0) + \sum_{j=0}^{n} b_{ij} D^j f(1) = 0 \ , \quad i = 1,\ldots,n+1$$

for $f \in C^{n+1}[0,1]$.

We further stipulate that the only solution of $Lf = 0$ satisfying (22) is the trivial solution (see Theorem 2 below). This fact guarantees the existence of a Green's function $G(x,\xi)$ with the property that $Lf = v$, for suitable v , can be uniquely solved with f satisfying (22), the solution admitting the integral representation

$$f(x) = \int_0^1 G(x,\xi)v(\xi)d\xi \quad .$$

Sign-regularity properties of $G(x,\xi)$ were studied in [3] (supplementing Karon [5], see also Krein [6]) for the situation of separated boundary condition, while in [4] the analogous problem for periodic and anti-periodic boundary conditions was considered. In what follows, we indicate a general framework under which the previous works may be subsumed following the analysis of Theorem 1.

We introduce the sequence of vectors

$$C^{(i)} = (a_{i0}, -a_{i1}, \ldots, (-1)^n a_{in}, b_{i0}, \ldots, b_{in}) \quad , \quad i = 1, \ldots, n+1$$

$$\bar{u}^{(i)} = (u_i(0), -D^1 u_i(0), \ldots, (-1)^n D^n u_i(0), u_i(1), \ldots, D^n u_i(1)) \quad ,$$

$$i = 0, 1, \ldots, n \quad ,$$

and the vector function $\bar{\phi}(\xi) = (\phi(0,\xi), D_x^1\phi(0,\xi), \ldots, D_x^n\phi(0,\xi),$ $\phi(1,\xi), D_x^1\phi(1,\xi), \ldots, D_x^n\phi(1,\xi))$, where D_x^μ signifies that the differential operation D^μ is performed with respect to the variable x . (Note that $D_x^\mu\phi(0,\xi) = 0$, $\mu = 0,1,\ldots,n$.)

An explicit expression of the Green's function is

$$G(x,\xi) = \frac{1}{w_n(\xi)\Delta} \begin{vmatrix} (C^{(1)},\bar{u}^{(0)}) & \ldots & (C^{(1)},\bar{u}^{(n)}) & (C^{(1)},\bar{\phi}(\xi)) \\ \vdots & & \vdots & \vdots \\ (C^{(n+1)},\bar{u}^{(0)}) & \ldots & (C^{(n+1)},\bar{u}^{(n)}) & (C^{(n+1)},\bar{\phi}(\xi)) \\ u_0(x) & \ldots & u_n(x) & \phi(x,\xi) \end{vmatrix}$$

where Δ is the $n+1 \times n+1$ minor of the above determinant extracted by deleting the last row and column. The quantity $(C^{(i)},\bar{u}^{(j)})$ stands for the inner product of the indicated vectors. The existence of $G(x,\xi)$ is equivalent to $\Delta \neq 0$.

Application of Sylvester determinant identity (see [1, p. 3]) produces

$$G\begin{pmatrix} x_1, \ldots, x_r \\ \xi_1, \ldots, \xi_r \end{pmatrix} = \frac{D}{\left(\prod_{i=1}^{r} w_n(\xi_i)\right)\Delta} .$$

where

$$D = \begin{vmatrix} (C^{(1)},\bar{u}^{(0)}) & \dots & (C^{(1)},\bar{u}^{(n)}) & (C^{(1)},\bar{\phi}(\xi_i)) & \dots & (C^{(1)},\bar{\phi}(\xi_r)) \\ \vdots & & \vdots & \vdots & & \vdots \\ (C^{(n+1)},\bar{u}^{(0)}) & \dots & (C^{(n+1)},\bar{u}^{(n)}) & (C^{(n+1)},\bar{\phi}(\xi_i)) & \dots & (C^{(n+1)},\bar{\phi}(\xi_r)) \\ u_0(x_1) & \dots & u_n(x_1) & \phi(x_1,\xi_1) & \dots & \phi(x_1,\xi_r) \\ \vdots & & \vdots & \vdots & & \vdots \\ u_0(x_r) & \dots & u_n(x_r) & \phi(x_r,\xi_1) & \dots & \phi(x_r,\xi_r) \end{vmatrix}$$

Let $\tilde{C}_p = ||\tilde{c}_{ij}^{(p)}||_{i=1\ \ j=0}^{n+1\ \ 2n+1}$, where

$$\tilde{c}_{ij}^{(p)} = \begin{cases} a_{ij}(-1)^{j+n+p} , & i = 1,\dots,n+1 \ ; \ j = 0,1,\dots,n \\ b_{i,2n+1-j} , & i = 1,\dots,n+1 \ ; \ j = n+1,\dots,2n+1 , \end{cases}$$

and $p = 0$ or 1 .

