TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 322, Number 1, November 1990

L^1-APPROXIMATION WITH CONSTRAINTS

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ABSTRACT. In this paper we study problems of best L^1-approximation to continuous functions from finite-dimensional subspaces under a variety of constraints. Included are problems of bounded coefficient approximation, approximation with interpolation, restricted range approximation, and restricted range and derivative approximation. Emphasis is placed on problems of uniqueness.

1. INTRODUCTION

The literature of approximation theory abounds with numerous results on best approximation under a variety of constraints. These results are mainly concerned with such problems in the L^2 or uniform (L^∞) norm. See, for example, the review articles by Chalmers [1] and Chalmers and Taylor [2], and the numerous references therein. In most of these problems, existence is easily established, and thus the questions considered are those of characterization, uniqueness, and algorithms for calculating best approximations.

Recently much progress has been made on the question of uniqueness of the best approximation in the L^1-norm, from finite-dimensional subspaces, where both the subspace and the approximated functions are continuous. Similar progress has also been made on the analogous one-sided approximation problem. In this paper we consider problems of characterization and uniqueness for best L^1-approximants to continuous functions, where the approximating sets are specified convex subsets of finite-dimensional subspaces. These convex subsets may or may not depend on the function being approximated. In §3 we consider the problem of bounded coefficient approximation, while in §4 we deal with best approximation under interpolatory constraints. In §5 and §6 we discuss the problem of best restricted range approximation. §5 contains the more theoretical aspects of this problem, while in §6 we consider various examples. In §7 we show how these results can be generalized to restricted range and derivative approximation.

2. BACKGROUND

Let us first fix some notation. K will denote a compact subset of $\mathbb{R}^d$ satisfying $K = \overline{\operatorname{int} K}$. We let μ be any nonatomic positive finite measure on K

Received by the editors November 23, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 41A29; Secondary 41A52, 41A50.

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0002-9947/90 $1.00 + $.25 per page

with the property that every real-valued $f \in C(K)$ (continuous functions on K) is μ-measurable, and such that if

$$\|f\|_1 := \int_K |f(x)|\, d\mu(x) = 0$$

for $f \in C(K)$, then $f = 0$, i.e., $\|\cdot\|_1$ is truly a norm on $C(K)$. For notational ease we denote the set of such measures by $\mathscr{A}$. We let $C_1(K, \mu)$ denote the linear space $C(K)$ equipped with norm $\|\cdot\|_1$. $C_1(K, \mu)$ is not a Banach space, i.e., it is not complete. However, it does have the important property that its dual space is $L^\infty(K, \mu)$, i.e.,

$$(C_1(K, \mu))^* = L^\infty(K, \mu).$$

U will always denote a finite-dimensional subspace of $C(K)$, and M a closed convex subset of U. To every $f \in C(K)$ there always exists a best $L^1(K, \mu)$-approximant from M. We first state two well-known criteria characterizing best approximants from M in $C_1(K, \mu)$. We recall the standard notation $Z(f) = \{x: f(x) = 0\}$, and $N(f) = K \backslash Z(f)$.

Theorem 2.1. *Let $f \in C(K)$. Then the following are equivalent.*

(i) *u^* is a best $L^1(K, \mu)$-approximant to f from M.*

(ii) *For every $u \in M$*

$$\int_K [\operatorname{sgn}(f - u^*)](u - u^*)\, d\mu \le \int_{Z(f-u^*)} |u - u^*|\, d\mu.$$

(iii) *There exists an $h \in L^\infty(K, \mu)$ satisfying*

(1) $|h(x)| = 1$, *all* $x \in K$,

(2) $\int_K hu^*\, d\mu \ge \int_K hu\, d\mu$, *all* $u \in M$,

(3) $\int_K h(f - u^*)\, d\mu = \|f - u^*\|_1$.

The only nonstandard statement in the above theorem is (1) of (iii). It is only here that one uses the fact that our measure is nonatomic. With this fact, the result follows by using Liapounoff's Theorem. See Phelps [10]. The characterization (ii) is the usual one given. However, we will need the stronger statement (iii) in much of what follows (cf. §3).

The question of when M is a *unicity set* for $C_1(K, \mu)$, i.e., when to each $f \in C(K)$ there exists a unique best $L^1(K, \mu)$-approximant from M, is a different and more difficult question. If $M = U$, then there are two known results characterizing unicity spaces. These are:

Theorem 2.2 (Cheney and Wulbert [3]). *U is a unicity space for $C_1(K, \mu)$ if and only if there does not exist an $h \in L^\infty(K, \mu)$ and a $u^* \in U$, $u^* \ne 0$, for which*

(1) $|h(x)| = 1$, *all* $x \in K$,

(2) $\int_K hu\, d\mu = 0$, *all* $u \in U$,

(3) $h|u^*| \in C(K)$.

Theorem 2.3 (Strauss [23]). *U is a unicity space for $C_1(K, \mu)$ if and only if the zero function is not a best $L^1(K, \mu)$-approximant to any $g \in U^*$, $g \neq 0$, where*

$$U^* = \{g\colon g \in C(K),\, |g| = |u| \text{ for some } u \in U\}.$$

The only if statement of both these related theorems follows from the fact that αu^* is a best $L^1(K, \mu)$-approximant to $h|u^*| \in U^*$ for all $\alpha \in [-1, 1]$, if h is as in Theorem 2.2.

If $M = \mathscr{U}(f) := \{u\colon u \in U,\, u \le f\}$, then it has been proved (see Strauss [24]) that to each $f \in C(K)$ there exists a unique best $L^1(K, \mu)$-approximant to f from $\mathscr{U}(f)$ (assuming $\mathscr{U}(f) \neq \varnothing$) if and only if for every $u \in U$, $u \neq 0$, the zero function is not a best $L^1(K, \mu)$-approximant to $|u|$ from $\mathscr{U}(|u|)$. Note the similarity to Theorem 2.3 in that the uniqueness question for all $f \in C(K)$ reduces to the uniqueness question on a set of "test" functions. There is also a different characterization if U contains a strictly positive function, in terms of quadrature formulae for U [12].

For arbitrary M as above, we have the recent result of Shi [18].

Theorem 2.4. *Let M be a closed convex subset of U. Then M is a unicity set for $C_1(K, \mu)$ if and only if there does not exist a $g \in C(K)$ and u_1, $u_2 \in M$, $u_1 \neq u_2$, such that*

(a) $(g - u_1)(g - u_2) = 0$,

(b) $(u_1 + u_2)/2$ *is a best $L^1(K, \mu)$-approximant to g from M.*

If $M = U$, then Theorem 2.3 is an immediate consequence of this theorem and is obtained by translating g by $(u_1 + u_2)/2$.

It has been noted that the various necessary and sufficient conditions delineated for U or M to be a unicity set are μ dependent. That is, M may be a unicity set for $C_1(K, \mu)$ for some measure but not a unicity set for other measures. As such, it is natural to ask for necessary and sufficient conditions on M implying that it is a unicity set for $C_1(K, \mu)$ for *all* "nice" measures μ. This problem in the case $M = U$ has been considered by Kroó [7], Sommer [19], and Pinkus [11]. The analogous problem for one-sided $L^1(K, \mu)$-approximation was considered by Pinkus and Totik [15] and Pinkus and Strauss [13].

To explain the result obtained in the case $M = U$, let us restrict our set of measures μ to

$$\mathscr{D} = \{d\mu = w(x)\,dx\colon w \in C(K),\, w > 0 \text{ on } K\}.$$

For each $u \in U$, $u \neq 0$, the (relatively) open set $K\backslash Z(u)$ $(= N(u))$ is the union of a possibly infinite, but necessarily countable number of open disjoint connected subsets of K, i.e., $K\backslash Z(u) = \bigcup_{i=1}^{m} A_i$, where the A_i are open, disjoint, and connected. For convenience we also introduce the following notation. $[K\backslash Z(u)]$ will denote the number of open connected disjoint components of $K\backslash Z(u)$, and for each $u \in U$,

$$U(u) = \{v\colon v \in U,\, v = 0 \text{ a.e. on } Z(u)\},$$

where the a.e. (almost everywhere) is with respect to Lebesgue measure.

We say that U satisfies *Property* A if to each $u \in U$, $u \neq 0$, with $K \backslash Z(u) = \bigcup_{i=1}^{m} A_i$, as above, and to each choice of $\varepsilon_i \in \{-1, 1\}$, $i = 1, \ldots, m$, there exists a $v \in U(u)$, $v \neq 0$, satisfying $\varepsilon_i v \geq 0$ on A_i, $i = 1, \ldots, m$. We can also state Property A in two other equivalent forms. Namely, U satisfies Property A if to each $g \in U^*$, $g \neq 0$, there exists a $u \in U$, $u \neq 0$, satisfying $u = 0$ a.e. on $Z(g)$, and $ug \geq 0$. Alternatively, U satisfies Property A if to each $u \in U$, $u \neq 0$, and $h \in L^\infty(K)$ with

(a) $|h(x)| = 1$, all $x \in K$,

(b) $h|u| \in C(K)$,

there exists a $v \in U$, $v \neq 0$, satisfying $hv \geq 0$.

The following result was proven with restrictions by Kroó [7] and Pinkus [11], and in this form by Kroó [8]. Schmidt [17] later proved a somewhat more general result.

Theorem 2.5. *U is a unicity space for $C_1(K, \mu)$ for all $\mu \in \mathscr{D}$ if and only if U satisfies Property* A.

