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LINEAR ALGEBRA AND ITS APPLICATIONS

An interlacing property of eigenvalues of strictly totally positive matrices

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Abstract

We prove results concerning the interlacing of eigenvalues of principal submatrices of strictly totally positive (STP) matrices. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

The central results concerning eigenvalues and eigenvectors of strictly totally positive (STP) matrices were proved by Gantmacher and Krein in their 1937 paper [1]. (An announcement appeared in 1935 in [2]. Chapter II of the book of Gantmacher and Krein [3] is a somewhat expanded version of this paper.) Among the results obtained in that paper is that an $n \times n$ STP matrix A has n positive, simple eigenvalues, and that the $n-1$ positive, simple eigenvalues of the principal submatrices obtained by deleting the first (or last) row and column of A strictly interlace those of the original matrix.

It has been known for many years that this interlacing property does not hold for all principal submatrices. An example may be found in the 1974 paper of Karlin and Pinkus [4]. Let

$$\lambda_1 > \cdots > \lambda_n > 0,$$

denote the eigenvalues of A, and

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$$\mu_1^{(k)} > \cdots > \mu_{n-1}^{(k)} > 0,$$

the eigenvalues of the principal submatrix of A obtained by deleting its kth row and column. It easily follows from the Perron theorem on positive matrices that

$$\lambda_1 > \mu_1^{(k)}$$

for each k. Moreover in 1985 Friedland [5] proved that

$$\mu_1^{(k)} > \lambda_2 \quad \text{and} \quad \mu_{n-1}^{(k)} > \lambda_n$$

for each k, $1 < k < n$. In this note we prove the following result.

Theorem 1. *Let A be an $n \times n$ STP matrix as above. Then for each k, $1 < k < n$,*

$$\lambda_{j-1} > \mu_j^{(k)} > \lambda_{j+1}, \qquad j = 1, \ldots, n-1$$

(*where $\lambda_0 = \lambda_1$*).

Before proving this result we introduce some notation and recall some known facts. For a non-zero vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n$, we define two notions of the number of sign changes of the vector $\mathbf{c}$. These are:
$S^-(\mathbf{c})$ – the number of sign changes in the sequence $c_1, \ldots, c_n$ with zero terms discarded.
$S^+(\mathbf{c})$ – the maximum number of sign changes in the sequence $c_1, \ldots, c_n$ where zero terms are arbitrarily assigned values $+1$ and -1.

For example

$$S^-(1,0,1,-1,0,1) = 2, \quad \text{and} \quad S^+(1,0,1,-1,0,1) = 4.$$

A matrix A is said to be an STP matrix if all its minors are strictly positive. STP matrices were independently introduced by Schoenberg in 1930, see [6], and by Krein and some of his colleagues in the 1930's. The main result of Gantmacher and Krein on the eigenvalues and eigenvectors of STP matrices is the following.

Theorem A. *Let A be an $n \times n$ STP matrix. Then A has n positive, simple eigenvalues. Let*

$$\lambda_1 > \cdots > \lambda_n > 0$$

denote these eigenvalues and $\mathbf{u}^k$ a real eigenvector (unique up to multiplication by a non-zero constant) associated with the eigenvalue λ_k. Then

$$q - 1 \leqslant S^-\left(\sum_{i=q}^{p} c_i \mathbf{u}^i\right) \leqslant S^+\left(\sum_{i=q}^{p} c_i \mathbf{u}^i\right) \leqslant p - 1$$

for each $1 \leqslant q \leqslant p \leqslant n$ *(and* c_i *not all zero). In particular* $S^-(\mathbf{u}^j) = S^+(\mathbf{u}^j) = j-1$ *for* $j = 1, \dots, n$*. In addition, the eigenvalues of the principal submatrices obtained by deleting the first (or last) row and column of A strictly interlace those of the original matrix.*

Proofs of this result may be found in Gantmacher, Krein [1,3], and in Ando [7].

A very important property of STP matrices is that of *variation diminishing*. This was Schoenberg's initial contribution to the theory. The particular result we need may be found in Gantmacher, Krein [3] and in Karlin [8], and is the following.

Theorem B. *Let A be an* $n \times m$ *STP matrix. Then for each vector* $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x} \neq \mathbf{0}$,

$$S^+(A\mathbf{x}) \leqslant S^-(\mathbf{x}).$$

2. Proof of Theorem 1

We first prove that for any k and $j = 1, \dots, n-1$,

$$\mu_j^{(k)} > \lambda_{j+1}.$$

Let $\mathbf{u} = (u_1, \dots, u_n)$ denote a real eigenvector of A with eigenvalue λ_{j+1}. That is,

$$A\mathbf{u} = \lambda_{j+1}\mathbf{u}.$$

Then from Theorem A,

$$S^+(\mathbf{u}) = S^-(\mathbf{u}) = j.$$

Let A_k denote the principal submatrix of A obtained by deleting the kth row and column. Let $\mathbf{v}$ denote a real eigenvector of A_k with eigenvalue $\mu_j^{(k)}$. That is,

$$A_k\mathbf{v} = \mu_j^{(k)}\mathbf{v}.$$

From Theorem A,

$$S^+(\mathbf{v}) = S^-(\mathbf{v}) = j-1.$$

From the vector $\mathbf{v} \in \mathbb{R}^{n-1}$ we construct the vector $\mathbf{v}' \in \mathbb{R}^n$ by simply inserting a 0 at the kth coordinate. That is, we will write (somewhat abusing notation)

$$\mathbf{v} = (v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n)^{\mathrm{T}}$$

and set

$$\mathbf{v}' = (v_1, \dots, v_{k-1}, 0, v_{k+1}, \dots, v_n)^{\mathrm{T}}.$$

Let $\mathbf{v}''$ be defined by

$$A\mathbf{v}' = \mu_j^{(k)}\mathbf{v}''.$$

Thus

$$\mathbf{v}'' = (v_1, \ldots, v_{k-1}, w_k, v_{k+1}, \ldots, v_n)^{\mathrm{T}}$$

for some easily calculated w_k.

