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Spectral Properties and Oscillation Theorems for Mixed Boundary-Value Problems of Sturm–Liouville Type

J. W. LEE

Department of Mathematics, Oregon State University, Corvallis, Oregon 97331

AND

A. PINKUS*,†

Mathematics Research Center, University of Wisconsin, Madison, Wisconsin 53705

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1. INTRODUCTION

This paper presents analogs, for certain mixed boundary-value problems, of the spectral and oscillatory properties exhibited by classical Sturm–Liouville systems. The usual analysis of the Sturm–Liouville eigenvalue problem is based on special ad hoc methods. In contrast, Gantmacher and Krein [3; see also references therein] showed that these fundamental spectral properties are direct consequences of the total positivity of the Green's function for the problem. They further showed that the total positivity of the Green's function is itself the mathematical expression of certain basic physical properties of vibrating mechanical systems, which are typically modeled by Sturm–Liouville systems. Subsequent to the work in [3], extensive studies have revealed several important classes of boundary-value problems with *separated* boundary conditions whose Green's functions are totally positive or sign regular. As in the classical Sturm–Liouville problem, these boundary-value problems exhibit a rich oscillation theory. Some principal contributors in this area are Gantmacher and Krein [3], Karlin [6–8], Karon [12], Krein [13], and Krein and Finkelstein [14].

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† Present Address: Department of Mathematics, Technion, Israel Institute of Technology, Haïfa, Israel.

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Karlin and Lee [9] studied the sign-consistency properties for *periodic* boundary-value problems. These results were later used by Lee [15, 16] to develop the spectral properties of these periodic problems. The analysis in [15, 16] extends easily to the case of antiperiodic boundary-value problems. Recently, Karlin and Pinkus [10] (see also Melkman [18]) showed that an important class of *mixed* boundary-value problems has Green's functions which are sign consistent for all even and/or odd orders. Special examples of this correspond to the above-mentioned periodic and antiperiodic boundary-value problems. In this paper, the spectral properties and oscillation theorems associated with the mixed boundary-value problems are developed.

2. Terminology and Preliminary Results

For given $w_i(x) > 0$, $x \in [0, 1]$, and $w_i \in C^{n-i}[0, 1]$, $i = 1,..., n$, define

$$L = D_n \cdots D_1, \tag{2.1}$$

where

$$(D_j u)(x) = \frac{d}{dx}\left[\frac{u(x)}{w_j(x)}\right], \qquad j = 1,..., n.$$

Thus L is a differential form of Pólya type W (disconjugate) on [0, 1]. The differential equation $Lu = 0$ has a basis of solutions

$$\begin{aligned} u_1(x) &= w_1(x), \\ u_2(x) &= w_1(x) \int_0^x w_2(t_1)\, dt_1, \\ &\ \vdots \\ u_n(x) &= w_1(x) \int_0^x w_2(t_1) \int_0^{t_1} w_3(t_2) \cdots \int_0^{t_{n-2}} w_n(t_{n-1})\, dt_{n-1} \cdots dt_1, \end{aligned} \tag{2.2}$$

which constitute an extended complete Tchebycheff (ECT) system on [0, 1]. (see [11]). Evidently,

$$D^{j-1}u_i(0) = w_i(0)\delta_{ij}, \qquad i, j = 1,..., n,$$

where $D^j = D_j \cdots D_1$, $j = 1,..., n$, and $D^0 = I$, the identity operator. The function

$$\phi_n(x; \xi) \begin{cases} 0, & x < \xi, \\ w_1(x) \int_\xi^x w_2(t_1) \int_\xi^{t_1} w_3(t_2) \cdots \int_\xi^{t_{n-2}} w_n(t_{n-1})\, dt_{n-1} \cdots dt_1, & \xi \leq x, \end{cases} \tag{2.3}$$

with $0 \leqslant x, \xi \leqslant 1$ is the fundamental solution for $Lu = 0$ determined by zero initial data at zero, and the characteristic jump discontinuity

$$D^{n-1}\phi_n(\xi+; \xi) - D^{n-1}\phi_n(\xi-; \xi) = w_n(\xi),$$

which corresponds to the requirement that the ordinary $(n-1)$st derivative of $\phi_n(x; \xi)$ exhibit a jump of $1/p_0(\xi)$ at $x = \xi$, where $L = p_0(x)(d^n/dx^n) + \cdots$.

Let A be an $m \times n$ matrix and $r \leqslant m, n$. Then

$$A\begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix}$$

denotes the determinant of the matrix obtained from A by deleting the rows and columns except for those labeled $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$, respectively. The matrix A is said to be *sign consistent of order r* (SC_r) if

$$\epsilon_r A\begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix} \geqslant 0$$

for all $1 \leqslant i_1 < \cdots < i_r \leqslant m$ and $1 \leqslant j_1 < \cdots < j_r \leqslant n$, where $\epsilon_r = +1$ or -1, dependent only upon r.

The following notation will be used. Given a matrix

$$C = \| A, B \|_{n \times 2n}$$

with both A and B $n \times n$ matrices, define

$$C^{(p)} = \| c_{ij}^{(p)} \|_{i=1, j=1}^{n, 2n} \tag{2.4}$$

by

$$c_{ij}^{(p)} = \begin{cases} a_{ij}(-1)^{j+p+n}, & i = 1, \ldots, n; \quad j = 1, \ldots, n, \\ b_{i, 2n+1-j}, & i = 1, \ldots, n; \quad j = n+1, \ldots, 2n, \end{cases}$$

where $p = 0$ or 1.

The matrix $C = \| A, B \|$ is said to satisfy *Postulate J* with respect to p if

(i) A and B are $n \times n$ matrices, and

(ii) $C^{(p)}$ has full rank and is SC_n.

Let $K(x, s)$ be a real-valued kernel defined on $J \times J$, where J is a real interval, and let

$$J_r = \{\mathbf{x}: \mathbf{x} = (x_1, \ldots, x_r), x_1 < \cdots < x_r, x_j \in J, j = 1, \ldots, r\}.$$

The function

$$K_{[r]}(\mathbf{x}, \mathbf{s}) = K\begin{pmatrix} x_1, \ldots, x_r \\ s_1, \ldots, s_r \end{pmatrix} = \det \| K(x_i, s_j) \|_{i,j=1}^{r}$$

defined on $J_r \times J_r$ is called *the compound kernel of order r* induced by K. The kernel $K(x, s)$ is said to be SC_r if $\epsilon_r K_{[r]}(\mathbf{x}, \mathbf{s}) \geq 0$ for $\mathbf{x}, \mathbf{s} \in J_r$, where $\epsilon_r = \pm 1$, dependent only on r.

Let $f_1, \ldots, f_r$ be defined on J. Then

$$(f_1 \wedge \cdots \wedge f_r)(\mathbf{x}) = \det \| f_i(x_j) \|_{i,j=1}^{r}$$

for $\mathbf{x} \in J_r$. Under suitably mild measurability and integrability assumptions, in particular when $f_1, \ldots, f_r$ are bounded and continuous on J and $K(x, s)$ is bounded and continuous on $J \times J$, the basic composition formula [7, p. 17] yields

$$K_{[r]}(f_1 \wedge \cdots \wedge f_r) = Kf_1 \wedge \cdots \wedge Kf_r, \tag{2.5}$$

where

$$(Kf)(x) = \int_J K(x, s) f(s)\, ds,$$

$$(K_{[r]}g)(\mathbf{x}) = \int_{J_r} K_{[r]}(\mathbf{x}, \mathbf{s})\, g(\mathbf{s})\, d\mathbf{s},$$

for g defined on J_r and $d\mathbf{s} = ds_1 \cdots ds_r$.

A family of real, continuous functions $\{f_1, \ldots, f_r\}$ is a Tchebycheff (T) system on J if $f_1 \wedge \cdots \wedge f_r$ maintains a fixed strict sign on J_r, or equivalently, never vanishes on J_r. Linear combinations of $f_1, \ldots, f_r$ are called f-polynomials. If f is a real continuous function on J, an isolated zero x_0 of f in the interior of J is called a *nodal zero* or *node* if f changes sign at x_0. All other zeros, including zeros at the endpoints of J, are called *nonnodal zeros*. We shall use this concept for $J = (0, 1)$, the open interval, and thus endpoints will not in fact concern us. Let $Z(f)$ denote the number of zeros of f in J where nonnodal zeros in the interior of J are counted twice and all other zeros are counted once. Let $N(f)$ denote the number of (distinct) nodes of f in J. If $\{f_1, \ldots, f_r\}$ is a T-system on J, then any f-polynomial satisfies $Z(f) \leq r - 1$.

The following version of Jentzsch's theorem will be used. We are concerned with the eigenvalue problem

$$\phi(x) = \lambda \int_J K(x, s)\, \phi(s)\, d\mu(s) \tag{2.6}$$

for the kernel $K(x, s)$ where $d\mu(s) = w(s)\, ds$ with $w(s) > 0$ and continuous on $\bar{J}$.