This result raises the natural question of which subspaces satisfy Property A. We know of two necessary conditions implied by Property A. To explain one of these conditions, we say that U *decomposes* if there exist nontrivial subspaces V and W of U such that $U = V \oplus W$, i.e., $U = V + W$ and $V \cap W = \{0\}$, such that $(K \backslash Z(v)) \cap (K \backslash Z(w)) = \varnothing$ for all $v \in V$ and $w \in W$. In other words, there exist disjoint subsets B and C of K such that every function in V vanishes identically off B, while every function in W vanishes identically off C. With this definition, we can now state the following result.

Theorem 2.6 (Pinkus and Wajnryb [16]). *If U satisfies Property* A, *then*

(1) $[K \backslash Z(u)] \leq \dim U(u)$, *for all* $u \in U$.

(2) *If* $Z(U) = \bigcap_{u \in U} Z(u)$, *and* $[K \backslash Z(U)] \geq 2$, *then U decomposes.*

If $K \subset \mathbb{R}$, then based on these results a full characterization of those U satisfying Property A may be given. From (2) of Theorem 2.6, it suffices to state this result for $K = [a, b]$.

Theorem 2.7 (Pinkus [11], [12]). *Let U be a finite-dimensional subspace of $C[a, b]$. Then the following are equivalent.*

(1) *U satisfies Property* A.

(2) $[[a, b] \backslash Z(u)] \leq \dim U(u)$ *for all* $u \in U$.

(3) (a) *If $[c, d]$, $a < c < d < b$, is a zero interval of $u \in U$, then*

(i) *there exists a $v \in U$ for which*

$$v(x) = \begin{cases} u(x), & a \leq x \leq c, \\ 0, & c < x \leq b, \end{cases}$$

(ii) *there exists a $w \in U$ for which*

$$w(x) = \begin{cases} 0, & a \leq x < d, \\ u(x), & d \leq x \leq b. \end{cases}$$

(b) *For* $a \le c < d \le b$, *let*

$$V_{c,d} = \{u: u \in U, u = 0 \text{ on } [a, c) \cup (d, b]\}.$$

Then each $V_{c,d}$ *is a WT-system (if* $V_{c,d} \neq \{0\}$*). Furthermore, if* $v \in V_{c,d}$ *has no zero interval in* $[c, d]$, *then* v *has at most* $\dim V_{c,d} - 1$ *distinct zeros in* (c, d).

For $K \subset \mathbb{R}^d$, $d \ge 2$, a full characterization is not yet known. In the case of one-sided $L^1(K, \mu)$-approximation, the answers to the analogous questions are somewhat different. Since we will need these results in §5, we review them here.

Let K and μ be as above. For each $f \in C(K)$, set

$$\mathscr{U}(f) = \{u: u \in U, u \le f\}.$$

If $\mathscr{U}(f) \neq \emptyset$, then a best $L^1(K, \mu)$-approximant to f from $\mathscr{U}(f)$ necessarily exists. We will say that U is a unicity set for $C_{1+}(K, \mu)$ if to each $f \in C(K)$ with $\mathscr{U}(f) \neq \emptyset$, there exists a unique best $L^1(K, \mu)$-approximant to f from $\mathscr{U}(f)$.

Theorem 2.8 (Pinkus and Totik [15]). *U is a unicity set for $C_{1+}(K, \mu)$ for all $\mu \in \mathscr{D}$ if and only if there exists a basis $u_1, \ldots, u_n$ for U such that*

(a) $u_i \ge 0$, $i = 1, \ldots, n$,
(b) $N(u_i) \cap N(u_j) = \emptyset$, *all* $i \neq j$.

This result says that the unicity set property holds in this one-sided case for all $\mu \in \mathscr{D}$ only if U has a very simple, trivial form. It is trivial in the sense that the approximation problem for U satisfying (b) reduces to n independent one-dimensional approximation problems.

The story, however, does not end here. If we restrict ourselves to continuously differentiable functions, then much more can be said if we add some additional minor assumptions. Before stating these assumptions, let us note that $f \in C^1(K)$, for K compact, if f has an extension to some open neighborhood of K, where f and its first partial derivatives are all continuous.

Assumption I. We assume that $U \subset C^1(K)$ and

(1) U contains a strictly positive function,
(2) K is a compact, convex subset of $\mathbb{R}^d$, with piecewise smooth boundary (and $K = \overline{\operatorname{int} K}$).

For $f \in C^1(K)$, we let $Z_1(f)$ denote the set of zeros of f on K for which the following hold:

(a) If $x \in \operatorname{int} K$, then all partial derivatives of f at x vanish.
(b) If $x \in \partial K$, then all directional derivatives to f at x vanish for all directions tangent to K at x.

If K is the interval $[c, d]$, then $Z_1(f)$ is simply the set of zeros of f in $[c, d]$, with the proviso that if $x \in (c, d)$, then $f'(x) = 0$.

Finally we say that $U \subset C^1(K)$ satisfies Property B if to each $u \in U$, $u \neq 0$, there exists a $v \in U$, $v \neq 0$, satisfying

(1) $Z_1(u) \subseteq Z_1(v)$,
(2) $v \geq 0$.

The following result was proven by Pinkus and Strauss [13].

Theorem 2.9. *Let $U \subset C^1(K)$, and Assumption* I *hold. For every $\mu \in \mathscr{D}$ and each $f \in C^1(K)$ with $\mathscr{U}(f) \neq \varnothing$, there exists a unique best $L^1(K, \mu)$-approximant to f from $\mathscr{U}(f)$ if and only if U satisfies Property* B.

3. Best approximation with coefficient constraints

Let $\dim U = n$, and $u_1, \ldots, u_n$ be a basis for U. Given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ satisfying $-\infty \leq \alpha_i < \beta_i \leq \infty$, $i = 1, \ldots, n$, set

$$M(\boldsymbol{\alpha}; \boldsymbol{\beta}) = \left\{ \sum_{i=1}^{n} a_i u_i : \alpha_i \leq a_i \leq \beta_i, \, i = 1, \ldots, n \right\}.$$

The problem we consider in this section is that of best approximating $f \in C(K)$ from $M(\boldsymbol{\alpha}; \boldsymbol{\beta})$ in the $L^1(K, \mu)$-norm. This same problem in the uniform norm was studied by Pinkus and Strauss [14].

For $u^* = \sum_{i=1}^n a_i^* u_i \in M(\boldsymbol{\alpha}; \boldsymbol{\beta})$, set

$$b_i^* = \begin{cases} 1, & a_i^* = \beta_i, \\ 0, & \alpha_i < a_i^* < \beta_i, \\ -1, & a_i^* = \alpha_i, \end{cases}$$

and

$$U(\mathbf{b}^*) = \left\{ \sum_{i=1}^{n} c_i u_i : c_i b_i^* \leq 0, \, i = 1, \ldots, n \right\}.$$

From Theorem 2.1, we easily obtain

Theorem 3.1. *Let $f \in C(K) \backslash M(\boldsymbol{\alpha}; \boldsymbol{\beta})$ and $\mu \in \mathscr{A}$. Then the following are equivalent:*

(i) *$u^* = \sum_{i=1}^n a_i^* u_i \in M(\boldsymbol{\alpha}; \boldsymbol{\beta})$ is a best $L^1(K, \mu)$-approximant to f from $M(\boldsymbol{\alpha}; \boldsymbol{\beta})$.*
(ii) *For all $u \in U(\mathbf{b}^*)$*

$$\int_K [\operatorname{sgn}(f - u^*)] u \, d\mu \leq \int_{Z(f-u^*)} |u| \, d\mu.$$

(iii) *There exists an* $h \in L^\infty(K, \mu)$ *satisfying*

$$(1) \qquad |h(x)| = 1, \quad \text{all } x \in K,$$

$$(2) \qquad \int_K hu_i\,d\mu \begin{cases} \geq 0, & \text{if } b_i^* = 1, \\ = 0, & \text{if } b_i^* = 0, \\ \leq 0, & \text{if } b_i^* = -1, \end{cases}$$

$$(3) \qquad \int_K h(f - u^*)\,d\mu = \|f - u^*\|_1 .$$

Statement (iii) provides us with the more precise information needed for the solution of the unicity problem. For this unicity problem, we also need an important addendum to (2) and (3) of (iii).