From Theorem B and the definitions of $\mathbf{v}$, $\mathbf{v}'$ and $\mathbf{v}''$ we have,

$$j - 1 = S^+(\mathbf{v}) \leqslant S^+(\mathbf{v}'') \leqslant S^-(\mathbf{v}') = S^-(\mathbf{v}) = j - 1.$$

Thus

$$S^+(\mathbf{v}'') = S^-(\mathbf{v}') = j - 1.$$

If $u_k = 0$, then the vector obtained by deleting u_k from $\mathbf{u}$ is an eigenvector of A_k with eigenvalue λ_{j+1}. Since this new vector also has exactly j sign changes, it follows from Theorem A that

$$\lambda_{j+1} = \mu_{j+1}^{(k)} < \mu_j^{(k)}.$$

If $w_k = 0$, then $\mathbf{v}' = \mathbf{v}''$ is an eigenvector of A with $j - 1$ sign changes. As such

$$\mu_j^{(k)} = \lambda_j > \lambda_{j+1}.$$

Thus we may assume that both u_k and w_k are non-zero.

Since $\mathbf{u}$ and $\mathbf{v}$ are defined up to multiplication by a non-zero constant we may assume, based on the above, that u_k and w_k are both positive. Set

$$c^* = \inf\{c\colon\ c > 0, S^-(c\mathbf{v}' + \mathbf{u}) \leqslant j - 1\}.$$

We claim that c^* is a well-defined positive number. This is a consequence of the following.

For all c sufficiently large, e.g., such that $c|v_i| > |u_i|$ if $v_i \neq 0$, we have

$$S^-(c\mathbf{v}' + \mathbf{u}) \leqslant S^+(c\mathbf{v}' + \mathbf{u}) \leqslant S^+(\mathbf{v}'') = j - 1.$$

(Here we have used the fact that $cv_k + u_k = u_k$ and w_k are of the same sign.) For all c sufficiently small and positive, e.g., such that $c|v_i| < |u_i|$ if $u_i \neq 0$, we have

$$S^-(c\mathbf{v}' + \mathbf{u}) \geqslant S^-(\mathbf{u}) = j.$$

From the definition of c^* and continuity properties of S^+ and S^-, it follows that

$$S^-(c^*\mathbf{v}' + \mathbf{u}) \leqslant j - 1$$

and

$$S^+(c\mathbf{v}' + \mathbf{u}) \geqslant j$$

for all $c \leqslant c^*$.

Now

$$A(c^*\mathbf{v}' + \mathbf{u}) = c^*\mu_j^{(k)}\mathbf{v}'' + \lambda_{j+1}\mathbf{u}.$$

Let

$$\hat{c} = \frac{c^*\mu_j^{(k)}}{\lambda_{j+1}} > 0.$$

From Theorem B,

$$S^+(\hat{c}\mathbf{v}'' + \mathbf{u}) \leqslant S^-(c^*\mathbf{v}' + \mathbf{u}) \leqslant j - 1.$$

Since $\hat{c}\mathbf{v}'' + \mathbf{u}$ and $\hat{c}\mathbf{v}' + \mathbf{u}$ differ only in the kth coordinate, where both are positive, it follows that

$$S^+(\hat{c}\mathbf{v}' + \mathbf{u}) = S^+(\hat{c}\mathbf{v}'' + \mathbf{u}) \leqslant j - 1.$$

This implies from the above that $\hat{c} > c^*$. Thus $\mu_j^{(k)} > \lambda_{j+1}$.

The proof of the reverse inequality

$$\lambda_{j-1} > \mu_j^{(k)}$$

for any k and $j = 2, \ldots, n-1$ is essentially the same.

Let $\mathbf{x}$ be a real eigenvector of A with eigenvalue λ_{j-1}. That is,

$$A\mathbf{x} = \lambda_{j-1}\mathbf{x}.$$

Note that

$$S^+(\mathbf{x}) = S^-(\mathbf{x}) = j - 2.$$

The vectors $\mathbf{v}$, $\mathbf{v}'$, and $\mathbf{v}''$ are as above. If $w_k = 0$, then $\mu_j^{(k)} = \lambda_j < \lambda_{j-1}$. If $x_k = 0$, then

$$\lambda_{j-1} = \mu_{j-1}^{(k)} > \mu_j^{(k)}.$$

We may thus assume that w_k and x_k are both positive. We set

$$a^* = \inf\{a\colon\ a > 0, S^-(a\mathbf{x} + \mathbf{v}') \leqslant j - 2\}$$

and now follow the previous proof. □

Since STP matrices are dense in the set of all totally positive (TP) matrices (matrices all of whose minors are nonnegative), and because of the continuity of the eigenvalues as functions of the matrix entries, we have:

Corollary 2. *Let A be an $n \times n$ TP matrix with eigenvalues*

$$\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$$

(listed to their algebraic multiplicity). Let

$$\mu_1^{(k)} \geqslant \cdots \geqslant \mu_{n-1}^{(k)} \geqslant 0,$$

denote the eigenvalues of the principal minor of A obtained by deleting its kth row and column. Then for any k, $1 < k < n$,

$$\lambda_{j-1} \geqslant \mu_j^{(k)} \geqslant \lambda_{j+1}, \quad j = 1, \ldots, n-1$$

(*where* $\lambda_0 = \lambda_1$).

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