Theorem 2. Assume $\tilde{C}_0$ is SC_{n+1} and of full rank. The differential operator L coupled with boundary conditions of the form (22) possesses a Green's function iff $\Delta \neq 0$ or, equivalently, iff there exist sets of indices $\{i_\mu\}_{\mu=1}^{s}$ and $\{j_\mu\}_{\mu=1}^{n+1-s}$ $(0 \leq i_1 < \dots < i_s \leq n , \ 0 \leq j_1 < \dots < j_{n+1-s} \leq n)$ such that for some $s = 0,1,\dots,n$

$$\tilde{C}_0\begin{pmatrix} 1 , & \dots & , n+1 \\ i_1,\dots,i_s , & 2n+1-j_{n+1-s},\dots,2n+1-j_1 \end{pmatrix} \neq 0 ,$$

and $j_\mu \leq i'_\mu$, $\mu = 1,\dots,n+1-s$, where $\{i'_\mu\}_{\mu=1}^{n+1-s}$ are complementary indices to $\{i_\mu\}_{\mu=1}^{s}$ in $\{0,1,\dots,n\}$.

Proof. Expanding Δ as in the proof of Theorem 1 produces

$$\Delta = \sum_{s=0}^{n+1} \sum_{\substack{0\le i_1<\dots<i_s\le n\\ 0\le j_1<\dots<j_{n+1-s}\le n}} \left[\prod_{\nu=1}^{s} w_{i_\nu}(0)\right]$$

$$\times \tilde{C}_0\begin{pmatrix} 1\,, & \dots & , n+1 \\ i_1,\dots,i_s, & 2n+1-j_{n+1-s},\dots, & 2n+1-j_1 \end{pmatrix}$$

$$\times K^*\begin{pmatrix} j_1,\dots,j_{n+1-s} \\ i'_1,\dots,i'_{n+1-s} \end{pmatrix} (-1)^{\frac{s(s-1)}{2} + \frac{(n+1-s)(n-s)}{2} + sn}$$

Since $(-1)^{\frac{s(s-1)}{2} + \frac{(n+1-s)(n-s)}{2} + sn} = (-1)^{\frac{n(n+1)}{2}}$,

$[\prod_{\nu=1}^{s} w_{i_\nu}(0)] > 0$, $\tilde{C}_0$ is SC_{n+1} , and $K^*\begin{pmatrix} j_1,\dots,j_{n+1-s} \\ i'_1,\dots,i'_{n+1-s} \end{pmatrix} \geq 0$

where strict positivity prevails iff $j_\mu \le i'_\mu$, $\mu = 1,\dots,n+1-s$, the result follows.

Theorem 3. *Where the Green's function exists* (*see Theorem* 2) *and* $\tilde{C}_p$ *is* SC_{n+1} *and of full rank, then* $G(x,\xi)$ *is* $SC_{2\ell-p}$, $\ell = 1,2\dots$, *and for each* ℓ , *there exist* $\{x_i\}_{i=1}^{2\ell-p}$, $\{\xi_i\}_{i=1}^{2\ell-p}$, $0 < x_1 < \dots < x_{2\ell-p} < 1$, $0 < \xi_1 < \dots < \xi_{2\ell-p} < 1$, *such that* $G\begin{pmatrix} x_1,\dots,x_{2\ell-p} \\ \xi_1,\dots,\xi_{2\ell-p} \end{pmatrix} \neq 0$.

Proof. Expanding D , it follows that

$$G\begin{pmatrix} x_1,\dots,x_{2\ell-p} \\ \xi_1,\dots,\xi_{2\ell-p} \end{pmatrix} = \frac{1}{[\prod_{i=1}^{2\ell-p} w_n(\xi_i)]\Delta} \sum_{s=0}^{n+1}$$

$$\times \sum_{\substack{0 \le i_1 < \ldots < i_s \le n \\ 0 \le j_1 < \ldots < j_{n+1-s} \le n}} \left[\prod_{\nu=1}^{s} w_{i_\nu}(0) \right]$$

$$\times \tilde{C}_p \begin{pmatrix} 1, & \ldots & , n+1 \\ i_1, \ldots, i_s, 2n+1-j_{n+1-s}, \ldots, 2n+1-j_1 \end{pmatrix}$$

$$\times K^* \begin{pmatrix} x_1, \ldots, x_{2\ell-p}, j_1, \ldots, j_{n+1-s} \\ i_1', \ldots, i_{n+1-s}', \xi_1, \ldots, \xi_{2\ell-p} \end{pmatrix}$$

$$\times (-1)^{\frac{s(s-1)}{2} + (2\ell-p)(n+1-s) + \frac{(n+1-s)(n-s)}{2} + sn+ps} .$$

Since $\Delta \ne 0$, $\left[\prod_{i=1}^{2\ell-p} w_n(\xi_i) \right] > 0$, $\left[\prod_{\nu=1}^{s} w_{i_\nu}(0) \right] > 0$, $\tilde{C}_p$ is SC_{n+1} , K^* is TP , and

$$(-1)^{\frac{s(s-1)}{2} + (2\ell-p)(n+1-s) + \frac{(n+1-s)(n-s)}{2} + sn+ps} =$$

$$= (-1)^{\frac{n(n+1)}{2} + p(n+1)}$$

, independent of s and ℓ , it follows that $G(x,\xi)$ is $SC_{2\ell-p}$, $\ell = 1,2,\ldots$ with a fixed predictable sign.