THEOREM A. *Assume the kernel $K(x, s)$ in (2.6) is nonnegative and continuous on $\bar{J} \times \bar{J}$ and that $K(x, s) > 0$ for all (resp., almost all) points (x, s) in some neighborhood of the diagonal $\{(x, x): x \in \bar{J}\}$. Then the kernel $K(x, s)$ has a positive eigenvalue λ_0 which is a simple root of the Fredholm determinant and which is strictly smaller in modulus than all other eigenvalues of $K(x, s)$. Furthermore,* *the corresponding eigenfunction may be chosen positive (resp., positive almost everywhere) on J.*

Remark 2.1. Jentzsch's [4] original proof may be modified in a straightforward manner to obtain Theorem A. The reasoning in [1] is, however, simpler.

Remark 2.2. Jentzsch's theorem (Theorem A) generalizes directly to kernels defined on the simplices J_r, $r = 2, 3, \ldots$.

There is a fundamental relation, Schur's theorem (Theorem B below) (see [21] or [3]), between the eigenvalue problem (2.6) for the kernel $K(x, s)$ and the corresponding eigenvalue problem

$$\Phi(\mathbf{x}) = \Lambda \int_{J_r} K_{[r]}(\mathbf{x}, \mathbf{s})\, \Phi(\mathbf{s})\, d\mu(\mathbf{s}) \tag{2.7}$$

for the rth compound kernel $K_{[r]}(\mathbf{x}, \mathbf{s})$, where $d\mu(\mathbf{s}) = d\mu(s_1) \cdots d\mu(s_r)$.

THEOREM B. *Let $K(x, s)$ be continuous on $J \times J$. Let $\lambda_0, \lambda_1, \lambda_2, \ldots$ be the set (possibly empty) of eigenvalues of $K(x, s)$ where each eigenvalue is listed according to its multiplicity as a root of $D(\lambda)$, the Fredholm determinant of $K(x, s)$. Then*

$$\lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_r}, \qquad 0 \leq i_1 < i_2 < \cdots < i_r,$$

are the totality of eigenvalues of $K_{[r]}(\mathbf{x}, \mathbf{s})$, and each such eigenvalue automatically occurs to its multiplicity as a root of $D_{[r]}(\lambda)$, the Fredholm determinant of $K_{[r]}(\mathbf{x}, \mathbf{s})$.

Remark 2.3. The assumption that $K(x, s)$ is continuous on $J \times J$ can be substantially relaxed; however, the result as stated is adequate for our purposes.

3. Sign-Consistent Green's Functions

Karlin and Pinkus [10] consider an important class of boundary-value problems with mixed boundary conditions. Examples of these boundary conditions include the separated boundary conditions commonly used in Sturm–Liouville problems as well as periodic and antiperiodic boundary conditions. In this section we summarize the sign-consistency results of [10] for certain Green's functions and present some refinements necessary for our analysis of the associated eigenvalue problem. Assume henceforth that $n > 1$.

Consider the differential operator $(L, \mathscr{B})$ specified by a differential form L of Pólya type W (see (2.1)) and a set of mixed boundary conditions

$$U_i(u) = \sum_{j=1}^{n} a_{ij} D^{j-1} u(0) + \sum_{j=1}^{n} b_{ij} D^{j-1} u(1) = 0, \qquad i = 1, 2, \ldots, n. \tag{3.1}$$

Subsequently, $\mathscr{B}$ will denote either the set of boundary conditions (3.1), or the set of functions in $C^n[0, 1]$ satisfying these conditions. It is a well-known fact (see, e.g., [2, Chap. 7] or [20, Chap. 1]) that the differential operator has a Green's function $G(x, s)$ iff

$$\Delta = \det \| U_i(u_j)\|_{i,j=1}^n \neq 0,$$

where $\{u_j\}_{j=1}^n$ are as given in (2.2). If $\Delta \neq 0$, then

$$G(x, s) = \frac{1}{\Delta} \det \left\| \begin{array}{c|c} \| U_i(u_j)\|_{i,j=1}^n & \| U_i[\phi_n(\cdot\,; s)]\|_{i=1}^n \\ \hline u_1(x),\dots, u_n(x) & \phi_n(x; s) \end{array} \right\|_{(n+1)\times(n+1)} \tag{3.2}$$

and an application of Sylvester's determinant identity (cf. [7, p. 3]) to (3.2) yields

$$G_{[r]}(\mathbf{x}, \mathbf{s}) = \frac{1}{\Delta} \det \left\| \begin{array}{c|c} \| U_i(u_j)\|_{i,j=1}^n & \| U_i[\phi_n(\cdot\,; s_j)]\|_{i=1,j=1}^{n,r} \\ \hline \| u_j(x_i)\|_{i=1,j=1}^{r,n} & \| \phi_n(x_i\,; s_j)\|_{i,j=1}^r \end{array} \right\|_{(n-r)\times(n+r)} \tag{3.3}$$

for $\mathbf{x}, \mathbf{s} \in J_r$, $J = [0, 1]$.

Let $C = \| A, B \|$ be the $n \times 2n$ matrix defined by adjoining A to B where $A = \| a_{ij} \|_{i,j=1}^n$ and $B = \| b_{ij} \|_{i,j=1}^n$ are from (3.1) and define $C^{(0)}$, $C^{(1)}$ as in Section 2. In addition, given indices $1 \leqslant j_1 < \cdots < j_s \leqslant n$ and $1 \leqslant k_1 < \cdots < k_{n-s} \leqslant n$ let M_μ denote the number of indices in the set $\{j_1, \dots, j_s, k_1, \dots, k_{n-s}\}$ which are less than or equal to μ. The next two results are reformulations of Theorems 2 and 3 in [10] and follow from the proofs therein.

Proposition 3.1. *A necessary condition for the differential operator $(L, \mathscr{B})$ to have a Green's function is that there exist an integer s, $0 \leqslant s \leqslant n$, and indices $1 \leqslant j_1 < \cdots < j_s \leqslant n$, $1 \leqslant k_1 < \cdots < k_{n-s} \leqslant n$, such that*

$$C^{(0)} \begin{pmatrix} 1, & \dots, & n \\ j_1, \dots, j_s, & 2n+1-k_{n-s}, \dots, & 2n+1-k_1 \end{pmatrix} \neq 0 \tag{3.4}$$

and $M_\mu \geqslant \mu$, $\mu = 1, \dots, n$. Furthermore, these conditions are also sufficient provided that $C^{(0)}$ is SC_n.

The main result in [10] necessary for this work is the following theorem.

Theorem 3.1. *If the Green's function $G(x, s)$ for $(L, \mathscr{B})$ exists and $C^{(p)}$ is SC_n, then $G(x, s)$ is SC_{2l-p} for $l = 1, 2, \dots$. Furthermore $G_{[2l-p]}(\mathbf{x}, \mathbf{s}) \neq 0$ iff the following holds: There exists at least one set of indices $1 \leqslant j_1 < \cdots < j_s \leqslant n$, $1 \leqslant k_1 < \cdots < k_{n-s} \leqslant n$ such that*

$$C^{(p)} \begin{pmatrix} 1, & \dots, & n \\ j_1, \dots, j_s, & 2n+1-k_{n-s}, \dots, & 2n+1-k_1 \end{pmatrix} \neq 0$$

and

(a) *if* $\min\{s, n-s\} \geqslant 2l - p$, *then*

$$M_\mu \geqslant (2l-p) + \mu, \qquad \mu = 2l - p + 1, \ldots, n,$$

while

(b) *if* $\min\{s, n-s\} < 2l - p$, *then*

$$x_{\mu-s} < s_\mu < x_{\mu+n-s}, \qquad \mu = 1, \ldots, 2l - p,$$

wherever these inequalities are meaningful.

Remark 3.1. Kalafaty [5] obtains results bearing on Theorem 3.1. It is essentially shown in [5] that if $C^{(p)}$ is SC_n then $G(x, s)$ is SC_{2l-p}, $l = 1, 2, \ldots$; however, the precise conditions determining when $G_{[2l-p]}(\mathbf{x}, \mathbf{s}) \not\equiv 0$ are not treated.

The next result is a consequence of Proposition 3.1 and Theorem 3.1.

PROPOSITION 3.2. *Assume* $(L, \mathscr{B})$ *has a Green's function* $G(x, s)$, $C^{(p)}$ *is* SC_n, *and*

$$C^{(p)}\begin{pmatrix} 1, & \ldots, & n \\ j_1, \ldots, j_s, 2n+1-k_{n-s}, & \ldots, & 2n+1-k_1 \end{pmatrix} \neq 0, \tag{3.5}$$

for some $1 \leqslant s \leqslant n-1$, *and some choice* $\{j_l\}_1^s$, $\{k_l\}_1^{n-s}$ *as above. Then* $G_{[2l-p]}(\mathbf{x}, \mathbf{x}) \neq 0$.

