Structure of Invertible (Bi)infinite Totally Positive Matrices*

C. de Boor
Mathematics Research Center
University of Wisconsin — Madison
Madison, Wisconsin 53706

Rong-qing Jia
Department of Mathematics
University of Wisconsin — Madison
Madison, Wisconsin 53706

and

A. Pinkus
Department of Mathematics
Technion
Haifa, Israel

Submitted by Hans Schneider

ABSTRACT

An l_∞-invertible nonfinite totally positive matrix A is shown to have one and only one "main diagonal." This means that exactly one diagonal of A has the property that all finite sections of A principal with respect to this diagonal are invertible and their inverses converge boundedly and entrywise to A^{-1}. This is shown to imply restrictions on the possible shapes of such a matrix. In the proof, such a matrix is also shown to have an l_∞-invertible LDU factorization. In addition, decay of the entries of such a matrix away from the main diagonal is demonstrated. It is also shown that a bounded sign-regular matrix carrying some bounded sequence to a uniformly alternating sequence must have all its columns in c_0.

INTRODUCTION

An investigation into the structure of invertible (bi)infinite totally positive (or, tp) matrices was begun in [B_2] and [$CDMS_1$], motivated by certain

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questions [B_1, M] in spline approximation. First results concerned banded matrices [B_2, $CDMS_2$] and strictly banded block Toeplitz matrices [$CDMS_1$]. Somewhat surprisingly, such restriction to banded matrices was found to be unnecessary in [BFP], where it was proved that a tp (or, more generally, a sign-regular) matrix A mapping $l_\infty(J)$ to $l_\infty(I)$ is onto if and only if it has a uniformly alternating sequence in its range. This latter condition means that, for some bounded x, $\sup_i (Ax)(i)(Ax)(i+1) < 0$.

A similar result had been proved in [B_3] for band matrices, following up a conjecture in [M]. The argument in [B_3] provided much additional information about the inverse of a banded tp matrix. In the present paper, we obtain the same information for an arbitrary l_∞-invertible tp matrix.

We prove (in Section 1) that an l_∞-invertible tp matrix $A \in \mathbb{R}^{I \times J}$ has a "main diagonal." By this we mean that all sufficiently large square submatrices of A taken from consecutive rows and columns and principal with respect to a fixed diagonal are invertible and their inverses are bounded uniformly. This implies that A^{-1} is the pointwise limit of these inverses. We even show that such convergence is monotone. As a consequence, $D^J A^{-1} D^I$ or its negative is again tp, with $D^K \in \mathbb{R}^{K \times K}$ the diagonal matrix for which $D^K(k) = (-1)^k$, all $k \in K$.

In addition, we show (in Section 2) the uniqueness of such a main diagonal. For the proof, we establish a result of independent interest, namely the existence and uniqueness of an invertible LDU factorization for A. By this we mean the possibility of writing A as the product LDU of bounded and boundedly invertible tp matrices, with L and its inverse unit lower triangular, D and its inverse diagonal, and U and its inverse unit upper triangular.

We also follow up some consequences of the existence of a main diagonal for the matrix A:

(1) A can have only certain shapes. Explicitly, we must have $J = I + r$ for some r. We settle (in Section 3) the related question of possible choices for I and J in case we only know that A is tp and onto.

(2) The main diagonal must be bounded away from zero. This leads to the observation (made in Section 4) that the entries of A must decay at least linearly away from the main diagonal. We leave open the intriguing question whether, in fact, this decay is exponential, as was shown for arbitrary matrices with a *banded* inverse in [D].

In the last section, we settle a conjecture from [BFP] whose consideration gave much impetus to the present paper. We prove that a sign-regular l_∞-invertible matrix $A \in \mathbb{R}^{I \times J}$ must already carry $c_0(J)$ to $c_0(I)$, i.e., all its columns must vanish at infinity.

We use the following notation and conventions.