Lemma 3.2. *Let* f, u^*, *and* h *be as in Theorem* 3.1(iii). *If* $\tilde{u} = \sum_{i=1}^n \tilde{a}_i u_i$ *is any other best* $L^1(K, \mu)$*-approximant to* f *from* $M(\boldsymbol{\alpha}; \boldsymbol{\beta})$, *then*

(a) $\int_K h(f - \tilde{u})\,d\mu = \|f - \tilde{u}\|_1$,

(b) *if* $\int_K hu_j\,d\mu \neq 0$ *for some* $j \in \{1, \dots, n\}$, *then*

$$\tilde{a}_j = a_j^* = \begin{cases} \beta_j, & \text{if } b_j^* = 1, \\ \alpha_j, & \text{if } b_j^* = -1. \end{cases}$$

Proof. From (1), (2), and (3) of (iii) of Theorem 3.1,

$$\|f - u^*\|_1 = \int_K h(f - u^*)\,d\mu \leq \int_K h(f - \tilde{u})\,d\mu \leq \|f - \tilde{u}\|_1 .$$

Since equality holds, we immediately obtain (a) and

$$\int_K h(u^* - \tilde{u})\,d\mu = 0.$$

That is,

$$\sum_{i=1}^n (a_i^* - \tilde{a}_i) \int_K hu_i\,d\mu = 0.$$

From (2) of (iii) of Theorem 3.1, we have that

$$(a_i^* - a_i) \int_K hu_i\,d\mu \geq 0$$

if $\alpha_i \leq a_i \leq \beta_i$ and $i \in \{1, \dots, n\}$. Thus

$$(a_i^* - \tilde{a}_i) \int_K hu_i\,d\mu = 0, \qquad i = 1, \dots, n.$$

If $\int_K hu_j\,d\mu \neq 0$, it follows that $\tilde{a}_j = a_j^*$ and (b) holds as a consequence of (2) of (iii) of Theorem 3.1. □

The seemingly innocuous Lemma 3.2 has a simple and important consequence. Set

$$I = \left\{ i \colon \int_K hu_i\,d\mu = 0 \right\}$$

and $J = \{1, \ldots, n\}\backslash I$. Every best $L^1(K, \mu)$-approximant to f from $M(\boldsymbol{\alpha}; \boldsymbol{\beta})$ of the form $\sum_{i=1}^n \tilde{a}_i u_i$ satisfies $\tilde{a}_j = a_j^*$ for all $j \in J$. Let

$$\overline{f} = f - \sum_{j\in J} a_j^* u_j .$$

Then $\tilde{u} = \sum_{i=1}^n \tilde{a}_i u_i$ is a best $L^1(K, \mu)$-approximant to f from $M(\boldsymbol{\alpha}; \boldsymbol{\beta})$ if and only if $\sum_{i\in I} \tilde{a}_i u_i$ is a best $L^1(K, \mu)$-approximant to $\overline{f}$ from $M_I(\alpha; \boldsymbol{\beta})$, where

$$M_I(\boldsymbol{\alpha}; \boldsymbol{\beta}) = \left\{ \sum_{i\in I} a_i u_i \colon \alpha_i \le a_i \le \beta_i ,\, i \in I \right\} .$$

Let

$$U_I = \operatorname{span}\{u_i \colon i \in I\} .$$

From Theorem 3.1(iii) and Lemma 3.2, we have

(1) $|h(x)| = 1$ all $x \in K$,

(2) $\int_K h u_i \, d\mu = 0$, all $i \in I$,

(3)

$$\text{(a)} \qquad \int_K h \left(\overline{f} - \sum_{i\in I} a_i^* u_i \right) d\mu = \left\| \overline{f} - \sum_{i\in I} a_i^* u_i \right\|_1 ,$$

$$\text{(b)} \qquad \int_K h \left(\overline{f} - \sum_{i\in I} \tilde{a}_i u_i \right) d\mu = \left\| \overline{f} - \sum_{i\in I} \tilde{a}_i u_i \right\|_1 .$$

From Theorem 2.1(iii), this implies that both $\sum_{i\in I} a_i^* u_i$ and $\sum_{i\in I} \tilde{a}_i u_i$ are best $L^1(K, \mu)$-approximants to $\overline{f}$ from U_I, and not only from $M_I(\boldsymbol{\alpha}; \boldsymbol{\beta})$. We have therefore proved:

Theorem 3.3. *Let* $N = \{i \colon \alpha_i = -\infty, \beta_i = \infty\}$. *If* $\operatorname{span}\{u_{i_1}, \ldots, u_{i_k}\}$ *is a unicity space for* $C_1(K, \mu)$ *for every choice of distinct* $\{i_1, \ldots, i_k\}$ *satisfying* $N \subseteq \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, *then* $M(\boldsymbol{\alpha}; \boldsymbol{\beta})$ *is a unicity set for* $C_1(K, \mu)$.

It is not clear that the converse result is valid in this generality. It is, however, true with a minor additional assumption.

Theorem 3.4. *Let* $N = \{i \colon \alpha_i = -\infty, \beta_i = \infty\}$. *Assume that for all* $i \notin N$, *we have* $-\infty < \alpha_i < \beta_i < \infty$. *If* $M(\boldsymbol{\alpha}; \boldsymbol{\beta})$ *is a unicity set for* $C_1(K, \mu)$, *then* $\operatorname{span}\{u_{i_1}, \ldots, u_{i_k}\}$ *is a unicity space for* $C_1(K, \mu)$ *for every choice of distinct* $\{i_1, \ldots, i_k\}$ *satisfying* $N \subseteq \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.

Proof. Let

$$N \subseteq \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\},$$

and assume that $U_k = \operatorname{span}\{u_{i_1}, \ldots, u_{i_k}\}$ is not a unicity space for $C_1(K, \mu)$. By definition there exists an $f \in C(K)$ and $u^1, u^2 \in U_k$, $u^1 \neq u^2$, such that both u^1 and u^2 are best $L^1(K, \mu)$-approximants to f from U_k. Since the set of best approximants is convex and $u^r + u$, $r = 1, 2$, are best $L^1(K, \mu)$-approximants to $f + u$ from U_k for each $u \in U_k$, we may assume that

$$u^1 = \sum_{j=1}^{k} a_{i_j}^1 u_{i_j}, \quad u^2 = \sum_{j=1}^{k} a_{i_j}^2 u_{i_j}$$

where $\alpha_{i_j} < a_{i_j}^r < \beta_{i_j}$, $r = 1, 2$; $j = 1, \dots, k$. From Theorem 2.1(iii), there exists an $h \in L^\infty(K, \mu)$ satisfying

(1) $|h(x)| = 1$, all $x \in K$,
(2) $\int_K hu_i \, d\mu = 0$, $j = 1, \dots, k$,
(3) $\int_K h(f - u^r) \, d\mu = \|f - u^r\|_1$, $r = 1, 2$.

Set $Q = \{1, \dots, n\} \backslash \{i_1, \dots, i_k\}$. For each $i \in Q$ we have $-\infty < \alpha_i < \beta_i < \infty$. Let $i \in Q$. If $\int_K hu_i \, d\mu = 0$ let $\gamma_i \in [\alpha_i, \beta_i]$. If $\int_K hu_i \, d\mu > 0$ set $\gamma_i = \beta_i$, while if $\int_K hu_i \, d\mu < 0$ set $\gamma_i = \alpha_i$. Define

$$\hat{u} = \sum_{i \in Q} \gamma_i u_i$$

and $\hat{f} = f + \hat{u}$, $\hat{u}^r = u^r + \hat{u}$, $r = 1, 2$. From Theorem 3.1(iii), we have that both $\hat{u}^1$ and $\hat{u}^2$ are best $L^1(K, \mu)$-approximants to $\hat{f}$ from $M(\boldsymbol{\alpha}; \boldsymbol{\beta})$. □

We can now put together Theorems 2.5, 3.3, and 3.4 to obtain the analogous necessary and sufficient conditions for when $M(\boldsymbol{\alpha}; \boldsymbol{\beta})$ is a unicity set in $C_1(K, \mu)$ for all $\mu \in \mathscr{D}$. Since Chebyshev (T-)systems satisfy Property A on $[a, b]$, we obtain, for example, that if $\{u_{i_1}, \dots, u_{i_k}\}$ is a T-system on $[a, b]$ for every choice of $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and $k = 1, \dots, n$, then $M(\boldsymbol{\alpha}; \boldsymbol{\beta})$ is a unicity set for $C_1(K, \mu)$ for all $\mu \in \mathscr{D}$, and for any choice of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as originally defined. A sequence of functions $\{u_1, \dots, u_n\}$, such that every subsequence is a T-system, is called a Descartes system.

4. Best approximation under interpolatory constraints

Assume that $\dim U = n$, and that we are given m distinct points $t_1, \dots, t_m$ in K. We wish to approximate $f \in C(K)$ from the set of $u \in U$ satisfying $u(t_1) = f(t_i)$, $i = 1, \dots, m$. For this problem to be well defined, we assume that this interpolation is always possible. That is, we assume that U is of dimension m over $\{t_1, \dots, t_m\}$. Equivalently, we can restate this condition as follows. Set

$$V = \{u \colon u \in U, \, u(t_i) = 0, \, i = 1, \dots, m\}.$$

Then V is a subspace of U of dimension $n - m$. If $m = n$, then these interpolating conditions uniquely determine the admissible u, and there is no approximation involved. As such, we always assume in what follows that $1 \leq m < n$.

For notational ease, set

$$M(f) = \{u \colon u \in U, \, u(t_i) = f(t_i), \, i = 1, \dots, m\}.$$

Our problem is to approximate f from $M(f)$. Obviously $M(f)$ depends on f. Nonetheless $M(f)$ is a closed convex subset of U, and as such, a best $L^1(K, \mu)$-approximant necessarily exists. It is important to note that if $\tilde{u} \in M(f)$, then

$$M(f) = \{\tilde{u} + v : v \in V\}.$$

Using this form of $M(f)$ we may apply Theorem 2.1 to obtain a characterization of best $L^1(K, \mu)$-approximants from $M(f)$.