Proof. Since the Green's function exists, (3.4) is valid for some $0 \leqslant s \leqslant n$ and $M_\mu \geqslant \mu$ for $\mu = 1, \ldots, n$, by Proposition 3.1. If $s \neq 0$ or n, then the result follows from Theorem 3.1(a) and (b). Assume this is not the case. Thus (3.4) is valid only for $s = 0$ or $s = n$. Assume its validity for $s = 0$. Since (3.5) must hold for some $1 \leqslant s \leqslant n-1$, it is easily shown that (3.5) is maintained for $s = 1$. Appealing again to Theorem 3.1, the result follows.

Remark 3.2. Assume that the Green's function $G(x, s)$ exists and $C^{(p)}$ is SC_n. By Theorem 3.1, $\sigma_{2l-p} G_{[2l-p]}(\mathbf{x}, \mathbf{s}) \geqslant 0$, where $\sigma_{2l-p} = \pm 1$, independent of $\mathbf{x}$ and $\mathbf{s}$. In fact, from [10, Remark 3], we see that σ_{2l-p} is also independent of l. Explicitly,

$$\sigma_{2l-p} = (-1)^{np+n(n-1)/2}\, \sigma_n(C^{(p)})\, \operatorname{sgn}(\Delta),$$

where $\sigma_n(C^{(p)})$ is the constant sign of the $n \times n$ minors of $C^{(p)}$. If $C^{(0)}$ is SC_n, then this further reduces to $\sigma_{2l} = +1$. If *both* $C^{(0)}$ and $C^{(1)}$ are SC_n, then

$$\sigma_{2l-p} = (-1)^{np}\, \sigma_n(C^{(p)})\, \sigma_n(C^{(0)}).$$

4. Existence of Eigenvalues

Consider the eigenvalue problem

$$(1/w_{n+1})Lu = \lambda u, \qquad 0 \leqslant x \leqslant 1, \quad u \in \mathscr{B}, \tag{4.1}$$

where $w_{n+1}(x) \neq 0$ and is continuous on $[0, 1]$, and $\mathscr{B}$ is specified as in (3.1). Assume $(L, \mathscr{B})$ has a Green's function $G(x, s)$. Then (4.1) is equivalent to the eigenvalue problem

$$u(x) = \lambda \int_0^1 G(x, s)\, u(s)\, w_{n+1}(s)\, ds. \tag{4.2}$$

The boundary conditions $\mathscr{B}$ whose matrix is denoted by $C = \| A, B \|$, as in Section 3, may be equivalent to a set of initial conditions. In this case the Green's function always exists, is a Volterra kernel, and hence has no eigenvalues. This situation is exceptional for boundary conditions satisfying Postulate J: If the Green's function for $(L, \mathscr{B})$ exists and the boundary conditions satisfy Postulate J for $p = 0$ or 1, then $G(x, s)$ has an infinite number of eigenvalues provided $\mathscr{B}$ is not equivalent to a set of initial conditions. Thus we shall exclude below the case when $\mathscr{B}$ is equivalent to a set of initial conditions. The following easily proved result is pertinent.

Proposition 4.1. *A set of boundary conditions $\mathscr{B}$ with matrix $C = \| A, B \|$ of rank n is equivalent to a set of initial conditions iff $A = 0$ or $B = 0$.*

This result should be considered with Proposition 3.2.

The main result of this section follows.

Theorem 4.1. *Assume that the boundary conditions $\mathscr{B}$ satisfy Postulate J for some p, are not equivalent to a set of initial conditions, and that $(L, \mathscr{B})$ has a Green's function $G(x, s)$. Then the eigenvalue problem* (4.1), *equivalently* (4.2), *has an infinite number of eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$. Furthermore, if $|\lambda_0| \leqslant |\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$, where each eigenvalue is listed according to its multiplicity as a root of the Fredholm determinant of the kernel $G(x, s)$, then the following holds.*

I. *If $C^{(0)}$ is SC_n, then*

(1) $0 < |\lambda_0| \leqslant |\lambda_1| < |\lambda_2| \leqslant |\lambda_3| < \cdots < |\lambda_{2l}| \leqslant |\lambda_{2l+1}| < \cdots$;

(2) $\lambda_{2l}\lambda_{2l+1} > 0$, $l = 0, 1, 2, \ldots$;

(3) *also, λ_{2l} is not real iff $\lambda_{2l+1} = \bar{\lambda}_{2l}$.*

II. *If $C^{(1)}$ is SC_n, then*

(1) $0 < \hat{\sigma}_1\lambda_0 < |\lambda_1| \leqslant |\lambda_2| < \cdots < |\lambda_{2l-1}| \leqslant |\lambda_{2l}| < \cdots$, *where* $\hat{\sigma}_1 = \operatorname{sgn} G(x, s)\, w_{n+1}(s)$;

(2) $\lambda_{2l}\lambda_{2l-1} > 0$, $l = 1, 2, \ldots$;

(3) *also, λ_{2l} is not real iff $\lambda_{2l-1} = \bar{\lambda}_{2l}$.*

Proof. Assume $C^{(p)}$ is SC_n and $w_{n+1} > 0$. Thus $\sigma_{2l-p}G_{[2l-p]}(\mathbf{x}, \mathbf{s}) \geq 0$ for all $(\mathbf{x}, \mathbf{s}) \in J_{2l-p} \times J_{2l-p}$, $J = [0, 1]$, for some $\sigma_{2l-p} = \pm 1$ independent of $\mathbf{x}$ and $\mathbf{s}$. Since $\mathscr{B}$ is not equivalent to a set of initial conditions, the hypothesis of Proposition 3.2 is easily seen to hold and so $\sigma_{2l-p}G_{[2l-p]}(\mathbf{x}, \mathbf{x}) > 0$.

Jentzsch's theorem (Theorem A) implies that $\sigma_{2l-p}G_{[2l-p]}(\mathbf{x}, \mathbf{s})$ has a positive simple eigenvalue, strictly smaller in modulus than all other eigenvalues of $\sigma_{2l-p}G_{[2l-p]}(\mathbf{x}, \mathbf{s})$. Schur's theorem (Theorem B) then implies that $G(x, s)$ has at least $(2l - p)$ eigenvalues, $l = 1, 2, \ldots$. Thus $G(x, s)$ has infinitely many eigenvalues and again by Theorems A and B

$$0 < \sigma_{2l-p}\lambda_0 \cdots \lambda_{2l-p-1} < |\lambda_0\lambda_1 \cdots \lambda_{2l-p-2}\lambda_{2l-p}|,$$

for $l = 1, 2, \ldots$. Thus, $|\lambda_{2l-p-1}| < |\lambda_{2l-p}|$ and $\lambda_{2l-p}\lambda_{2l-p+1} > 0$, where we have used the fact that $\sigma_{2l-p}\sigma_{2l-p+2} = 1$ (see Remark 3.2). Note that for $p = 0$, $\sigma_{2l} = +1$ and thus $\lambda_0\lambda_1 > 0$. For $p = 1$, $\sigma_1\lambda_0 > 0$. Thus both (1) and (2) of Cases I and II obtain. The fact that complex eigenvalues occur in conjugate pairs follows from the fact that the Fredholm determinant of $G(x, s)$,

$$D(\lambda) = 1 + \sum_{r=1}^{\infty} \frac{(-\lambda)^r}{r!} \int_{J_r} G_{[r]}(\mathbf{x}, \mathbf{x})\, d\mu(\mathbf{x}),$$

is an entire function with real coefficients. The proof is complete. (When $w_{n+1} < 0$ replace λ by $-\lambda$ and apply the results just proved.)

The following examples illustrate the breadth of applicability of Theorem 4.1.

EXAMPLES. (a) *Periodic boundary conditions.* For the periodic boundary conditions

$$D^j u(0) = D^j u(1), \qquad j = 0, 1, \ldots, n-1,$$

$$C^{(p)} = \left\| \begin{array}{cccc|cccc} (-1)^{n+p+1} & & & & & & & (-1) \\ & (-1)^{n+p+2} & & & & & (-1) & \\ & & \ddots & & & \iddots & & \\ & & & (-1)^p & (-1) & & & \end{array} \right\|_{n\times 2n}.$$

It is easily verified that $C^{(1)}$ is SC_n and of full rank while $C^{(0)}$ is not. Periodic boundary conditions are not equivalent to initial conditions, and as shown in [9, 10], $(L, \mathscr{B})$ has a Green's function iff $\prod_{i=1}^{n}(w_i(1) - w_i(0)) \neq 0$. With this assumption, Theorem 4.1, Case II, is applicable.

(b) *Antiperiodic boundary conditions.* For the antiperiodic boundary conditions

$$D^j u(0) = -D^j u(1), \qquad j = 0, 1, \ldots, n-1,$$

$C^{(0)}$ is SC_n and of rank n while $C^{(1)}$ is not. Furthermore, it is easily shown that the assumptions of Theorem 4.1 hold (see [9, 10]) so that Case I is applicable.