We use lowercase letters to denote elements of $\mathbb{R}^I$, i.e., real-valued functions (or sequences) on some integer interval (or, more generally, some integer set) I, with $x(i)$ the ith entry, or value at i, of the sequence x. If K is a subset of I, then x_K denotes the restriction of x to K. The ith unit sequence is denoted by e^i and defined by

$$e^i(j) := \delta_{ij}, \qquad \text{all } i, j.$$

Further, for $x \in \mathbb{R}^I$,

$$|x|(i) := |x(i)|, \qquad \text{all } i.$$

We use the customary abbreviation $l_\infty(I)$ for the normed linear space of all sequences x on I with

$$\|x\| := \|x\|_\infty := \sup_i |x(i)|$$

finite, and use $c_0(I)$ for its closed linear subspace of all bounded sequences x vanishing at infinity, i.e., for which $\{i \in I: |x(i)| \geqslant \alpha\}$ is finite for all positive α. We use $l_1(I)$ for the normed linear space of all absolutely summable sequences x on I, i.e., with

$$\|x\|_1 := \sum_i |x(i)|$$

finite, and we make use of the fact that $l_\infty(I)$ is the dual of $l_1(I)$, which, in turn, is the dual of $c_0(I)$.

Let also J be an integer interval (or, more generally, a set of integers). We are interested in $A \in \mathbb{R}^{I \times J}$ which carries $l_\infty(J)$ to $l_\infty(I)$, and we identify such a matrix A with the linear map from $l_\infty(J)$ to $l_\infty(I)$ induced by it. Such A is necessarily bounded, and we denote by $\|A\|$ its map norm. If A is one-one and onto, it is boundedly invertible. We will then say that A is l_∞-invertible, for short. As with sequences,

$$|A|(i,j) := |A(i,j)|, \qquad \text{all } i, j.$$

We write $A(\cdot, j)$ for the sequence which makes up the jth column of A, and $A(i, \cdot)$ for the ith row. We call the sequence $A(\cdot, \cdot + r)$ the rth band or diagonal of A.

If $K \times L \subseteq I \times J$, then we denote by $A_{K,L}$ the restriction of A to $K \times L$. At times, it is also convenient to use the alternate notation

$$A\begin{bmatrix} K \\ L \end{bmatrix} := A_{K,L}$$

and even

$$A[K] := A_{K,K}.$$

On replacing square brackets by round brackets, we get to the determinant:

$$A(\cdots) := \det A[\cdots].$$

Thus $A(i,i)$ and $A(i)$ both denote the ith diagonal entry of A—the latter involving an abuse of notation, since $A(\{i\})$ would have been correct.

We recall that A is sr_k (: = sign-regular of order k) if, for some $\varepsilon_1,\ldots,\varepsilon_k \in \{-1,1\}$ and for all $K \times L \subseteq I \times J$ with $|K| = |L| \leqslant k$,

$$\varepsilon_{|K|} A\binom{K}{L} \geqslant 0,$$

where $|K|$ denotes the cardinality of the index set K. If all $\varepsilon_i = 1$, then we use tp_k (: = totally positive of order k) instead of sr_k. If A is sr_k (tp_k) for all k, then A is simply called sr (tp). We intend to make use of Sylvester's determinant identity (see, e.g., [K, p. 3]) and Hadamard's inequality (see, e.g., [K, p. 88]).

1. EXISTENCE OF A MAIN DIAGONAL

We establish that any l_∞-invertible tp matrix A must have a main diagonal. By this we mean that A is the stable limit of its finite sections which are principal with respect to a particular diagonal. It is this diagonal that we then call the "main diagonal." We use the definite article here in anticipation of the uniqueness established in the next section.

THEOREM 1. *If the matrix $A \in \mathbb{R}^{I\times J}$ is tp and l_∞-invertible, then, for some r, and for all finite intervals $M \subseteq I$, $A_{M,M+r}$ is invertible and $(A_{M,M+r})^{-1}$ converges, monotonely in each entry, to A^{-1} as $M \to I$.*

Proof. Since A is onto, there exists a bounded sequence x such that

$$(Ax)(i) = (-1)^i, \qquad \text{all } i.$$

With this, the argument for Theorem 1 of [BFP] and, especially, the Claim proved there, provides us, for each finite interval $L \subseteq I$, with an index set $K \subseteq J$ such that $\|A_{L,K}{}^{-1}\| \leq \|x\|$. [In the Claim, choose $S = l_1$ (hence $S^* = l_\infty$), choose $I = L$, choose $u(i) = (-1)^i$, all i, and choose for s an extremal for $F_v : \Sigma\alpha_i A(i,\cdot) \mapsto \Sigma\alpha_i v(i)$ with $v = u$.]