To establish the existence of $\{x_i\}_{i=1}^{2\ell-p}$ and $\{\xi_i\}_{i=1}^{2\ell-p}$ guaranteeing $G\begin{pmatrix} x_1, \ldots, x_{2\ell-p} \\ \xi_1, \ldots, \xi_{2\ell-p} \end{pmatrix} \ne 0$, we note that $\Delta \ne 0$ (in both the cases $p = 0$ and $p = 1$) implies the existence of an $s \in \{0,1,\ldots,n+1\}$, $\{i_\mu\}_{\mu=1}^{s}$, and $\{j_\mu\}_{\mu=1}^{n+1-s}$ such that $j_\mu \le i_\mu'$, $\mu = 1,\ldots,n+1-s$. Returning to Theorem 1, conditions (7) reveal that there then exists an s such that the second part of the condition (7) is satisfied.

Choosing $\{x_i\}_{i=1}^{2\ell-p}$, $\{\xi_i\}_{i=1}^{2\ell-p}$ to satisfy the remaining interlacing conditions, the result follows.

Remark 3. From Theorem 1, explicit conditions under which $G\begin{pmatrix} x_1,\ldots,x_{2\ell-p} \\ \xi_1,\ldots,\xi_{2\ell-p} \end{pmatrix} \neq 0$ can be readily recorded.

The sign for $G\begin{pmatrix} x_1,\ldots,x_{2\ell-p} \\ \xi_1,\ldots,\xi_{2\ell-p} \end{pmatrix}$ is calculated to be

$\delta(-1)^{\frac{n(n+1)}{2}+p(n+1)}[\varepsilon(\tilde{C}_p)]$, where $\delta = \operatorname{sgn}\Delta$, and $\varepsilon(\tilde{C}_p)$ is the sign associated with $\tilde{C}_p$. In the case $p = 0$, this reduces to $+1$.

Example (a). Separated boundary conditions. The results are explicitly stated in [3]. Both $\tilde{C}_0$ and $\tilde{C}_1$ are SC_{n+1} and of full rank.

Example (b). Periodic boundary conditions. $D^j f(0) = D^j f(1)$, $j = 0,1,\ldots,n$. $G(x,\xi)$ is $SC_{2\ell-1}$, $\ell = 1,2,\ldots$ iff $G(x,\xi)$ exists. Direct calculation shows that $\Delta = \prod_{i=0}^{n} [w_i(0)-w_i(1)]$. The assumption of $\Delta \neq 0$ is thus equivalent to $w_i(0) \neq w_i(1)$, $i = 0,1,\ldots,n$. (In [4], $w_i(x) = \exp(\int_0^x \gamma_i(t)dt)$. There the condition is consistent with ours as $\int_0^1 \gamma_i(t)dt \neq 0$, $i = 0,1,\ldots,n$.)

Example (c). Anti periodic boundary conditions. $D^j f(0) = -D^j f(1)$, $j = 0,1,\ldots,n$. $G(x,\xi)$ is $SC_{2\ell}$, and $\Delta = \prod_{i=0}^{n} [w_i(0) + w_i(1)] > 0$.

REFERENCES

1. Karlin, S. Total Positivity, Vol. I, Stanford Univ. Press, Stanford, Calif. 1968.

2. Karlin, S. Best quadrature formulas and splines. J. Approx. Theory 4: 59-90. 1971.

3. Karlin, S. Total positivity, interpolation by splines, and Green's functions of differential operators. J. Approx. Theory 4: 91-112. 1971.

4. Karlin S. and J.W. Lee. Periodic Boundary-Value Problems with cyclic totally positive Green's functions and applications to periodic spline theory. J. Diff. Eqns. 8: 374-396. 1970.

5. Karon, J. The sign-regularity properties of a class of Green's functions for ordinary differential equations. J. Diff. Eqns. 6: 484-502. 1969.

6. Krein, M. Sur les fonctions de Green nonsymetriques oscillatoires des operateurs differentiels ordinaires. C.R. Acad. Sci. URSS 15: 643-646. 1939.

7. Melkman, A.A. Interpolation by splines satisfying mixed boundary conditions. Isr. J. Math. 19: 369-381. 1974.

8. Melkman, A.A. The Budan Fourier theorem for splines. Isr. J. Math. 19: 256-263. 1974.