(c) *Separated boundary conditions.* Let $C = \| A, B \|$ specify separated boundary conditions. Then $C^{(p)}$ has the form

$$c_{ij}^{(p)} \begin{cases} a_{ij}(-1)^{j+n+p}, & i = 1,\ldots, r, \quad j = 1,\ldots, n; \\ b_{i,2n+1-j}\,, & i = r+1,\ldots, n, \quad j = n+1,\ldots, 2n; \\ 0, & \text{otherwise}. \end{cases}$$

A simple linear-dependence argument implies that the only possible nonzero subdeterminants of $C^{(p)}$ are

$$C^{(p)} \begin{pmatrix} 1, & \ldots, & n \\ j_1,\ldots, j_r\,, 2n+1-k_{n-r}\,,\ldots, & 2n+1-k_1 \end{pmatrix}$$
$$= (-1)^{r(p+n)}\,(-1)^{(n-r)(n-r-1)/2}\,\tilde{A}\begin{pmatrix} 1,\ldots, r \\ j_1\,,\ldots, j_r \end{pmatrix} B \begin{pmatrix} r+1,\ldots, n \\ k_1\,,\ldots, k_{n-r} \end{pmatrix},$$

where $1 \leqslant j_1 < \cdots < j_r \leqslant n$, $1 \leqslant k_1 < \cdots < k_{n-r} \leqslant n$, and $\tilde{A} = \| a_{ij}(-1)^j \|_{r \times n}$, $B = \| b_{ij} \|_{(n-r)\times n}$. Clearly $C^{(p)}$ is SC_n for both $p = 0$ and $p = 1$ and of full rank iff $\tilde{A}$ is SC_r of full rank and B is SC_{n-r} of full rank. Furthermore, the separated boundary conditions are not equivalent to initial conditions iff $0 < r < n$.

Separated boundary conditions satisfying these stipulations occur frequently in mechanical oscillation problems and Sturm–Liouville problems (see [3, 7]). Assuming that $\tilde{A}$ is SC_r and B is SC_{n-r} it follows from Proposition 3.1 that $(L, \mathscr{B})$ has a Green's function iff there exist indices $1 \leqslant j_1 < \cdots < j_r \leqslant n$ and $1 \leqslant k_1 < \cdots < k_{n-r} \leqslant n$ such that

$$A\begin{pmatrix} 1,\ldots, r \\ j_1\,,\ldots, j_r \end{pmatrix} \neq 0, \qquad B\begin{pmatrix} r+1,\ldots, n \\ k_1\,,\ldots, k_{n-r} \end{pmatrix} \neq 0,$$

and $M_\mu \geqslant \mu$, $\mu = 1,\ldots, n$. If these conditions are met both Case I and Case II of Theorem 4.1 apply. Thus,

$$0 < \sigma_1\lambda_0 < \sigma_1\lambda_1 < \sigma_1\lambda_2 < \cdots,$$

where $\sigma_1 = \operatorname{sgn} G(x, s)\, w_{n+1}(s)$.

(d) Consider the eigenvalue problem

$$-e^x \left(\frac{d}{dx}\, e^{-2x} \left(\frac{d}{dx}\, e^x\right)\right) u = \lambda u,$$
$$u(0) = u(1), \qquad u'(0) = u'(1). \tag{4.3}$$

The differential equation simplifies to $-u'' + u = \lambda u$ while

$$C^{(p)} = \left\| \begin{matrix} (-1)^{p+1} & 0 & 0 & -1 \\ (-1)^p & (-1)^p & -1/e & 1 \end{matrix} \right\|.$$

$C^{(0)}$ is not SC_2 but $C^{(1)}$ is. Furthermore the Green's function exists. The eigenvalues for (4.3) are

$$1,\ 1 + 4\pi^2,\ 1 + 4\pi^2, \ldots,\ 1 + 4\pi^2 n^2,\ 1 + 4\pi^2 n^2, \ldots .$$

Thus equality can hold in $\lambda_{2l-1} \leqslant \lambda_{2l}$ for each l in Theorem 4.1, Case II. Replacing the periodic boundary conditions by antiperiodic boundary conditions provides a similar example for Case I.

(e) Consider the eigenvalue problem

$$\begin{gathered} -u'' = \lambda u, \\ -u'(0) + u(1) = 0, \qquad u'(0) + u'(1) = 0. \end{gathered} \tag{4.4}$$

Here

$$C^{(p)} = \left\| \begin{matrix} 0 & (-1)^{p+1} & 0 & 1 \\ 0 & (-1)^p & 1 & 0 \end{matrix} \right\|,$$

$C^{(0)}$ is SC_2, while $C^{(1)}$ is not. The characteristic equation determining the eigenvalues of (4.4) is

$$1 + \cos \lambda^{1/2} - \lambda^{1/2} \sin \lambda^{1/2} = 0. \tag{4.5}$$

Hence

$$\cos \frac{\lambda^{1/2}}{2} \left(\cos \frac{\lambda^{1/2}}{2} - \lambda^{1/2} \sin \frac{\lambda^{1/2}}{2} \right) = 0,$$

$$\cos \frac{\lambda^{1/2}}{2} = 0 \qquad \text{or} \qquad \lambda^{1/2} \tan \frac{\lambda^{1/2}}{2} = 1.$$

The first equation yields eigenvalues

$$\nu_n = (2n + 1)^2 \pi^2, \qquad n = 0, 1, 2, \ldots,$$

with eigenfunctions,

$$v_n(x) = -(2n + 1)\pi \cos(2n + 1)\pi x + \sin(2n + 1)\pi x. \tag{4.6}$$

The second equation, $\lambda^{1/2} \tan(\lambda^{1/2}/2) = 1$, has only real zeros because it is the characteristic equation for the self-adjoint eigenvalue problem on $[0, \frac{1}{2}]$

$$\begin{gathered} -\varphi'' = \lambda \varphi, \\ \varphi'(0) = 0, \\ \varphi'(\tfrac{1}{2}) + \varphi(\tfrac{1}{2}) = 0. \end{gathered}$$

The zeros $\mu_0, \mu_1, \mu_2, \ldots$ of $\lambda^{1/2}\tan(\lambda^{1/2}/2) = 1$ satisfy

$$\begin{aligned} 2n\pi < (\mu_n)^{1/2} < (2n+1)\pi, \qquad & n = 0, 1, 2, \ldots, \\ 0 < (\mu_n)^{1/2} - 2n\pi \to 0 \qquad & \text{as} \quad n \uparrow \infty, \end{aligned}$$

and corresponding eigenfunctions are

$$u_n(x) = (\mu_n)^{1/2}\cos(\mu_n)^{1/2}x + \sin(\mu_n)^{1/2}x. \tag{4.7}$$

The totality of eigenvalues of (4.4) is

$$0 < \mu_0 < \nu_0 < \mu_1 < \nu_1 < \cdots < \mu_n < \nu_n < \cdots,$$

which shows that equality need never hold for Theorem 4.1, Case I, even when only $C^{(0)}$ is SC_n.

Under the hypotheses of Theorem 4.1 stronger results are obtained when *both* $C^{(0)}$ and $C^{(1)}$ are SC_n, as is clear from Example (c) (separated boundary conditions). We now prove the rather surprising result that if both $C^{(0)}$ and $C^{(1)}$ are SC_n of rank n, then the boundary conditions are, in fact, equivalent to separated boundary conditions.

PROPOSITION 4.2. *If the matrix C is of rank n and both $C^{(0)}$ and $C^{(1)}$ are SC_n, then the boundary conditions $\mathscr{B}$ are equivalent to separated boundary conditions.*

Remark 4.1. By the term "equivalent to separated boundary conditions" we mean that the boundary conditions may be rewritten as separated boundary conditions. For example, the boundary conditions $u(0) = 0$, $u(0) + u(1) = 0$ can be rewritten as $u(0) = 0$, $u(1) = 0$.

We prove Proposition 4.2 via two lemmas.

LEMMA 4.1. *If the conditions of Proposition 4.2 hold, then there is exactly one s, $0 \leqslant s \leqslant n$, such that*

$$C\begin{pmatrix} 1, & \ldots, & n \\ j_1, \ldots, j_s, & k_1, \ldots, & k_{n-s} \end{pmatrix} \neq 0 \tag{4.8}$$

for some choices of indices $\{j_i\}_{i=1}^{s}$ and $\{k_i\}_{i=1}^{n-s}$ satisfying $1 \leqslant j_1 < \cdots < j_s \leqslant n < k_1 < \cdots < k_{n-s} \leqslant 2n$.

Proof. Since C has rank n there exist at least one s, $0 \leqslant s \leqslant n$, and corresponding indices for which (4.8) holds.