Extend each $A_{L,K}{}^{-1}$ to a matrix $C^L \in \mathbb{R}^{J \times I}$ by taking its value to be zero off $K \times L$. Since, for each i, $\|C^L(\cdot,i)\| \leq \|x\|$, we are permitted to choose a sequence of integer intervals L approaching I for which, for each $i \in I$, $C^L(\cdot,i)$ converges weak* [i.e., pointwise on $l_1(J)$] to some $y^i \in l_\infty(J)$. Then, for each $m \in I$, the sequence $A(m,\cdot)$ being in $l_1(J)$, we have

$$(Ay^i)(m) = \lim_{L \to I} \sum_s A(m,s)C^L(s,i) = \lim_{L \to I} \begin{Bmatrix} e^i(m), & m \in L \\ 0, & m \notin L \end{Bmatrix} = e^i(m),$$

showing that $Ay^i = e^i$, and therefore $y^i = A^{-1}(\cdot,i)$, all i. We conclude that $C^L(\cdot,i)$ converges pointwise on l_1 and so, in particular, pointwise to $A^{-1}(\cdot,i)$.

The remainder of the proof relies on the following known fact.

LEMMA 1. *If $B \in \mathbb{R}^{n \times n}$ is tp and invertible, then, for any interval $M \subseteq \{1,\dots,n\}$, so is $C := B[M] = B_{M,M}$, and*

$$0 \leq (-)^{i+j}C^{-1}(i,j) \leq (-)^{i+j}B^{-1}(i,j), \qquad \textit{all} \quad i,j \in M.$$

The nonsingularity of C is a consequence of Hadamard's inequality. The rest of the lemma follows by repeated application of the special case $M = \{1,\dots,n-1\}$. For this case, use the checkerboard nature of the inverse of a tp matrix and the rank-one correction formula for an inverse (called the Woodbury-Sherman-Morrison formula by numerical analysts), which in this case states that

$$C^{-1}(i,j) = B^{-1}(i,j) - \frac{B^{-1}(i,n)B^{-1}(n,j)}{B^{-1}(n)}, \qquad i,j = 1,\dots,n-1.$$

Of course, this identity also follows from Cramer's rule and Sylvester's determinant identity using $B\begin{bmatrix} \backslash j, \backslash n \\ \backslash i, \backslash n \end{bmatrix}$ as pivot block.

As a corollary, we find that, for an invertible tp $B \in \mathbb{R}^{n \times n}$, we have $1/B(i) \leq B^{-1}(i)$, i.e., $1 \leq B(i)B^{-1}(i)$, all i, and hence

$$\|B^{-1}\|^{-1} \leq B(i), \qquad \text{all } i.$$

This shows that the (main) diagonal entries of an invertible tp matrix are bounded away from zero.

Returning now to the invertible matrices $A_{L,K}$ obtained earlier, we conclude that their (main) diagonal entries must all be bounded below by $1/\|x\|$. Assuming for simplicity of notation that $0 \in I$, it follows that, for every L containing 0, the index j^L for which $A(0, j^L)$ lies on the (main) diagonal of $A_{L,K}$ must come from the set $\{j \in J: A(0,j) \geq 1/\|x\|\}$, and this set is *finite* since $A(0,\cdot) \in l_1$. We are therefore permitted to choose a further subsequence of intervals L increasing to I (and all containing 0) for which $j^L = r$ for some fixed integer r. Since J and L are intervals and $A_{K,L}$ is square, this implies that $L + r \subseteq J$ for all such L.