Theorem 4.1. *Let $f \in C(K)$. Under the previous assumptions, the following are equivalent*:

(a) *u^* is a best $L^1(K, \mu)$-approximant to f from $M(f)$.*

(b) *For every $v \in V$*

$$\left|\int_K \operatorname{sgn}(f - u^*) v \, d\mu\right| \le \int_{Z(f-u^*)} |v| \, d\mu.$$

(c) *There exists an $h \in L^\infty(K, \mu)$ satisfying*

  (1) $|h| = 1$, *all* $x \in K$,

  (2) $\int_K hv \, d\mu = 0$, *all* $v \in V$,

  (3) $\int_K h(f - u^*) \, d\mu = \|f - u^*\|_1$.

We also have the following result.

Theorem 4.2. *To each $f \in C(K)$, there exists a unique best $L^1(K, \mu)$-approximant from $M(f)$ if and only if V is a unicity space for $C_1(K, \mu)$.*

Proof. ($\Leftarrow$) Let $f \in C(K)$ and $\tilde{u} \in M(f)$. Since V is a unicity space for $C_1(K, \mu)$, there exists a unique best $L^1(K, \mu)$-approximant $v^* \in V$ to $f - \tilde{u}$. The function $u^* = \tilde{u} + v^*$ is necessarily the unique best $L^1(K, \mu)$-approximant to f from $M(f)$.

($\Rightarrow$) Assume V is not a unicity space for $C_1(K, \mu)$. From Theorem 2.2 there exists an $h \in L^\infty(K, \mu)$ and a $v^* \in V$, $v^* \neq 0$, such that $h|v^*| \in C(K)$. From Theorem 2.1, it may be easily checked that αv^* is a best $L^1(K, \mu)$-approximant to $f = h|v^*|$ from V, for every $\alpha \in [-1, 1]$. Since $v^*(t_i) = 0$, $i = 1, \dots, m$, we also have $f(t_i) = 0$, $i = 1, \dots, m$, and therefore $M(f) = V$. We have constructed an $f \in C(K)$ with more than one best $L^1(K, \mu)$-approximant from $M(f)$.

Remark 4.1. The latter half of the proof of Theorem 4.2 epitomizes one of the principles implied by Theorems 2.2 and 2.3. Namely, to check the unicity space property of a finite-dimensional subspace V of $C(K)$, it suffices to check it only on a set of "test" functions. Moreover, these "test" functions f necessarily satisfy $f(x) = 0$ if $v(x) = 0$ for all $v \in V$. Since $v(t_i) = 0$, $i = 1, \dots, m$, for all $v \in V$, it is sufficient to verify the unicity space property for those $f \in C(K)$ satisfying $f(t_i) = 0$, $i = 1, \dots, m$, i.e., $f \in C(K)$ for which $M(f) = V$.

Putting together Theorems 4.2 and 2.5, we obtain

Theorem 4.3. *For each* $\mu \in \mathscr{D}$ *there exists a unique best* $L^1(K, \mu)$*-approximant to every* $f \in C(K)$ *from* $M(f)$ *if and only if* V *satisfies Property* A.

The consequence of this result is generally negative in character. For example, if $K = [a, b]$ and at least one of the points $\{t_i\}_{i=1}^m$ lies in (a, b), then either $[K \backslash Z(V)] \geq 2$ or all functions in V have their support between two consecutive t_i's. If the former occurs, then it follows from Theorem 2.6 that V does not satisfy Property A unless V decomposes. Thus if U is a T-system on (a, b), at least one of the points $\{t_i\}_{i=1}^m$ lies in (a, b), and $1 \leq m < n$, then V does not decompose and therefore does not satisfy Property A. There thus exist a $\mu \in \mathscr{D}$ and an $f \in C(K)$ without a unique best $L^1(K, \mu)$-approximant from $M(f)$.

This result is in marked contrast to the corresponding problem in the uniform norm (see Paszkowski [9]), where it is shown that uniqueness of the best uniform approximation to f from $M(f)$ always holds if U is a T-system.

5. Best restricted range approximation: Theory

We assume that we are given two functions b, $t \in C(K)$ which satisfy $b(x) < t(x)$ for all $x \in K$. Set

$$M(b, t) = \{u: u \in U, b \leq u \leq t\}.$$

The problem now under consideration is that of finding best $L^1(K, \mu)$-approximants to $f \in C(K)$ from $M(b, t)$. One characterization of the best $L^1(K, \mu)$-approximants is given by Theorem 2.1 with $M = M(b, t)$. However, this result is insufficient for our purposes. We first give a more particular characterization theorem, which is a consequence of Theorem 2.1(iii). To this end we will make the following assumption without which the next result is not valid. We assume there exists a $\hat{u} \in U$ satisfying $b(x) < \hat{u}(x) < t(x)$ for all $x \in K$. We write this as $\operatorname{int} M(b, t) \neq \varnothing$. Our characterization theorem generalizes results of Duffin and Karlovitz [4] and Gehner [5].

Theorem 5.1. *Let* $M(b, t)$ *be as just described and* $\dim U = n$. *Then* u^* *is a best* $L^1(K, \mu)$*-approximant to* $f \in C(K)$ *from* $M(b, t)$ *if and only if there exist distinct points* $x_1, \ldots, x_m \in Z(u^* - b)$ *and* $y_1, \ldots, y_r \in Z(t - u^*)$, $0 \leq m + r \leq n$, *positive numbers* $\{\alpha_i\}_{i=1}^m$ *and* $\{\beta_i\}_{i=1}^r$, *and an* $h \in L^\infty(K, \mu)$ *satisfying*

(1) $|h(x)| = 1$, *all* $x \in K$,
(2) $\int_K hu\, d\mu + \sum_{i=1}^m \alpha_i u(x_i) - \sum_{i=1}^r \beta_i u(y_i) = 0$, *all* $u \in U$,
(3) $\int_K h(f - u^*)\, d\mu = \|f - u^*\|_1$.

Proof. ($\Leftarrow$) Assume (1), (2), and (3) hold. From the definition of the $\{x_i\}_{i=1}^m$, $\{y_i\}_{i=1}^r$, $\{\alpha_i\}_{i=1}^m$ and $\{\beta_i\}_{i=1}^r$, it follows from (2) that

(2′) $\int_K hu^*\, d\mu \geq \int_K hu\, d\mu$, all $u \in M(b, t)$.

Theorem 2.1(iii), and (1), (2′), and (3) imply that u^* is a best $L^1(K, \mu)$-approximant to f from $M(b, t)$.

($\Rightarrow$) Assume u^* is a best $L^1(K, \mu)$-approximant to $f \in C(K)$ from $M(b, t)$. Since $M(b, t)$ is a closed convex set, we have from Theorem 2.1(iii) the existence of an $h \in L^\infty(K, \mu)$ satisfying (1), (3), and

(2″) $\int_K hu^*\, d\mu \geq \int_K hu\, d\mu$, all $u \in M(b, t)$.

Our aim is to prove (2) of the statement of the theorem. To this end, let $u_1, \dots, u_n$ be any basis for U, and set

$$P = \{(u_1(x), \dots, u_n(x): x \in Z(t - u^*)\} \cup \{(-u_1(x), \dots, -u_n(x)): x \in Z(u^* - b)\}.$$

Let Q denote the closed convex cone generated by P. (If $P = \varnothing$, set $Q = \{0\}$.) Furthermore, set

$$\mathbf{c} = \left(\int_K hu_1\, d\mu, \dots, \int_K hu_n\, d\mu\right).$$

If $\mathbf{c} \notin Q$ there exists a hyperplane strictly separating $\mathbf{c}$ from Q. Since Q is a cone, we may assume that this hyperplane passes through the origin. Thus there exists an $\mathbf{a} = (a_1, \dots, a_n) \neq \mathbf{0}$ for which

$$\sum_{j=1}^n a_j \int_K hu_j\, d\mu > 0 \geq \sum_{j=1}^n a_j q_j$$

for all $\mathbf{q} = (q_1, \dots, q_n) \in Q$.

Set $\tilde{u} = \sum_{j=1}^n a_j u_j$. Then $\int_K h\tilde{u}\, d\mu > 0$, while $\tilde{u}(x) \leq 0$ for all $x \in Z(t - u^*)$ and $\tilde{u}(x) \geq 0$ for all $x \in Z(u^* - b)$.

Recall that $\hat{u} \in U$ satisfies $b < \hat{u} < t$. Therefore, for some $\delta > 0$, δ sufficiently small, $v = \tilde{u} - \delta(u^* - \hat{u})$ satisfies $v < 0$ on $Z(t - u^*)$, $v > 0$ on $Z(u^* - b)$, and $\int_K hv\, d\mu > 0$. A standard compactness argument implies that $u^* + \varepsilon v \in M(b, t)$ for $\varepsilon > 0$, ε sufficiently small. But $\int_K h(u^* + \varepsilon v)\, d\mu > \int_K hu^*\, d\mu$ contradicting (2″). Thus $\mathbf{c} \in Q$.

Since $\mathbf{c} \in Q$, there exists $\{x_i\}_{i=1}^m$, $\{y_i\}_{i=1}^r$, $\{\alpha_i\}_{i=1}^m$, and $\{\beta_i\}_{i=1}^r$ as in the statement of the theorem with $0 \leq m + r \leq n$ satisfying (2). □

As a corollary to this result we have the more easily stated, but less useful

Corollary 5.2. *Let $M(b, t)$ be as described earlier and* $\dim U = n$. *Then u^* is a best $L^1(K, \mu)$-approximant to $f \in C(K)$ from $M(b, t)$ if and only if*

$$\int_K [\operatorname{sgn}(f - u^*)] u\, d\mu \leq \int_{Z(f - u^*)} |u|\, d\mu \tag{5.1}$$

for all $u \in U$ satisfying $u \leq 0$ on $Z(t - u^*)$ and $u \geq 0$ on $Z(u^* - b)$.