Assume that there exist an $s' > s$ and indices $\{j_i'\}_{i=1}^{s'}$, $\{k_i'\}_{i=1}^{n-s'}$ ordered as above such that

$$C\begin{pmatrix} 1, & \ldots, & n \\ j_1', \ldots & j_{s'}', & k_1', \ldots, k_{n-s'}' \end{pmatrix} \neq 0. \tag{4.9}$$

Let C_l denote the lth column vector of the matrix C, $l = 1,\ldots, 2n$. Then (4.8) and (4.9) imply

$$\{C_{j_m'}\}_{m=1}^{s'} \in \operatorname{span}\{C_{j_1}, \ldots, C_{j_s}, C_{k_1}, \ldots, C_{k_{n-s}}\}.$$

Since $s' > s$, and $\{C_{j_m'}\}_{m=1}^{s'}$ is a linearly independent set, there exists an $m_0 \in \{1,\ldots, s'\}$ such that

$$C_{j'_{m_0}} \notin \operatorname{span}\{C_{j_1}, \ldots, C_{j_s}\}$$

and, hence, the matrix with column vectors

$$\{C_{j_i}\}_{i=1}^{s}, \qquad C_{j'_{m_0}}, \qquad \{C_{k_i}\}_{i=1, i\neq i_0}^{n-s}$$

must be nonsingular for some $i_0 \in \{1,\ldots, n-s\}$. Ordering and renumbering, it follows that there exist indices

$$1 \leqslant \hat{j}_1 < \cdots < \hat{j}_{s+1} \leqslant n < \hat{k}_1 < \cdots < \hat{k}_{n-s-1} \leqslant 2n$$

such that

$$C\begin{pmatrix} 1, & \ldots, & n \\ \hat{j}_1, \ldots, \hat{j}_{s+1}, & \hat{k}_1, \ldots, & \hat{k}_{n-s-1} \end{pmatrix} \neq 0. \tag{4.10}$$

Then for $l_i = 3n + 1 - k_{n-s+1-i}$, $i = 1,\ldots, n-s$, and $\hat{l}_i = 3n + 1 - \hat{k}_{n-s-i}$, $i = 1,\ldots, n-s-1$, we have

$$C^{(1)}\begin{pmatrix} 1, & \ldots, & n \\ j_1, \ldots, j_s, & l_1, \ldots, & l_{n-s} \end{pmatrix} = (-1)^s\, C^{(0)}\begin{pmatrix} 1, & \ldots, & n \\ j_1, \ldots, j_s, & l_1, \ldots, & l_{n-s} \end{pmatrix},$$

$$C^{(1)}\begin{pmatrix} 1, & \ldots, & n \\ \hat{j}_1, \ldots, \hat{j}_{s+1}, & \hat{l}_1, \ldots, & \hat{l}_{n-s-1} \end{pmatrix} = (-1)^{s+1}\, C^{(0)}\begin{pmatrix} 1, & \ldots, & n \\ \hat{j}_1, \ldots, \hat{j}_{s+1}, & \hat{l}_1, \ldots, & \hat{l}_{n-s-1} \end{pmatrix},$$

with all determinants nonzero by (4.8) and (4.10). This shows that $C^{(0)}$ and $C^{(1)}$ cannot both be SC_n and proves the lemma.

LEMMA 4.2. *If* (4.8) *holds for exactly one* s, $0 \leqslant s \leqslant n$, *then the boundary conditions* $\mathscr{B}$ *determined by* $C = \|A, B\|$ *are equivalent to separated boundary conditions.*

Proof. Let $\{j_i\}_{i=1}^{s}$ and $\{k_i\}_{i=1}^{n-s}$ be ordered indices such that (4.8) holds. Since the interchange of rows and the addition of linear combinations of one row to another in C in no way affect the boundary conditions $\mathscr{B}$, we may assume that the matrix which gives rise to the determinant in (4.8) is the identity, i.e., $C_{j_i} = e_i$ $(i = 1,\ldots, s)$ and $C_{k_i} = e_{s+i}$ $(i = 1,\ldots, n-s)$, where e_i is the standard ith coordinate basis vector in n-space. Let $j \notin \{j_1, ..., j_s\}$, $1 \leqslant j \leqslant n$. Then

$$C_j = \sum_{i=1}^{s} \alpha_i C_{j_i} + \sum_{i=1}^{n-s} \beta_i C_{k_i},$$

and if $\beta_i \neq 0$ for some i, the uniqueness of s is contradicted. Thus, $\beta_i = 0$, $i = 1, ..., n - s$,

$$C_j = \sum_{i=1}^{s} \alpha_i C_{j_i} = \sum_{i=1}^{s} \alpha_i e_i,$$

and hence $c_{lj} = 0$ for $l = s + 1, ..., n$. Similarly if $k \notin \{k_1, ..., k_{n-s}\}$, $n + 1 \leqslant k \leqslant 2n$, the first s components of C_k are zero. Hence the boundary conditions $\mathscr{B}$ are equivalent to boundary conditions specified by a matrix of the form

$$\left\| \begin{array}{c|c} A_1 & 0 \\ \hline 0 & B_1 \end{array} \right\|$$

with A_1 $s \times n$ of rank s and B_1 $(n - s) \times n$ of rank $n - s$; i.e., $\mathscr{B}$ is equivalent to separated boundary conditions.

Remark 4.2. Assume $C = \| A, B \|$ is such that $C^{(p)}$ is SC_n. Let $\tilde{C} = \| -A, B \|$ and $\tilde{\tilde{C}} = \| A, -B \|$. Then $\tilde{C}^{(q)}$ and $\tilde{\tilde{C}}^{(q)}$ are SC_n where $p \neq q$, $p, q \in \{0, 1\}$. Furthermore, if $C = \| I, B \|$, then $C^{(p)}$ is SC_n iff $(-1)^p B$ is totally positive, i.e., SC_k for all $k = 1, ..., n$ with $\epsilon_k = 1$, $k = 1, ..., n$. If $C = \| A, I \|$, set $\hat{A} = \| \hat{a}_{ij} \|$, $\hat{a}_{ij} = a_{ij}(-1)^{i+j}$. Then $C^{(p)}$ is SC_n iff $(-1)^p \hat{A}$ is totally positive.

5. Oscillation Properties of Eigenfunctions

Throughout this section, we shall assume that the hypotheses of Theorem 4.1 hold for the eigenvalue problem (4.1). Let

$$\lambda_0, \lambda_1, \lambda_2, ...$$

be the eigenvalues of (4.1) enumerated as in Theorem 4.1, and let

$$u_0, u_1, u_2, ... \tag{5.1}$$

be a corresponding sequence of independent eigenfunctions and/or generalized eigenfunctions. Suppose λ_l is a simple eigenvalue and u_l is a corresponding eigenfunction. If λ_l is real, choose u_l real. If λ_l is nonreal, then $\bar{\lambda}_l$ is also a simple eigenvalue, and $\bar{u}_l$ is chosen for its eigenfunction. If λ_l is not simple, it must be real with multiplicity 2, say $\lambda_l = \lambda_{l+1}$. Either u_l and u_{l+1} are both eigenfunctions (chosen real) for $\lambda_l = \lambda_{l+1}$ or, if a generalized eigenfunction occurs, u_l and u_{l+1} may be chosen real satisfying

$$\begin{aligned} \lambda_l \int_0^1 G(x, s)\, u_l(s)\, w_{n+1}(s)\, ds &= u_l(x), \\ \lambda_l \int_0^1 G(x, s)\, u_{l+1}(s)\, w_{n+1}(s)\, ds &= u_l(x) + u_{l+1}(x) \end{aligned} \tag{5.2}$$

(see [15, p. 597, Eq. (1.9)]). Throughout this section the sequence (5.1) is assumed to satisfy the preceding conditions.

THEOREM 5.1. *Let the hypotheses of Theorem* 4.1 *hold. Then there is a* (*complex*) *constant* $\hat{\sigma}_{2l-p}$ *such that*

$$\hat{\sigma}_{2l-p}(u_0 \wedge u_1 \wedge \cdots \wedge u_{2l-p-1}) > 0$$

on J^o_{2l-p}, $J^o = (0, 1)$. *Furthermore, if* λ_{2j-p} *is nonreal, let* $v_{2j-p} = \operatorname{Re}(u_{2j-p})$ *and* $v_{2j-p+1} = \operatorname{Im}(u_{2j-p})$, *while if* λ_k *is real, let* $v_k = u_k$. *Then*

$$\{v_0, v_1, \ldots, v_{2l-p-1}\}$$

is a T-*system on* (0, 1) *satisfying the boundary conditions* $\mathscr{B}$.