Since $A_{L,K}{}^{-1}$ (or, more precisely, its extension C^L to all of $J \times I$) converges pointwise to A^{-1}, yet $C^L(j,\cdot) = 0$ for all $j \notin K$, it follows that every $j \in J$ is eventually in every K. Thus, for every interval $M \subseteq I$, there exists L such that $M + r \subseteq K$. By Lemma 1, $A_{M,M+r}$ is invertible and $\|A_{M,M+r}{}^{-1}\| \leq \|A_{L,K}{}^{-1}\| \leq \|x\|$. Further, by that lemma, $M \subseteq N$ implies that

$$0 \leq (-)^{i+j} A_{M,M+r}{}^{-1}(i+r,j) \leq (-)^{i+j} A_{N,N+r}{}^{-1}(i+r,j) \qquad \text{for} \quad i,j \in M.$$

We conclude that $A_{M,M+r}{}^{-1}$ (or, more precisely, its extension to all of $J \times I$) converges, monotonely in each entry, to a bounded matrix $C \in R^{J \times I}$, which, by a repeat of the earlier argument, is necessarily A^{-1}. ∎

COROLLARY 1. *The main diagonal of A is bounded below by $\|A^{-1}\|^{-1}$. More precisely, $A(i, i+r)A^{-1}(i+r, i) \geq 1$ for all i.*

COROLLARY 2. *The matrix $\left((-)^{i+j}A^{-1}(i+r,j)\right)$ is tp.*

The second corollary improves on Proposition 1 of [BFP] in which the same conclusion is only reached under the additional assumption that the columns of A are all in $c_0(I)$. (See also Section 5.)

Theorem 1 implies that *an l_∞-invertible tp matrix A has a main diagonal* in the sense introduced in $[B_1]$: For some r and for all sufficiently large intervals M, $A_{M,M+r}$ is invertible and $\limsup_{M \to I} \|A_{M,M+r}{}^{-1}\| < \infty$. This means that one can approximate the solution x of the nonfinite linear system $Ax = b$ by the solution $x^M := A_{M,M+r}{}^{-1} b_M$ of the finite system $A_{M,M+r} x^M = b_M$, at least when b and x are in c_0. We show in Section 5 that $x \in c_0$ if and only if $b \in c_0$.

Finally, Theorem 1 implies that $I + r = J$ for some r. Hence it is no restriction to assume, after a shift, that $I = J$. If I has a first or a last element, this then forces band 0 to be the main diagonal.

2. EXISTENCE OF STABLE *LDU* FACTORIZATION AND UNIQUENESS OF MAIN DIAGONAL

The *LDU* factorization of a matrix A of order n is a basic tool in the solution of linear algebraic systems. The matrix A is written, if possible, in the form LDU, with L lower triangular, D diagonal, and U upper triangular. Both L and U are required to be *unit* triangular, i.e., to have all their diagonal entries equal to 1, and this insures uniqueness of the factorization (see the proof of Lemma 2 below). Such a factorization can be obtained in case all upper left principal minors of A are nonzero. By Hadamard's inequality, this condition is satisfied for an invertible tp A. Further, if $LDU = A$, then $(DU)(i,\cdot)$ is the particular linear combination of the first i rows of A which vanishes at $1,2,\dots,i-1$ and in which $A(i,\cdot)$ has coefficient 1. Therefore

$$(DU)(i,\cdot) = \frac{A\begin{pmatrix}1,\dots,i-1, & i\\ & \cdot\end{pmatrix}}{A(1,\dots,i-1)},$$

since the right-hand side has the same properties. Also,

$$A(1,\dots,i) = L(1,\dots,i)D(1,\dots,i)U(1,\dots,i) = D(1)\cdots D(i);$$

hence

$$D(i) = \frac{A(1,\dots,i)}{A(1,\dots,i-1)}.$$

Therefore, finally,

$$L(\cdot,j) = \frac{A\begin{pmatrix}1,\dots,j-1, & \cdot\\ & j\end{pmatrix}}{A(1,\dots,j)}.$$

We have recalled these well-known facts to facilitate discussion of an *LDU* factorization of a *nonfinite* matrix. We intend to obtain such a factorization by a limiting process and need therefore to consider the behavior of the *LDU* factorization of A as we add rows and columns to A.

Let now $A \in \mathbb{R}^{I\times J}$. We say that the matrices L, D, and U provide a *(normalized) LDU factorization for* A provided $A = LDU$, L is unit lower triangular with band 0 as its rightmost (nontrivial) band, D has band 0 as its only nontrivial band, and U is unit upper triangular. We call such a factorization (boundedly) *invertible* if each factor is bounded and boundedly invertible, with L^{-1} and U^{-1} being again triangular.