Proof. If u^* is a best $L^1(K, \mu)$-approximant to $f \in C(K)$ from $M(b, t)$ then (5.1) follows from (1), (2), and (3) of Theorem 5.1. If (5.1) holds, then we use Theorem 2.1(ii) to obtain our result. □

Given $\mu \in \mathscr{A}$, let us now ask when $M(b, t)$ is a unicity space for $C_1(K, \mu)$. As above, we assume that $\operatorname{int} M(b, t) \neq \varnothing$. We can now state the following analog of Theorem 2.2.

Theorem 5.3. *Let $M(b, t)$ be as above and $\dim U = n$. $M(b, t)$ is a unicity set for $C_1(K, \mu)$ if and only if there do not exist $u_1, u_2 \in M(b, t)$, $u_1 \neq u_2$, and*

(a) $\{x_i\}_{i=1}^m \subseteq Z(u_1 - b) \cap Z(u_2 - b)$,
(b) $\{y_i\}_{i=1}^r \subseteq Z(t - u_1) \cap Z(t - u_2)$,
(c) $\alpha_i > 0$, $i = 1, \ldots, m$, *and* $\beta_i > 0$, $i = 1, \ldots, r$,

where $0 \leq m + r \leq n$, *and an* $h \in L^\infty(K, \mu)$ *for which*

(1) $|h(x)| = 1$, *all* $x \in K$,
(2) $\int_K hu\, d\mu + \sum_{i=1}^m \alpha_i u(x_i) - \sum_{i=1}^r \beta_i u(y_i) = 0$, *all* $u \in U$,
(3) $h|u_1 - u_2| \in C(K)$.

Proof. ($\Rightarrow$) Assume there exist u_1, $u_2 \in M(b, t)$ with $\{x_i\}_{i=1}^m$, $\{y_i\}_{i=1}^r$, $\{\alpha_i\}_{i=1}^m$, $\{\beta_i\}_{i=1}^r$, and h satisfying (a)–(c) and (1)–(3). Set

$$f(x) = \frac{h(x)|u_1(x) - u_2(x)|}{2} + \frac{u_1(x) + u_2(x)}{2}.$$

Then $f \in C(K)$. Furthermore, as is easily checked,

$$h(f - u_i) \geq 0, \qquad i = 1, 2,$$

on all of K. Thus

(3′) $\int_K h(f - u_i)\, d\mu = \|f - u_i\|_1$, $i = 1, 2$.

Using (1), (2), and (3′), we obtain from Theorem 5.1 that both u_1 and u_2 are best $L^1(K, \mu)$-approximants to f from $M(b, t)$. Thus $M(b, t)$ is *not* a unicity set for $C_1(K, \mu)$.

($\Leftarrow$) Assume $M(b, t)$ is not a unicity set for $C_1(K, \mu)$. There exists an $f \in C(K)$ and u_1, $u_2 \in M(b, t)$, $u_1 \neq u_2$, such that both u_1 and u_2 are best $L^1(K, \mu)$-approximants to f from $M(b, t)$. Since $(u_1 + u_2)/2$ is also a best $L^1(K, \mu)$-approximant to f from $M(b, t)$, we easily see that

$$2\left|\left(f - \left(\frac{u_1 + u_2}{2}\right)\right)(x)\right| = |(f - u_1)(x)| + |(f - u_2)(x)|$$

for all $x \in K$. Thus

$$Z\left(f - \frac{u_1 + u_2}{2}\right) \subseteq Z(f - u_1) \cap Z(f - u_2).$$

Because $(u_1 + u_2)/2$ is a best $L^1(K, \mu)$-approximant to f from $M(b, t)$ we have $\{x_i\}_{i=1}^m$, $\{y_i\}_{i=1}^r$, $\{\alpha_i\}_{i=1}^m$, $\{\beta_i\}_{i=1}^r$ and h satisfying the various conditions given in Theorem 5.1 with respect to $(u_1 + u_2)/2$. Thus, (1) and (2) of this theorem hold. Since u_1, $u_2 \in M(b, t)$ we immediately obtain (a)–(c). It remains to prove (3) of this theorem.

From Theorem 5.1, we have

$$\int_K h\left(f-\frac{u_1+u_2}{2}\right)d\mu = \left\|f-\frac{u_1+u_2}{2}\right\|_1 .$$

Thus, using (1) we have $h = \operatorname{sgn}(f-(u_1+u_2)/2)\mu$ a.e. on $N(f-(u_1+u_2)/2)$. We alter h on a set of μ-measure zero so that $h = \operatorname{sgn}(f-(u_1+u_2)/2)$ on all of $N(f-(u_1+u_2)/2)$, while maintaining (1) and (2). Consider $h|u_1-u_2|$. If $h|u_1-u_2|$ is discontinuous at a point $x \in K$, then x is necessarily a point of discontinuity of h. Every point of discontinuity of h occurs, by construction, at a zero of $f-(u_1+u_2)/2$. But

$$Z(f-(u_1+u_2)/2) \subseteq Z(f-u_1) \cap Z(f-u_2) \subseteq Z(u_1-u_2) = Z(|u_1-u_2|),$$

and $|u_1-u_2|$ is continuous. Thus, $h|u_1-u_2|$ is continuous at each point of discontinuity of h, implying that $h|u_1-u_2| \in C(K)$. □

The characterization of a unicity set as determined by Theorem 5.3 is generally difficult if not impossible to verify. An easier sufficient condition is the following:

Proposition 5.4. *Let $M(b, t)$ be as shown before and* $\dim U = n$. *If for each $g \in U^*$, $g \neq 0$, and every set of $\{x_i\}_{i=1}^m$, $\{y_i\}_{i=1}^r \subseteq Z(g)$ with $0 \leq m+r \leq n$, there exists a $u \in U$ satisfying $u(x_i) \geq 0$, $i = 1, \dots, m$; $u(y_i) \leq 0, 1, \dots, r$, and*

$$(5.2) \qquad \int_K [\operatorname{sgn} g]u\, d\mu > \int_{Z(g)} |u|\, d\mu,$$

then $M(b, t)$ is a unicity set for $C_1(K, \mu)$.

Proof. If $M(b, t)$ is not a unicity set for $C_1(K, \mu)$, then we have the existence of $\{x_i\}_{i=1}^m$, $\{y_i\}_{i=1}^r$, $\{\alpha_i\}_{i=1}^m$, $\{\beta_i\}_{i=1}^r$, h, u_1 and u_2 satisfying (a)–(c) and (1)–(3) of Theorem 5.3. From (1) and (3), $g = h|u_1-u_2| \in U^*$, $g \neq 0$, and $h = \operatorname{sgn} g$ on $N(g)$. By construction $\{x_i\}_{i=1}^m$, $\{y_i\}_{i=1}^r \subseteq Z(g)$. Let $u^* \in U$ satisfy (5.2) with respect to this g and these $\{x_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^r$. From (2)

$$\int_K hu^*\, d\mu = -\sum_{i=1}^m \alpha_i u^*(x_i) + \sum_{i=1}^r \beta_i u^*(y_i) \leq 0.$$

Thus

$$\int_K [\operatorname{sgn} g]u^*\, d\mu + \int_{Z(g)} hu^*\, d\mu \leq 0$$

which implies that

$$\int_K [\operatorname{sgn} g]u^*\, d\mu \leq -\int_{Z(g)} hu^*\, d\mu \leq \int_{Z(g)} |u^*|\, d\mu.$$

This contradicts the fact that u^* satisfies (5.2). □

The conditions of Theorem 5.3 and Proposition 5.4 very much depend on the choice of $\mu \in \mathscr{A}$. We wish to obtain conditions on U and $M(b, t)$ that would imply that $M(b, t)$ is a unicity set for $C_1(K, \mu)$ for all $\mu \in \mathscr{D}$. To this end, we first introduce a condition we call Property C. As always, $\operatorname{int} M(b, t) \neq \varnothing$.

Definition 5.1. We say that $M(b, t)$ satisfies Property C (with respect to b and t), if for any $u_1, u_2 \in M(b, t)$, $u_1 \neq u_2$, with $\{x_i\}_{i=1}^m \subseteq Z(u_1 - b) \cap Z(u_2 - b)$, $\{y_i\}_{i=1}^r \subseteq Z(t - u_1) \cap Z(t - u_2)$, $0 \leq m + r \leq n$, $K \backslash Z(u_1 - u_2) = \bigcup_{i=1}^k A_i$, and every choice of $\varepsilon_i \in \{-1, 1\}$, $i = 1, \dots, k$, there exists a $v \in U$, $v \neq 0$ satisfying

(1) $v = 0$ a.e. on $Z(u_1 - u_2)$,
(2) $\varepsilon_i v \geq 0$ on A_i, $i = 1, \dots, k$,
(3) $v(x_i) \geq 0$, $i = 1, \dots, m$; $v(y_i) \leq 0$, $i = 1, \dots, r$.

The a.e. of (1) is with respect to Lebesgue measure. There is a simple equivalent definition of Property C that we will also use.