Proof. By Schur's theorem, the eigenvalue of $G_{[2l-p]}(\mathbf{x}, \mathbf{s})$ of minimum modulus is $\lambda_0\lambda_1 \cdots \lambda_{2l-p-1}$. By Jentzsch's theorem this eigenvalue is real, simple, of sign σ_{2l-p}, and has a real eigenfunction which does not vanish on J^o_{2l-p}. Since

$$\lambda_0\lambda_1 \cdots \lambda_{2l-p-1} G_{[2l-p]}(u_0 \wedge \cdots \wedge u_{2l-p-1}) = u_0 \wedge \cdots \wedge u_{2l-p-1},$$

where $G_{[2l-p]}$ denotes the integral operator with kernel $G_{[2l-p]}(\mathbf{x}, \mathbf{s})$, and $u_0 \wedge \cdots \wedge u_{2l-p-1} \not\equiv 0$ because $u_0, \ldots, u_{2l-p-1}$ are linearly independent, it follows that

$$\hat{\sigma}_{2l-p}(u_0 \wedge u_1 \wedge \cdots \wedge u_{2l-p-1}) > 0 \tag{5.3}$$

on J^o_{2l-p} for some constant $\hat{\sigma}_{2l-p}$. A short calculation using (5.3) yields

$$\tilde{\sigma}_{2l-p}(v_0 \wedge v_1 \wedge \cdots \wedge v_{2l-p-1}) > 0,$$

where $\tilde{\sigma}_{2l-p} = (-2i)^r \hat{\sigma}_{2l-p}$ and r is the number of pairs of complex conjugate eigenvalues in $\{\lambda_0, \ldots, \lambda_{2l-p-1}\}$. Consequently $\{v_0, v_1, \ldots, v_{2l-p-1}\}$ is a T-system and the theorem is proved.

The real sequence $v_0, v_1, v_2, \ldots$ (see Theorem 5.1) of real and imaginary parts of the eigenfunctions $u_0, u_1, u_2, \ldots$ belonging to the Green's function $G(x, s)$ exhibits oscillation properties analogous to those of the classical Sturm–Liouville eigenvalue problem. These properties emanate from the fact that

$\{v_j\}_{j=0}^{2l-p-1}$ is a T-system on (0, 1) for $l = 1, 2, \ldots$, and from certain orthogonality properties of $v_0, v_1, \ldots$.

Let $G^*(x, s) = G(s, x)$ be the Green's function for the adjoint problem to (4.1). Then $G^*(x, s)$ has precisely the same eigenvalues (to multiplicity) as $G(x, s)$, and the generalized eigenfunctions of $G^*(x, s)$, say $\{u_l^*\}_0^\infty$, which we choose according to the same conventions set forth at the beginning of this section for $\{u_l\}_0^\infty$, must satisfy

$$\int_0^1 u_l(s)\, \overline{u_m^*(s)}\, w_{n+1}(s)\, ds = 0 \qquad \text{if} \quad \lambda_l \neq \bar{\lambda}_m . \tag{5.4}$$

Let $\{v_j^*\}_0^\infty$ be obtained from $\{u_j^*\}_0^\infty$ just as $\{v_j\}_0^\infty$ was obtained from $\{u_j\}_0^\infty$. Since $G^*(x, s)$ clearly has the same sign-consistency properties as $G(x, s)$, the argument of Theorem 5.1 establishes that $\{v_0^*, v_1^*, \ldots, v_{2l-p-1}^*\}$, $l = 1, 2, \ldots$, is a T-system on (0, 1) satisfying the boundary conditions adjoint to $\mathscr{B}$. Also, (5.4) and the fact that nonreal eigenvalues occur in conjugate pairs yield the orthogonality relation:

If $|\lambda_l| \neq |\lambda_m|$, then

$$\int_0^1 v_l(s)\, v_m^*(s)\, w_{n+1}(s)\, ds = 0, \tag{5.5}$$

where $v_l(s)$ [resp., $v_m^*(s)$] is the real or the imaginary part of an eigenfunction of $G(x, s)$ [resp., $G^*(x, s)$] belonging to λ_l [resp., λ_m].

THEOREM 5.2. *Assume that the hypotheses of Theorem* 4.1 *hold. Then the following obtain.*

Case I. If $C^{(0)}$ *is* SC_n, *then*

(a) *for each* $0 \leqslant k \leqslant 2l - 1$ *the zeros of*

$$v = \sum_{j=k}^{2l-1} a_j v_j \qquad \left(\sum a_j^2 > 0,\ a_j \text{ real}\right)$$

satisfy

$$2[k/2] \leqslant N(v) \leqslant Z(v) \leqslant 2l - 1;$$

consequently, v_{2l-2} *and* v_{2l-1} *have either* $(2l - 2)$ *or* $(2l - 1)$ *nodes in* (0, 1), *and no other zeros, and*

(b) *the nodes of* v_{2l-2} *and* v_{2l-1} *strictly interlace.*

Case II. *If* $C^{(1)}$ *is* SC_n, *then*

(a) *for each* $0 \leqslant k \leqslant 2l$ *the zeros of*

$$v = \sum_{j=k}^{2l} a_j v_j \qquad \left(\sum a_j^2 > 0,\ a_j \text{ real}\right)$$

satisfy

$$2[(k-1)/2]+1 \leqslant N(v) \leqslant Z(v) \leqslant 2l;$$

consequently v_{2l-1} *and* v_{2l} *have either* $(2l-1)$ *or* $2l$ *nodes in* $(0, 1)$ *and no other zeros. Also* v_0 *has no zeros in* $(0, 1)$, *and*

(b) *the nodes of* v_{2l-1} *and* v_{2l} *strictly interlace.*

Remark 5.1. In case $G(x, s)$ has real spectrum, in particular if the eigenvalue problem is self-adjoint, $v_j = u_j$ for $j = 0, 1, 2,...$ and Theorem 5.2 describes the oscillation properties of the eigenfunctions of $G(x, s)$. See also Theorem 5.3 below.

Proof. We shall prove only Case I. The proof of Case II follows in an entirely analogous manner. Recall that N and Z count zeros in $(0, 1)$.

The inequality $Z(v) \leqslant 2l-1$ holds because $\{v_0, v_1,..., v_{2l-1}\}$ is a T-system (Theorem 5.1) on $(0, 1)$. The inequality for $N(v)$ follows from the orthogonality relation (5.5): Suppose that $v = \sum_{j=k}^{2l-1} a_j v_j$ ($\sum a_j^2 > 0$, a_j real) has nodes

$$0 < \xi_1 < \cdots < \xi_r < 1, \qquad r = N(v).$$

1. If r is *odd*, $r = 2m-1$ say, form a nontrivial v^*-polynomial

$$v^*(x) = \sum_{j=0}^{2m-1} b_j v_j^*(x)$$

which has nodal zeros (sign changes) at $\xi_1,..., \xi_{2m-1}$. Since $\{v_0^*,..., v_{2m-1}^*\}$ is a T-system, v^* has no other zeros in $(0, 1)$. By construction

$$\int_0^1 v(s)\, v^*(s)\, w_{n+1}(s)\, ds \neq 0.$$

However, if $k > r = 2m-1$, then since $|\lambda_{2m-1}| < |\lambda_{2m}|$ (Theorem 4.1) the orthogonality conditions (5.5) imply that the above integral is zero, a contradiction. Thus, $k \leqslant r = N(v)$.

2. If r is *even*, $r = 2m$ say, form the v^*-polynomial

$$v^{*(n)}(x) = \sum_{j=0}^{2m+1} b_j^{(n)} v_j^*(x)$$

which has nodal zeros at $\xi_0^{(n)}, \xi_1,..., \xi_{2m}$ $(0 < \xi_0^{(n)} < \xi_1)$ and satisfies $\sum_{j=0}^{2m+1} |b_j^{(n)}| = 1$. $v^{*(n)}$ has no additional zeros in $(0, 1)$ by the Tchebycheff property of $\{v_0^*,..., v_{2m+1}^*\}$. Let $\xi_0^{(n)} \downarrow 0$ as $n \uparrow \infty$. A simple compactness argument implies the existence of a v^*-polynomial

$$v^*(x) = \sum_{j=0}^{2m+1} b_j v_j^*(x), \qquad \sum_{j=0}^{2m+1} |b_j| = 1,$$

which has nodal zeros at $\xi_1, \ldots, \xi_{2m}$. Since any sign change of v^* in $(0, 1)$ is the limit of zeros of the $v^{*(n)}$'s as $n \to \infty$ through an appropriate subsequence, and $Z(v^*) \leqslant 2m$, it follows that v^* has only the zeros $\xi_1, \ldots, \xi_{2m}$ in $(0, 1)$. As above, a contradiction follows if $2m + 1 = r + 1 < k$. Thus, $k - 1 \leqslant N(v)$.

Thus if $N(v)$ is odd, $N(v) \geqslant k$, while if $N(v)$ is even, $N(v) \geqslant k - 1$. This is equivalent to $N(v) \geqslant 2[k/2]$. The remaining part of Case I(a) follows from the first part and the fact that $Z(v) - N(v)$ is always a nonnegative *even* integer.

The proof of Case I(b) is set forth as a sequence of lemmas. The arguments follow those used by Gantmacher and Krein [3, pp. 215–217] with suitable modifications required by our weaker hypotheses. As an immediate consequence of Case I(a), we have, since $Z(v) - N(v)$ is always even.

LEMMA 5.1. *The zeros of the v-polynomial*

$$v = av_{2l-2} + bv_{2l-1} \qquad (a^2 + b^2 > 0)$$

satisfy

$$2l - 2 \leqslant N(v) \leqslant Z(v) \leqslant 2l - 1.$$

Hence, v has only nodal zeros in $(0, 1)$.