We are not certain that having a "good" diagonal (i.e., a diagonal all of whose principal minors are nonzero) is sufficient for having an LDU factorization; it may not even be sufficient to have a main diagonal. But if A is also tp, then having a main diagonal is sufficient for having a boundedly invertible LDU factorization. This insures that the main diagonal is unique, as we will see.

LEMMA 2. *If L, D, U and L_1, D_1, U_1 are both invertible normalized LDU factorizations for A (as maps from l_∞ to l_∞), then $L = L_1$, $D = D_1$, $U = U_1$.*

Proof. Let band r be the leftmost band of U, and band r_1 the leftmost band of U_1, and assume without loss that $r \leqslant r_1$. Then, by assumption,

$$L_1^{-1} L D = D_1 U_1 U^{-1},$$

and the left-hand side is lower triangular with band 0 as its rightmost band, and equal to the interesting diagonal of D, while the right-hand side is upper triangular with band $r_1 - r$ as its leftmost band, and equal to the interesting diagonal of D_1. Since $r_1 \geqslant r$, it follows that $r_1 = r$, hence $D = D_1$, and further $L_1^{-1} L = 1 = U_1 U^{-1}$. ∎

THEOREM 2. *If A is l_∞-invertible and tp, and hence has a main diagonal, say band r, then A has an invertible normalized LDU factorization L, D, U, with band r the leftmost band of U.*

Proof. Assume first that $r = 0$. Further, assume without loss that $A_n := A[-n, \ldots, n]$ is a typical principal section for that main diagonal of A. Let L_n, D_n, U_n be the LDU factorization for A_n. Then

$$D_n(i) = \frac{A(-n, \ldots, i)}{A(-n, \ldots, i-1)} = \frac{1}{A[-n, \ldots, i]^{-1}(i)}$$

and, by Lemma 1 and Theorem 1,

$$A[-n, \ldots, i]^{-1}(i) \leqslant A_n^{-1}(i) \leqslant A^{-1}(i),$$

and $A[-n,\dots,i]^{-1}(i)$ increases as n grows. We conclude that

$$\|D_n^{-1}\| \leqslant \|A_n^{-1}\|,$$

and D_n decreases monotonely to some invertible diagonal matrix D for which $D(i) \geqslant \|A^{-1}\|^{-1}$, all i. Therefore, D is boundedly invertible. Next consider D_nU_n. We saw earlier in this section that

$$(D_nU_n)(i,j) = \frac{A\begin{pmatrix} -n,\dots,i-1, \begin{smallmatrix} i \\ j \end{smallmatrix} \end{pmatrix}}{A(-n,\dots,i-1)}.$$

Use Sylvester's determinant identity on

$$A\begin{bmatrix} -n-1,\dots,i-1, \begin{smallmatrix} i \\ j \end{smallmatrix} \end{bmatrix}$$

with pivot block $A[-n,\dots,i-1]$ to conclude that

$$\begin{aligned} A\begin{pmatrix} -n-1,\dots,i-1, \begin{smallmatrix} i \\ j \end{smallmatrix} \end{pmatrix} & A(-n,\dots,i-1) \\ &= A(-n-1,\dots,i-1)A\begin{pmatrix} -n,\dots,i-1, \begin{smallmatrix} i \\ j \end{smallmatrix} \end{pmatrix} \\ &\quad - A\begin{pmatrix} -n,\dots,i \\ -n-1,\dots,i-1 \end{pmatrix} A\begin{pmatrix} -n-1,\dots,i-1 \\ -n,\dots,j \end{pmatrix}, \end{aligned}$$

which, by tp, implies that

$$0 \leqslant (D_{n+1}U_{n+1})(i,j) \leqslant (D_nU_n)(i,j).$$

We conclude that D_nU_n decreases monotonely to some matrix V, necessarily tp and upper triangular; therefore U_n converges to the tp unit upper triangular matrix $U := D^{-1}V$. Further, $D_nU_n \leqslant A_n$; hence

$$\|U_n\| \leqslant \|A_n\|\|D_n^{-1}\| \leqslant \|A_n\|\|A_n^{-1}\|; \qquad \text{thus} \quad \|U\| \leqslant \|A\|\|A^{-1}\|.$$