Definition 5.2. We say that $M(b, t)$ satisfies Property C (with respect to b and t) if for any $u_1, u_2 \in M(b, t)$, $u_1 \neq u_2$, with $\{x_i\}_{i=1}^m \subseteq Z(u_1 - b) \cap Z(u_2 - b)$, and $\{y_i\}_{i=1}^r \subseteq Z(t - u_1) \cap Z(t - u_2)$, $0 \leq m + r \leq n$, and any $h \in L^\infty(K)$ satisfying

(a) $|h(x)| = 1$, all $x \in K$,
(b) $h|u_1 - u_2| \in C(K)$,

there exists a $v \in U$, $v \neq 0$, for which

(1) $v = 0$ a.e. on $Z(u_1 - u_2)$,
(2) $hv \geq 0$ a.e.,
(3) $v(x_i) \geq 0$, $i = 1, \dots, m$; $v(y_i) \leq 0$, $i = 1, \dots, r$.

Condition (1) of Definition 5.2 is actually a consequence of the other conditions; see the proof of Theorem 5.5. The importance of Property C is a result of this next theorem.

Theorem 5.5. *$M(b, t)$ is a unicity set for $C_1(K, \mu)$ for every $\mu \in \mathscr{D}$ if and only if $M(b, t)$ satisfies Property* C (*with respect to b and t*).

Proof. ($\Rightarrow$) Assume Property C does not hold. As such, using Definition 5.2, there exist $u_1, u_2 \in M(b, t)$, $u_1 \neq u_2$, points $\{x_i\}_{i=1}^m \subseteq Z(u_1 - b) \cap Z(u_2 - b)$ and $\{y_i\}_{i=1}^r \subseteq Z(t - u_1) \cap Z(t - u_2)$, $0 \leq m + r \leq n$, as well as an $h \in L^\infty(K)$ satisfying (a) and (b) of Definition 5.2, and such that no $v \in U$, $v \neq 0$, satisfies (1)–(3) therein.

We will prove the existence of $\tilde{\mu} \in \mathscr{D}$, $\tilde{\alpha}_i \geq 0$, $i = 1, \dots, m$, $\tilde{\beta}_i \geq 0$, $i = 1, \dots, r$, and an $h \in L^\infty(K, \tilde{\mu})$ (not quite the previous h) such that (a)–(c) and (1)–(3) of Theorem 5.3 are satisfied.

To this end, set

$$\widetilde{U} = \{u \colon u \in U, u = 0 \text{ a.e. on } Z(u_1 - u_2)\},$$

i.e., $\widetilde{U} = U(u_1 - u_2)$. $\widetilde{U}$ is a subspace of U of dimension s, where $1 \leq s \leq n$. Let

$$\widetilde{U} = \operatorname{span}\{v_1, \dots, v_s\} \quad \text{and} \quad \widetilde{W} = \operatorname{span}\{v_{s+1}, \dots, v_n\}$$

where the $\{v_i\}_{i=1}^n$ span U. Note that the $v_{s+1}, \dots, v_n$ are linearly independent on $Z(u_1 - u_2)$. We alter h on $Z(u_1 - u_2)$ by setting it equal to $\tilde{h}$ thereon, where $\tilde{h}$ is defined by

(1) $|\tilde{h}(x)| = 1$, all $x \in Z(u_1 - u_2)$,

(2) $\int_{Z(u_1-u_2)} \tilde{h}w\,dx = 0$, all $w \in \widetilde{W}$.

Such an $\tilde{h}$ necessarily exists. Note that this new h still satisfies (a) and (b) of Definition 5.2, and has no effect on (1)–(3) thereof.

Let $B \subseteq \mathbb{R}^n$ be defined by

$$B = \left\{ \left(\int_K hv_j\,d\mu + \sum_{i=1}^m \alpha_i v_j(x_i) - \sum_{i=1}^r \beta_i v_j(y_i) \right)_{j=1}^n : \mu \in \mathscr{D},\ \alpha_i,\ \beta_i \geq 0 \right\}.$$

Since $\mathscr{D}$ is a convex cone, the set B is a convex cone. If $\mathbf{0} \in \partial B$, then there exists an $\mathbf{a} = (a_1, \dots, a_n) \neq \mathbf{0}$ that defines a support hyperplane to B at $\mathbf{0}$. Set $v = \sum_{j=1}^n a_j v_j$. Then $v \neq 0$ and

$$\int_K hv\,d\mu + \sum_{i=1}^m \alpha_i v(x_i) - \sum_{i=1}^r \beta_i v(y_i) \geq 0$$

for all $\mu \in \mathscr{D}$ and α_i, $\beta_i \geq 0$. This implies that $hv \geq 0$ a.e. on K, $v(x_i) \geq 0$, $i = 1, \dots, m$, and $v(y_i) \leq 0$, $i = 1, \dots, r$. Now $v = \tilde{u} + \tilde{w}$ where $\tilde{u} \in \widetilde{U}$ and $\tilde{w} \in \widetilde{W}$. Furthermore, $hv = \tilde{h}\tilde{w}$ on $Z(u_1 - u_2)$. Thus $\tilde{h}\tilde{w} \geq 0$ a.e. on $Z(u_1 - u_2)$. This contradicts (2) in the definition of $\tilde{h}$ unless $\tilde{w} = 0$. Thus $v = \tilde{u} \in \widetilde{U}$. That is, $v = 0$ a.e. on $Z(u_1 - u_2)$ and $hv \geq 0$ a.e. This contradicts our assumption that Property C does not hold. Thus $\mathbf{0} \notin \partial B$.

Since B is a convex cone, we have $\mathbf{0} \in B$. Thus there exists a $\tilde{\mu} \in \mathscr{D}$ and $\tilde{\alpha}_i$, $\tilde{\beta}_i \geq 0$ such that

$$\int_K hu\,d\tilde{\mu} + \sum_{i=1}^m \tilde{\alpha}_i u(x_i) - \sum_{i=1}^r \tilde{\beta}_i u(y_i) = 0$$

for all $u \in U$. From Theorem 5.3 it follows that $M(b, t)$ is not a unicity set for $C_1(K, \tilde{\mu})$.

($\Leftarrow$) Assume Property C holds. If $M(b, t)$ is not a unicity set for $C_1(K, \mu)$, some $\mu \in \mathscr{D}$, we have the existence of u_1, u_2, $\{x_i\}$, $\{y_i\}$, $\{\alpha_i\}$, $\{\beta_i\}$, and h satisfying (a)–(c) and (1)–(3) of Theorem 5.3. Let $v \in U$, $v \neq 0$, satisfy (1)–(3) of Definition 5.2 with respect to these terms. This immediately contradicts (2) of Theorem 5.3. Thus $M(b, t)$ is a unicity set for $C_1(K, \mu)$ for every $\mu \in \mathscr{D}$. □

While Property C is the correct characterization, it is also generally difficult to verify for a specific b and t. However, we can totally characterize those U for which $M(b, t)$ satisfies Property C for all admissible b and t.

Theorem 5.6. *Assume* $\dim U = n$. *Then* $M(b, t)$ *satisfies Property* C *for all* $b, t \in C(K)$ *for which* $\operatorname{int} M(b, t) \neq \varnothing$ *if and only if there exists a basis* $v_1, \ldots, v_n$ *for* U *such that*

(1) $N(v_i)$ *is connected for each* i,

(2) $N(v_i) \cap N(v_j) = \varnothing$ *for all* $i \neq j$.

Proof. ($\Rightarrow$) Assume $M(b, t)$ satisfies Property C for all $b, t \in C(K)$ for which $\operatorname{int} M(b, t) \neq \varnothing$. Then there must exist a unique best approximant to t from $\mathscr{U}(t) = \{u: u \in U, u \leq t\}$, for every $t \in C(K)$. From Theorem 2.8 this implies that there exists a basis $v_1, \ldots, v_n$ for U satisfying (2). Since there must also exist a unique best approximant to every $f \in C(K)$ from all of U, it follows from Theorem 2.6 that $[K \backslash Z(u)] \leq \dim U(u)$ for all $u \in U$. Let u be any of the basis elements $v_1, \ldots, v_n$. Since $\dim U(u) = 1$, we obtain (1).

($\Leftarrow$) If U has a basis $v_1, \ldots, v_n$ satisfying (2), then it is easily see that $M(b, t)$ is a unicity set for $C_1(K, \mu)$ if and only if

$$M_i(b, t) = \{u: u \in V_i, u \in M(b, t)\}$$

is a unicity set for $C_1(K, \mu)$, where $V_i = \operatorname{span}\{v_i\}$, $i = 1, \ldots, n$. Conditions (1), (2), and (3) of Definition 5.1 are easily verified in this case. For if u_1, $u_2 \in M_i(b, t)$, $u_1 \neq u_2$, then since $Z(u_1 - b) \cap Z(u_2 - b) = Z(u_1 - u_2) = Z(v_i)$, we have $v = 0$ on $Z(u_1 - u_2)$ for all $v \in V_i$, with the similar result for $Z(t - u_1) \cap Z(t - u_2)$. Thus, (3) holds. Furthermore, from (1), $K \backslash Z(u_1 - u_2) = N(v_i)$ and this is a connected set. Thus, (1) and (2) of Definition 5.1 are obtained by setting the v therein to be v_i or $-v_i$. □

The subspaces satisfying the conditions of Theorem 5.6 are uninteresting as in the case of one-sided approximation. However, paralleling Theorems 2.8 and 2.9, we should also consider the case where differentiability is present. Using the arguments of Theorem 5.6, we can prove:

Proposition 5.7. *Let* $U \subset C^1(K)$ *and Assumption* I *hold. If* $M(b, t)$ *satisfies Property* C *for all* $b, t \in C^1(K)$ *with* $\operatorname{int} M(b, t) \neq \varnothing$, *then* U *must satisfy Property* A *and Property* B.