In what follows, let ϕ denote either v_{2l-2} or v_{2l-1} while ψ denotes v_{2l-1} or v_{2l-2}.

LEMMA 5.2. *Let* $0 < \xi_1 < \cdots < \xi_r < 1$, $r = N(\phi)$, *be the nodes of* ϕ *in* $(0, 1)$. *Then the function*

$$h = \psi/\phi$$

is strictly monotone on $I_i = (\xi_i, \xi_{i+1})$, $i = 0, 1, \ldots, r$; $\xi_0 = 0$, $\xi_{r+1} = 1$.

Proof. From Lemma 5.1, h cannot be constant on any interval of positive length. Thus, if h is not strictly monotone on I_i, h has a relative extremum at some point x_i of I_i. However, this would imply that the v-polynomial $\psi(x) - h(x_i)\phi(x)$ has a nonnodal zero at x_i, contradicting Lemma 5.1. Thus h is strictly monotone.

LEMMA 5.3. *$h(x)$ has a zero in each* I_i, $i = 1, \ldots, r - 1$.

Proof. Since $h(x)$ is monotone in each I_i, $i = 0, 1, \ldots, r$, the limits

$$\lim_{x \to \xi_i^-} h(x) = l_i^- \qquad \text{and} \qquad \lim_{x \to \xi_i^-} h(x) = l_i^+$$

both exist as extended real numbers for $i = 1, \ldots, r$. We shall show that none of the $\{l_i^+\}_{i=1}^r$ and $\{l_i^-\}_{i=1}^r$ is finite.

Neither l_i^- nor l_i^+ is finite when ξ_i is not a zero of ψ. We are concerned with one of the following four cases which may occur only if ξ_i is a zero of ψ.

(i) Exactly one of the l_i^- and l_i^+ is finite.

(ii) Both l_i^- and l_i^+ are finite and unequal.

(iii) $l_i^- = l_i^+$ (finite) and h is monotone near ξ_i.

(iv) $l_i^- = l_i^+$ (finite) and h is monotone in opposite senses for $x < \xi_i$ and $x > \xi_i$ but near ξ_i.

We show that (i)–(iv) are incompatible with Lemma 5.1. If (i) or (ii) holds choose c in the open interval determined by l_i^- and l_i^+, while if (iii) holds set $c = l_i^- = l_i^+$. Then the v-polynomial $v = \psi - c\phi$ has a nonnodal zero at ξ_i, as is easily seen from the fact that $\phi(\xi_i) = \psi(\xi_i) = 0$ and ϕ changes sign at ξ_i.

To contradict (iv), assume $h \leq c = l_i^+ = l_i^-$ in I_{i-1} and I_i. Consider the polynomial $v_\epsilon = \psi - (c - \epsilon)\phi$, $\epsilon > 0$. The polynomial $v = \psi - c\phi$ has at least $(2l - 2)$ nodes in $(0, 1)$ one of which is at ξ_i. Since $v_\epsilon = v + \epsilon\phi$, for $\epsilon > 0$ sufficiently small, v_ϵ must maintain at least $(2l - 3)$ nodes in $(0, 1)$ bounded away from ξ_i. However, by construction it is easily seen that for $\epsilon > 0$, sufficiently small, v_ϵ has two "new" nodes (one less than ξ_i and one greater than ξ_i) as well as the node at ξ_i. Thus for $\epsilon > 0$, sufficently small, v_ϵ has at least $2l$ nodes, a contradiction.

LEMMA 5.4. *The nodes of* v_{2l-2} *and* v_{2l-1} *strictly interlace in* $(0, 1)$.

Proof. If $N(v_{2l-1}) \neq N(v_{2l-2})$ then since $|N(v_{2l-2}) - N(v_{2l-1})| = 1$, the result is an immediate consequence of Lemma 5.3. Assume $N(v_{2l-2}) = N(v_{2l-1})$ and $l > 1$. If v_{2l-2} and v_{2l-1} have a common node ξ, then if η is an adjacent node of v_{2l-1} (such a node exists because $l > 1$), v_{2l-2} must have a node ζ between ξ and η by Lemma 5.3. But v_{2l-1} must then have a node between ζ and ξ, contradicting the definition of η. Thus the lemma holds for $l > 1$. Assume $l = 1$ so that $N(v_0), N(v_1) \leq 1$. If v_0 and v_1 have a common node ξ, then $h = v_0/v_1$ has equal, infinite (see the proof of Lemma 5.3) left and right limits at ξ. Thus there exists a constant c such that $v_0 - cv_1$ has three nodes, a contradiction.

This completes the proof of Theorem 5.2.

EXAMPLES. (a) Consider the eigenvalue problem

$$-u'' = \lambda u,$$
$$u(0) + u(1) = 0,$$
$$u'(0) + u'(1) = 0.$$

These boundary conditions are antiperiodic and the problem has eigenvalues $\lambda_n = (2n+1)^2\pi^2$, $n = 0, 1, 2,\ldots$, each with multiplicity 2 and corresponding eigenfunctions $\cos(2n+1)\pi x$ and $\sin(2n+1)\pi x$. The interlacing properties guaranteed by Theorem 5.2, Case I, are easily verified in this case.

(b) Consider the eigenvalue problem

$$\begin{aligned} -u'' + u &= \lambda u, \\ u(0) - u(1) &= 0, \\ u'(0) - \tfrac{1}{2}u'(1) &= 0. \end{aligned}$$

Then (see Example (d), Sect. 4)

$$C^{(p)} = \begin{Vmatrix} (-1)^{p+1} & 0 & 0 & -1 \\ (-1)^p & (-1)^p & -1/2e & \tfrac{1}{2} \end{Vmatrix},$$

and so $C^{(1)}$ is SC_2 but $C^{(0)}$ is not. The eigenvalues of this problem are $\lambda_n = 1 + 4\pi^2[(n+1)/2]^2$, $n = 0, 1, 2,\ldots$. The eigenvalue $\lambda_0 = 1$ has an eigenfunction $u_0(x) = 1$ and the double eigenvalue $\lambda_{2n} = \lambda_{2n-1}$, $n \geqslant 1$, has an eigenfunction and a generalized eigenfunction given, respectively, by

$$\begin{aligned} u_{2n-1}(x) &= \cos 2\pi n x, \\ u_{2n}(x) &= (x+1)\sin 2\pi n x. \end{aligned}$$

The interlacing properties of Theorem 5.2, Case II, are easily confirmed. This example also shows that generalized eigenfunctions can occur in Theorem 5.2.

(c) In eigenvalue problem (4.4), $C^{(0)}$ is SC_2 while $C^{(1)}$ is not. The eigenfunctions given in (4.6) and (4.7), when properly ordered, must have interlacing zeros as in Theorem 5.2, Case I. In this case, Theorem 5.2 seems to be the easiest way to verify the interlacing of the nodes.

For completeness, we include the following consequence of Theorems 4.1, 5.1, and 5.2 which pertains to the case of separated boundary conditions (Proposition 4.2) (see [6, 12, 13]).

THEOREM 5.3. *Assume that the hypotheses of Theorem* 4.1 *hold for* both $p = 0$ *and* $p = 1$. *Then the eigenvalues of* $G(x, s)$ *are all real and simple, and the eigenfunctions* $\{u_l\}_0^\infty$ *satisfy*:

(1) $\{u_0, u_1, \ldots, u_l\}$ *is a* T-*system on* $(0, 1)$ *satisfying the boundary conditions* $\mathscr{B}$ *for each* $l = 0, 1, 2,\ldots$.

(2) *For* $0 \leqslant k \leqslant l$, *the zeros of*

$$u = \sum_{j=k}^{l} a_j u_j \qquad \left(\sum a_j^2 > 0,\ a_j \text{ real}\right)$$

satisfy

$$k \leqslant N(u) \leqslant Z(u) \leqslant l.$$

(3) *The eigenfunction u_l has exactly l nodes in* $(0, 1)$ *and no other zeros there, and the nodes of u_l and u_{l-1} strictly interlace in* $(0, 1)$.

6. Composite Differential Operators

Many important differential operators are naturally expressed as compositions of lower-order operators. The applicability of the results of Sections 3–5 to such a composite operator may be inferred by an examination of its factors, as is shown in Theorem 6.1. Theorem 6.2 refines the results of Theorem 6.1 in the important special case where the differential form is self-adjoint.

Let $(L, \mathscr{B})$ be an nth-order differential form of Pólya type W with boundary conditions specified by the matrix $C = \| A, B \|$, as in the previous sections. Moreover, we shall also assume that $w_j \in C^{n+m+1-j}[0, 1]$, $j = 1,..., n$.