The corresponding argument shows that L_n converges to some tp unit lower triangular matrix L with $\|L\| \leqslant \|A\|\,\|A^{-1}\|$. Finally, $A_n^{-1} = U_n^{-1}D_n^{-1}L_n^{-1}$;

hence also $\|L_n^{-1}\|, \|U_n^{-1}\| \leqslant \|A_n\| \|A_n^{-1}\| \leqslant \|A\| \|A^{-1}\|$. Indeed, $|A_n^{-1}|$ is again tp and, because of the checkerboard nature of inverses of tp matrices, $|U_n^{-1}|$, $|D_n^{-1}|$, and $|L_n^{-1}|$ give the UDL factorization for $|A_n^{-1}|$ (i.e., the factorization obtained by starting Gauss elimination at the lower right corner rather than the upper left) and hence satisfy the same bounds. For, if $B^*(i,j) := B(-i,-j)$, all i,j—i.e., the order of both the rows and the columns is reversed—then tp is preserved as well as the max-row-sum norm, while, with M, E, V an LDU factorization for B, the matrices M^*, E^*, and V^* provide a UDL factorization for B^*. It follows that the inverse of any principal section of L_n or U_n is also bounded by $\|A\| \|A^{-1}\|$ (since it is the corresponding principal section of L_n^{-1} or U_n^{-1}). Since L_n and U_n converge pointwise to L and U, respectively, this implies that any section of L or U principal with respect to band 0 is invertible, with the inverse of that section bounded by $\|A\| \|A^{-1}\|$. This shows that also L and U (and, of course D) have band 0 as main diagonal, and are, in particular, invertible.

If now $r \neq 0$, then we obtain such an LDU factorization (with each factor having band 0 as main diagonal) for the matrix AE^{-r}, with E the shift, $(Ea)(i) := a(i+1)$, all i. But then $A = LD(UE^r)$ does it. ■

COROLLARY. *If also band s of A is a main diagonal for A, then $s = r$.*

Proof. By Theorem 2, A has an invertible normalized LDU factorization with band r as the leftmost nontrivial band for the upper triangular factor, as well as one with band s as the leftmost nontrivial band of the upper triangular factor. But, by Lemma 2, these two factors must be the same. ■

3. THE SHAPE OF A NONFINITE TP MATRIX

The fact that any invertible tp matrix $A \in \mathbb{R}^{I \times J}$ has a main diagonal implies that $I + r = J$ for some r. In particular, it rules out certain combinations of I and J which might, offhand, be possible. For example, the choice $I = -\mathbb{N}$ is not possible in conjunction with the choice $J = \mathbb{N}$. If we merely know that A is onto or one-one, certain other combinations, though certainly not all, are possible.

In this context, the only properties of the integer intervals I and J which matter are whether or not they have a first and/or last element. In effect, there are just four choices for I and J:

$$\text{finite}, \mathbb{N}, -\mathbb{N}, \mathbb{Z}.$$

If J is finite, then A cannot be onto unless $|J| \geqslant |I|$. The same conclusion holds in spirit when J is nonfinite.

THEOREM 3. *Let $A \in \mathbb{R}^{I \times J}$ be tp and carry $l_\infty(J)$ onto $l_\infty(I)$. Let I and J each be one of $\mathbb{N}$, $-\mathbb{N}$, or $\mathbb{Z}$. Then $I \subseteq J$.*