See §2 for the definitions of Assumptions I and Properties A and B. It is an open question as to whether Assumption I and Properties A and B imply Property C.

6. Best restricted range approximation: Examples

In this section, we give examples of $M(b, t)$ satisfying Property C. We will restrict ourselves to the case of an interval, i.e., $K = [c, d]$, and assume that $U \subset C^1(K)$, $b, t \in C^1(K)$ and $\operatorname{int} M(b, t) \neq \varnothing$.

We first give a sufficient condition and then delineate some subspaces satisfying this condition. For ease of notation, we introduce the following terminology.

Definition 6.1. For any set of points $W \subseteq [c, d]$, we let $|W|$ denote the number of points in W, and $I(W)$ denote the number of points in W where every point in $W \cap (c, d)$ is counted twice.

Definition 6.2. Let $f \in C^1[c, d]$. Then $Z^*(f)$ counts the number of zeros of f in $[c, d]$, where a zero $x \in (c, d)$ is counted twice if $f(x) = f'(x) = 0$. The analogous definitions hold for the open and half-open intervals. Similarly $\widetilde{Z}(f)$ counts the number of zeros in the same way as $Z^*(f)$, but zeros at endpoints may be counted twice if both the function and its derivative vanish there.

Thus for any $f \in C^1[c, d]$, $Z^*(f) \le \widetilde{Z}(f)$. For a given $U \subset C[c, d]$, and any $c \le e < f \le d$, we also define

$$V_{e,f} = \{u \colon u \in U,\ u = 0 \text{ on } [c, e) \cup (f, d]\}.$$

For notational ease, we also set

$$[[e, f]] = \begin{cases} [e, f], & \text{if } c = e,\ f = d, \\ [e, f), & \text{if } c = e,\ f < d, \\ (e, f], & \text{if } c < e,\ f = d, \\ (e, f), & \text{if } c < e,\ f < d. \end{cases}$$

We can now state:

Theorem 6.1. *Let $U \subset C^1[c, d]$, $\dim U = n$. Assume U satisfies Property* A, *and if $u \in V_{e,f}$ has no zero interval in $[e, f]$, then*

$$Z^*(u|_{[[e,f]]}) \le \dim V_{e,f} - 1. \tag{6.1}$$

Then $M(b, t)$ satisfies Property C *for all $b, t \in C^1[c, d]$ with* $\operatorname{int} M(b, t) \ne \varnothing$.

Proof. We are given $b, t \in C^1[c, d]$ with $\operatorname{int} M(b, t) \ne \varnothing$. Let $u_1, u_2 \in M(b, t)$, $u_1 \ne u_2$. Assume $\{x_i\}_{i=1}^m \subseteq Z(u_1 - b) \cap Z(u_2 - b)$, $\{y_i\}_{i=1}^r \subseteq Z(t - u_1) \cap Z(t - u_2)$ where $0 \le m + r \le n$, and $h \in L^\infty(K)$ satisfies

(a) $|h(x)| = 1$, all $x \in [c, d]$,

(b) $h|u_1 - u_2| \in C[c, d]$.

We will construct a $v \in U$, $v \ne 0$, satisfying (1)–(3) of Definition 5.2.

Suppose that $[e, f]$ is a maximal interval on which $u_1 - u_2$ has no zero interval. That is, $u_1 - u_2$ has no zero interval on $[e, f]$, if $c < e$ then $u_1 - u_2$ vanishes identically on $(e - \varepsilon, e)$ for some $\varepsilon > 0$, and if $f < d$ then $u_1 - u_2$ vanishes identically on $(f, f + \varepsilon)$ for some $\varepsilon > 0$. Such an $[e, f]$ exists as a consequence of Property A. Furthermore from Theorem 2.7(3), there exists a $w \in V_{e,f}$ such that $w = u_1 - u_2$ on $[e, f]$.

Let us assume, for convenience, that $c < e < f < d$, i.e., $[[e, f]] = (e, f)$. Set

$$R = \{\{x_i\}_{i=1}^m \cup \{y_i\}_{i=1}^r\} \cap (e, f).$$

Since $b, t \in C^1[c, d]$, we have $w(x) = w'(x) = 0$ for all $x \in R$. let

$$S = \{x \colon x \in (e, f),\ w(x) = 0,\ x \notin R\}.$$

By assumption

$$I(R) + |S| \le \dim V_{e,f} - 1 .$$

For $\varepsilon > 0$, ε small, set

$$T_\varepsilon = \{x - \varepsilon, x + \varepsilon : x \in R, h \text{ does not change sign at } x\} \cup \{x : x \in R \cup S, h \text{ changes sign at } x\} .$$

Then

$$l = |T_\varepsilon| \le I(R) + |S| \le \dim V_{e,f} - 1 .$$

Since $V_{e,f}$ is a weak Chebyshev (WT-) system (see Theorem 2.7 (3)) it follows from Sommer and Strauss [20] and Stockenberg [21] that $V_{e,f}$ contains an $(l+1)$-dimensional WT-system $\widetilde{V}$. From a theorem of Jones and Karlovitz [6], there exists a $\tilde{v}_\varepsilon \in \widetilde{V} \subseteq \widetilde{V}_{e,f}$, $\|\tilde{v}_\varepsilon\|_\infty = 1$, which changes sign weakly at each of the points of T_ε. Let $\tilde{v}$ be a limit point of $\tilde{v}_\varepsilon$ as $\varepsilon \downarrow 0$. By construction $\tilde{v} \ne 0$, $\delta h \tilde{v} \ge 0$ for some $\delta \in \{-1, 1\}$, and $\tilde{v}(x) = 0$ for all $x \in R$. Set $v = \delta \tilde{v}$. Then v satisfies (1)–(3) of Definition 5.2, i.e., Property C holds.

The other cases, i.e., $c = e$ and/or $f = d$, are similarly treated. □

Corollary 6.2. *Let* $U \subset C^1[c, d]$, $\dim U = n$. *If*

$$Z^*(u) \le n - 1 \tag{6.2}$$

for all $u \in U$, $u \ne 0$, *then* $M(b, t)$ *satisfies Property* C *for all* b, $t \in C^1[c, d]$ *with* $\operatorname{int} M(b, t) \ne \emptyset$.

A subspace $U \subset C^1[c, d]$ of dimension n is said to be an extended Chebyshev system of order 2 (ET_2-system) if $\widetilde{Z}(u) \le n - 1$ for all $u \in U$, $u \ne 0$. Since $Z^*(f) \le \widetilde{Z}(f)$ for all $f \in C^1[c, d]$, we immediately obtain that if U is an ET_2-system, then it satisfies (6.2). Polynomials are examples of ET_2-systems.

We now give two additional examples of subspaces satisfying the conditions of Theorem 6.1.

Example 6.1. *Polynomial splines with simple, fixed knots.* Let $n \ge 3$ and $\xi_0 = c < \xi_1 < \cdots < \xi_k < d = \xi_{k+1}$. Set

$$S = \left\{ s : s(x) = \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{k} b_i (x - \xi_i)_+^{n-1} , a_i, b_i \in \mathbb{R} \right\} .$$

S is the space of polynomial splines of degree $n - 1$ with the simple, fixed knots $\{\xi_i\}_{i=1}^k$. Since $n \ge 3$, we have $S \subset C^1[c, d]$. Pinkus [12] proved that S satisfies Property A. It is also well known that if $s \in V_{e,f}$ has no zero interval in $[e, f]$, then $e = \xi_i$ and $f = \xi_j$, and

$$Z^*(s|_{[[\xi_i, \xi_j]]}) \le \dim V_{\xi_i, \xi_j} - 1 .$$

Thus, S satisfies the conditions of Theorem 6.1.

Example 6.2. *Piecing together ET_2-systems.* Let $c = e_0 < \cdots < e_k < e_{k+1} = d$. On each interval $I_i = [e_{i-1}, e_i]$, $i = 1, \ldots, k+1$, let $U_i \subset C^1(I_i)$, $\dim U_i = n_i$, $n_i \geq 2$. Assume that the constant function is in U_i, and

$$U_i' = \{u' : u \in U\}$$

is a T-system on I_i. (Thus, U_i is an ET_2-system on I_i.) Let $U \subset C^1[c, d]$ be the subspace defined by $U|_{I_i} = U_i$, $i = 1, \ldots, k+1$. We claim that U satisfies the conditions of Theorem 6.1.

From [12], we have that U satisfies Property A. It remains to prove (6.1). If $u \in V_{e,f}$ has no zero interval in $[e, f]$, then $e = e_i$ and $f = e_j$ for some $0 \leq i < j \leq k+1$. It therefore suffices to prove (6.1) only for such e and f. Set $V_{e_i, e_j} = V_{i,j}$, $0 \leq i < j \leq k+1$. From [12], we have

(i) $\dim V_{0,k+1} = n_1 + \cdots + n_{k+1} - 2k$.
(ii) For $1 \leq j \leq k$, $\dim V_{0,j} = n_1 + \cdots + n_j - 2j$.
(iii) For $1 \leq i \leq k$, $\dim V_{i,k+1} = n_{i+1} + \cdots + n_{k+1} - 2(k+1-i)$.
(iv) For $1 \leq i < j \leq k$, $\dim V_{i,j} = n_{i+1} + \cdots + n_j - 2(j-i+1)$.