Likewise, let $(L^+, \mathscr{B}^+)$ be an mth-order differential form where

$$L^+ = D_m^+ \cdots D_1^+, \qquad (D_j^+ u)(x) = \frac{d}{dx}\left[\frac{u(x)}{w_j^+(x)}\right], \qquad j = 1,..., m,$$

and $w_j^+(x) > 0$, $x \in [0, 1]$, $w_j^+ \in C^{m+1-j}[0, 1]$, with associated boundary conditions determined by the matrix

$$C^+ = \| A^+, B^+ \|_{m \times 2m}$$

of rank m.

Let $N = L^+L$, and $\mathscr{B}^+\mathscr{B}$ be the set of boundary conditions

$$\sum_{j=1}^{n} a_{ij} D^{j-1}u(0) + \sum_{j=1}^{n} b_{ij} D^{j-1}u(1) = 0, \qquad i = 1,..., n,$$

$$\sum_{j=1}^{m} a_{i-n,j}^+ D^{n+j-1}u(0) + \sum_{j=1}^{m} b_{i-n,j}^+ D^{n+j-1}u(1) = 0, \qquad i = n + 1,..., n + m,$$

specified by the matrix

$$E = \left\| \begin{matrix} A & 0 & B & 0 \\ 0 & A^+ & 0 & B^+ \end{matrix} \right\|_{(n+m)\times 2(n+m)},$$

where $D_{n+j} = D_j^+$, $j = 1,..., m$, and $D^j = D_j \cdots D_1$, $j = 1,..., n + m$.

THEOREM 6.1. *Assume $C^{(p)}$ and $C^{+(p)}$ satisfy Postulate J for the same p and that either $\mathscr{B}$ or $\mathscr{B}^+$ is not equivalent to a set of initial conditions. If $(N, \mathscr{B}^+\mathscr{B})$ has a Green's function, then the conclusion of Theorems* 4.1, 5.1, *and* 5.2 *obtain for the eigenvalue problem*

$$(1/w)\, Nu = \lambda u, \qquad u \in \mathscr{B}^+\mathscr{B},$$

where $w(x)$ is continuous and nonvanishing on $[0, 1]$. *Also,* $(N, \mathscr{B}^+\mathscr{B})$ *has a Green's function* $H(x, s)$ *iff both* $(L, \mathscr{B})$ *and* $(L^+, \mathscr{B}^+)$ *have Green's functions, say* $G(x, s)$ *and* $G^+(x, s)$, *respectively, in which case*

$$H(x, s) = \int_0^1 G(x, t)\, G^+(t, s)\, dt. \tag{6.1}$$

Proof. The proof is a result of the application of the basic composition formula [7, p. 17]. From (6.1),

$$H_{[2l-p]}(\mathbf{x}, \mathbf{s}) = \int_{J_{2l-p}} G_{[2l-p]}(\mathbf{x}, \mathbf{t})\, G^+_{[2l-p]}(\mathbf{t}, \mathbf{s})\, d\mathbf{t}.$$

Since $G(x, s)$ and $G^+(x, s)$ are SC_{2l-p}, $l = 1, 2, \ldots$, so is $H(x, s)$. The fact that $H_{[2l-p]}(\mathbf{x}, \mathbf{x}) \neq 0$ follows in a similar fashion.

Remark 6.1. It should be noted that one can also prove directly that under the above assumptions, $E^{(p)}$ is SC_{n+m} and of rank $n + m$.

An important case of Theorem 6.1 is the case when $L^+ = L^*$ and $\mathscr{B}^+ = \mathscr{B}^*$, the adjoint differential form and boundary conditions to L and $\mathscr{B}$, respectively. Let $C = \| A, B \|$ denote the boundary conditions associated with $(L, \mathscr{B})$, and let $C_* = \| A_*, B_* \|$ denote the adjoint boundary conditions associated with $(L^*, \mathscr{B}^*)$. While the analysis is rather lengthy, an explicit form of C_* may be exhibited and, as is shown in [17] (see also [19]), if $C^{(p)}$ is SC_n and of rank n, then $C_*^{(p)}$ is SC_n and of rank n. From Theorem 6.1 we obtain the following.

THEOREM 6.2. *Assume* $(L, \mathscr{B})$ *has a Green's function* $G(x, s)$, *and* $C^{(p)}$ *satisfies Postulate J. Then the self-adjoint differential operator* $(L^*L, \mathscr{B}^*\mathscr{B})$ *has a Green's function* $H(x, s)$ *which is* SC_{2l-p} *for* $l = 1, 2, \ldots$. *Thus the conclusions of Theorem* 4.1, 5.1, *and* 5.2 *apply and since* $H(x, s)$ *is symmetric and positive definite, the spectrum of the associated eigenvalue problem is positive.*

7. EXTENSIONS

The spectral results for the eigenvalue problem (4.1) developed in Sections 4 and 5 can be cast in a more general setting as described below. No proofs will be given because the reasoning used in Sections 4 and 5 can be applied with inessential changes.

Let $J = (0, 1)$ and $K(x, s)$ be a real, continuous kernel on $\bar{J} \times \bar{J}$ for which

$$\begin{aligned} \epsilon_{2l-p} K_{[2l-p]}(\mathbf{x}, \mathbf{s}) &\geq 0 \quad \text{for} \quad \mathbf{x}, \mathbf{s} \in J_{2l-p}, \\ \epsilon_{2l-p} K_{[2l-p]}(\mathbf{x}, \mathbf{x}) &> 0 \quad \text{for} \quad \mathbf{x} \in J_{2l-p}, \end{aligned} \tag{7.1}$$

where $p = 0$ or 1, $\epsilon_{2l-p} = +1$ or -1 dependent only on $2l - p$, and $l = 1, 2, \ldots$. Consider the eigenvalue problem

$$\phi(x) = \lambda \int_J K(x, s)\, \phi(s)\, d\mu(s) \tag{7.2}$$

for the kernel $K(x, s)$, where $d\mu(s) = w(s)\, ds$ and w is a positive, continuous weight function on J.

THEOREM 7.1. *Assume* (7.1) *holds for* $p = 0$ *or* 1. *Then the eigenvalue problem* (7.2) *has an infinite sequence of eigenvalues* $\lambda_0, \lambda_1, \lambda_2, \ldots$. *Moreover, if* $|\lambda_0| \leqslant |\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$, *where each eigenvalue is listed according to its multiplicity as a root of the Fredholm determinant of* $K(x, s)$, *then the following holds.*

I. *If* $p = 0$, *then*

(1) $0 < |\lambda_0| \leqslant |\lambda_1| < |\lambda_2| \leqslant |\lambda_3| < \cdots < |\lambda_{2l}| \leqslant |\lambda_{2l+1}| < \cdots$;

(2) $\epsilon_{2l}\epsilon_{2l+2}\lambda_{2l}\lambda_{2l+1} > 0$ *for* $l = 0, 1, 2, \ldots$ *where* $\epsilon_0 = -1$;

(3) *also,* λ_{2l} *is nonreal iff* $\lambda_{2l+1} = \bar{\lambda}_{2l}$.

II. *If* $p = 1$, *then*

(1) $0 < \epsilon_1\lambda_0 < |\lambda_1| \leqslant |\lambda_2| < \cdots < |\lambda_{2l-1}| \leqslant |\lambda_{2l}| < \cdots$;

(2) $\epsilon_{2l-1}\epsilon_{2l+1}\lambda_{2l-1}\lambda_{2l} > 0$, $l = 1, 2, \ldots$;

(3) *also,* λ_{2l} *is nonreal iff* $\lambda_{2l-1} = \bar{\lambda}_{2l}$.

Parts (2) and (3) of Cases I and II yield the following interesting result, guaranteeing that the spectrum of the kernel $K(x, s)$ is real.

COROLLARY 7.1. *If in Theorem* 7.1, *Case* I, $\epsilon_{2l}\epsilon_{2l+2} < 0$ *for* $l = 0, 1, 2, \ldots$ (*i.e., if successive compounds in* (7.1) *alternate in sign*), *then* $K(x, s)$ *has only real spectrum. Likewise if* $\epsilon_{2l-1}\epsilon_{2l+1} < 0$ *for* $l = 1, 2, \ldots$ *in Theorem* 7.1, *Case* II, *then* $K(x, s)$ *has only real spectrum.*

On the basis of Theorem 7.1 and the reasoning of Section 5, we can establish the analogs of Theorems 5.1 and 5.2 for the kernel $K(x, s)$. Let $\{u_i\}_0^\infty$ be the set of eigenfunctions of $K(x, s)$ corresponding to $\{\lambda_i\}_0^\infty$ and chosen as in Section 5. Let $\{v_i\}_0^\infty$ be the real sequence constructed from the real and imaginary parts of the eigenfunction $\{u_i\}_0^\infty$ as in Theorem 5.1. Then

$$\{v_0, v_1, \ldots, v_{2l-p-1}\}$$

is a T-system on $(0, 1)$ for $l = 1, 2, \ldots$, and when $p = 0$ (resp., $p = 1$) the conclusions of Theorem 5.2, Case I (resp., Case II), hold for the sequence $\{v_i\}_0^\infty$.

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