If any of these values is negative, then we understand it to mean that $\dim V_{i,j} = 0$.

Similarly, if

$$V_{i,j}' = \{u' : u \in U, u' = 0 \text{ on } [c, e_i) \cup (e_j, d]\}$$

then $V_{i,j}'$ is a WT-system and

(i) $\dim V_{0,k+1}' = n_1 + \cdots + n_{k+1} - 2k - 1$.
(ii) For $1 \leq j \leq k$, $\dim V_{0,j}' = n_1 + \cdots + n_j - 2j$.
(iii) For $1 \leq i \leq k$, $\dim V_{i,k+1}' = n_{i+1} + \cdots + n_{k+1} - 2(k+1-i)$.
(iv) For $1 \leq i < j \leq k$, $\dim V_{i,j}' = n_{i+1} + \cdots + n_j - 2(j-i+1) + 1$.

These facts are also contained in [12]. Note that the relationship between $\dim V_{i,j}$ and $\dim V_{i,j}'$ is not the same for all i and j.

In what follows we will need a result of Stockenberg [22]. Namely, assume $U \subset C[a, b]$ is a WT-system of dimension n, and for every $x \in (a, b)$ there exists a $u \in U$ satisfying $u(x) \neq 0$. Then if $u^* \in U$ has no zero interval in (a, b), we have that u^* has at most $n-1$ distinct zeros in (a, b).

With these results we can proceed to prove (6.1).

Case 1. $i = 0$, $j = k+1$. u has no zero interval in $[c, d]$. Let $Z^*(u) = m$ on $[c, d]$. If u' has no zero interval in $[c, d]$, then $u' \in V_{0,k+1}'$, and u' has at least $m-1$ distinct zeros in (c, d). Using the above result of Stockenberg, applied to u', we get

$$m - 1 \leq \dim V_{0,k+1}' - 1 = \dim V_{0,k+1} - 2.$$

Thus $m = Z^*(u) \leq \dim V_{0,k+1} - 1$. If u' has zero intervals in $[c, d]$, then the inequality is even sharper. There is simply a bit more bookkeeping involved.

Case 2. $i = 0$, $1 \le j \le k$. $u \in V_{0,j}$ and u has no zero interval in $[e_0, e_j]$. Let $Z^*(u|_{[e_0, e_j)}) = m$. Assume u' has no zero interval in $[e_0, e_j]$. Since $u(e_j) = u'(e_j) = 0$, u' has at least m distinct zeros in $[e_0, e_j)$. Thus

$$m \le \dim V'_{0,j} - 1 = \dim V_{0,j} - 1,$$

and $Z^*(u|_{[e_0, e_j)}) \le \dim V_{0,j} - 1$. Again if u' has zero intervals in $[e_0, e_j]$ the result also holds.

The remaining two cases are similarly proven. □

7. Best restricted range and derivative approximation

The techniques of the previous two sections may be generalized to apply to problems of restricted range and derivatives approximation. For ease of notation, we will restrict ourselves to the range of the approximant and its first derivative only. Since the methods of proof are almost totally analogous, we will simply review and state the main results.

As previously, we assume that $U \subset C^1(K)$, $\dim U = n$. Here we are given four functions b_0, t_0, b_1, $t_1 \in C(K)$, where $b_0(x) < t_0(x)$ and $b_1(x) < t_1(x)$ for all $x \in K$. Set

$$M(b_0, t_0; b_1, t_1) = \{u: u \in U, b_0 \le u \le t_0, b_1 \le u' \le t_1\}.$$

We assume in what follows that $\operatorname{int} M(b_0, t_0; b_1, t_1) \ne \varnothing$, i.e., there exists a $\hat{u} \in U$ satisfying $b_0(x) < \hat{u}(x) < t_0(x)$ and $b_1(x) < \hat{u}'(x) < t_1(x)$ for all $x \in K$. Our first result is a generalization of Theorem 5.1.

Theorem 7.1. *Let* $M(b_0, t_0; b_1, t_1)$ *be as just described, and* $\dim U = n$. *Then* u^* *is a best* $L^1(K, \mu)$*-approximant to* $f \in C(K)$ *from* $M(b_0, t_0; b_1, t_1)$ *if and only if there exist distinct points*

(a) $\{x_i\}_{i=1}^m \subseteq Z(u^* - b_0)$,
(b) $\{y_i\}_{i=1}^r \subseteq Z(t_0 - u^*)$,
(c) $\{z_i\}_{i=1}^s \subseteq Z(u^{*\prime} - b_1)$,
(d) $\{w_i\}_{i=1}^t \subseteq Z(t_1 - u^{*\prime})$

with $0 \le m + r + s + t \le n$, *positive numbers* $\{\alpha_i\}_{i=1}^m$, $\{\beta_i\}_{i=1}^r$, $\{\gamma_i\}_{i=1}^s$ *and* $\{\sigma_i\}_{i=1}^t$, *and an* $h \in L^\infty(K, \mu)$ *satisfying*

(1) $|h(x)| = 1$, *all* $x \in K$,
(2) $\int_K hu\,d\mu + \sum_{i=1}^m \alpha_i u(x_i) - \sum_{i=1}^r \beta_i u(y_i) + \sum_{i=1}^s \gamma_i u'(z_i) - \sum_{i=1}^t \sigma_i u'(w_i) = 0$, *all* $u \in U$,
(3) $\int_K h(f - u^*)\,d\mu = \|f - u^*\|_1$.

Paralleling Theorem 5.3, we obtain this next result, which characterizes the $M(b_0, t_0; b_1, t_1)$ that are unicity sets for $C_1(K, \mu)$.

Theorem 7.2. *Let* $M(b_0, t_0; b_1, t_1)$ *be as shown before, and* $\dim U = n$. $M(b_0, t_0; b_1, t_1)$ *is a unicity set for* $C_1(K, \mu)$ *if and only if there do not exist*

$u_1, u_2 \in M(b_0, t_0; b_1, t_1)$, $u_1 \neq u_2$, *and*

(a) $\{x_i\}_{i=1}^m \subseteq Z(u_1 - b_0) \cap Z(u_2 - b_0)$,
(b) $\{y_i\}_{i=1}^r \subseteq Z(t_0 - u_1) \cap Z(t_0 - u_2)$,
(c) $\{z_i\}_{i=1}^s \subseteq Z(u_1' - b_1) \cap Z(u_2' - b_1)$,
(d) $\{w_i\}_{i=1}^t \subseteq Z(t_1 - u_1') \cap Z(t_1 - u_2')$,
(e) *nonnegative numbers* $\{\alpha_i\}$, $\{\beta_i\}$, $\{\gamma_i\}$, $\{\sigma_i\}$,

where $0 \leq m + r + s + t \leq n$, *and an* $h \in L^\infty(K, \mu)$ *for which*

(1) $|h(x)| = 1$, *all* $x \in K$,
(2) $\int_K hu\,d\mu + \sum_{i=1}^m \alpha_i u(x_i) - \sum_{i=1}^r \beta_i u(y_i) + \sum_{i=1}^s \gamma_i u'(z_i) - \sum_{i=1}^t \sigma_i u'(w_i) = 0$, *all* $u \in U$,
(3) $h|u_1 - u_2| \in C(K)$.

Analogous to Property C we have

Definition 7.1. We say that $M(b_0, t_0; b_1, t_1)$ satisfies Property D (with respect to b_0, t_0, b_1, and t_1) if for any $u_1, u_2 \in M(b_0, t_0; b_1, t_1)$, $u_1 \neq u_2$, with $\{x_i\}$, $\{y_i\}$, $\{z_i\}$ and $\{w_i\}$ as in Theorem 7.2, $0 \leq m + r + s + t \leq n$, and any $h \in L^\infty(K)$ satisfying

(a) $|h(x)| = 1$, all $x \in K$,
(b) $h|u_1 - u_2| \in C(K)$,

there exists a $v \in U$, $v \neq 0$, for which

(1) $v = 0$ a.e. on $Z(u_1 - u_2)$,
(2) $hv \geq 0$ a.e.,
(3) (a) $v(x_i) \geq 0$, $i = 1, \ldots, m$,
(b) $v(y_i) \leq 0$, $i = 1, \ldots, r$,
(c) $v'(z_i) \geq 0$, $i = 1, \ldots, s$,
(d) $v'(w_i) \leq 0$, $i = 1, \ldots, t$.

Finally we obtain

Theorem 7.3. *$M(b_0, t_0; b_1, t_1)$ is a unicity set for $C_1(K, \mu)$ for every $\mu \in \mathscr{D}$ if and only if $M(b_0, t_0; b_1, t_1)$ satisfies Property* D *(with respect to b_0, t_0, b_1, and t_1).*

We note, without proof, that both algebraic polynomials and splines of degree at least three with simple, fixed knots are such that $M(b_0, t_0; b_1, t_1)$ satisfies Property D on an interval $[c, d]$ for all $b_0, t_0, b_1, t_1 \in C^1[c, d]$, if $\operatorname{int} M(b_0, t_0; b_1, t_1) \neq \varnothing$.

Acknowledgment. The authors wish to thank Professor A. Kroó for various helpful comments.

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