Proof. There is nothing to prove unless $J = \mathbb{N}$ or $-\mathbb{N}$. Assume without loss of generality that $J = \mathbb{N}$. Assume by way of contradiction that $I = -\mathbb{N}$ or $\mathbb{Z}$. A being onto, we may pick a bounded x for which $Ax = \left((-1)^i\right)$. Pick any $i \in I$. Then, since $A(i, \cdot) \in l_1(J)$, we may pick n so that

$$j > n \quad \text{implies} \quad A(i,j) < \|x\|^{-1}.$$

Since $I = -\mathbb{N}$ or $\mathbb{Z}$, the interval $L := [i-n, i]$ is in I. By [BFP] (as used at the beginning of the proof for Theorem 1 above), we may pick a subset K of J for which $\|A_{L,K}{}^{-1}\| \leqslant \|x\|$. But then, with $K =: \{k_0 < \cdots < k_n\}$, we must have

$$A(i, k_n) \geqslant \|x\|^{-1}, \qquad \text{yet} \quad k_n > n,$$

and this is a contradiction to our choice of n. ∎

4. DECAY AWAY FROM THE MAIN DIAGONAL

The fact that an l_∞-invertible tp matrix A has a main diagonal insures, by Corollary 1 of Theorem 1, that it has a diagonal which is bounded away from zero. This implies at least linear decay away from that diagonal. More precisely, we have the following

THEOREM 4. *Let $A \in \mathbb{R}^{I \times I}$ be tp$_2$ and carry l_∞ to itself. If $d := \inf A(i) > 0$, then $\sup_i A(i, i+s) = O(1/|s|)$ as $|s| \to \infty$.*

Proof. Let $s \geqslant 0$. Since A is tp$_2$, we have

$$A(i,i+k)A(i+k,i+s) \geqslant A(i,i+s)A(i+k,i+k), \qquad k = 1,\ldots,s,$$

or

$$\frac{\sum_{k=1}^{s} A(i,i+k) \max_{k=1,\ldots,s} A(i+k,i+s)}{\min_{k=1,\ldots,s} A(i+k)} \geqslant sA(i,i+s).$$

This implies that $\|A\|_{1,\infty}\|A\|/d \geq |s| \sup_i A(i, i+s)$, using the fact that $\max_{i,j}|A(i,j)| = \|A\|_{1,\infty}$ while $\sup_i \Sigma_k |A(i,k)| = \|A\|$. An analogous argument proves the inequality for negative s. ■

COROLLARY. *If $A \in \mathbb{R}^{I\times J}$ is tp and l_∞-invertible, then, for some r,*

$$A(i, i+s) \leq \frac{\|A\|_{1,\infty}\kappa(A)}{|s-r|}, \qquad \textit{all } i, s.$$

Proof. By Corollary 1 to Theorem 1, we may choose $d = \|A^{-1}\|^{-1}$; hence $\|A\|/d = \|A\|\,\|A^{-1}\| =: \kappa(A)$. ■

In case A^{-1} is also banded, we know from [D] that the entries of A decay exponentially away from the main diagonal of A. Explicitly [B_1],

$$A(i, i+s) \leq \mathrm{const}_{\kappa(A), m}\left(\frac{\kappa(A)-1}{\kappa(A)+1}\right)^{|s-r|/m}$$

if A^{-1} is m-banded. In such a case, A^{-1} also decays exponentially, in a trivial way. It would be nice to know whether, in fact, *all* l_∞-invertible tp matrices decay exponentially away from their main diagonal.

5. INVERTIBLE SR MATRICES

It is not possible to extend the above results concerning main diagonals to invertible sr matrices. This is illustrated by the particular matrix B obtained from the identity matrix in $\mathbb{R}^{\mathbb{Z}\times\mathbb{Z}}$ by reversing the order of the rows. Every band has singular principal minors of arbitrarily large orders. Of course, one could weaken the notion of main diagonal to demand only that there be *some* sequence of principal submatrices converging stably to A. It is possible to show that an l_∞-invertible sr matrix has such main diagonals. For the particular matrix B, though, *every* band becomes main in this weaker sense, i.e., the distinction becomes empty. This is, of course, not surprising, since for B there is really nothing to distinguish one band from another.

Some of the work reported on in the preceding sections was initially based on the realization that the following conjecture is true.

CONJECTURE [BFP]. *The columns of a tp l_∞-invertible matrix are already in c_0.*

Indeed, this conjecture is now a simple consequence of Theorem 4. It is pointed out in [BFP] that the discussion there of the inverse of a sr matrix could be considerably shortened if this conjecture were to hold for sr matrices. We now prove the following generalization of the conjecture.

THEOREM 5. *Let $S \subseteq \mathbb{R}^J$ be a normed sequence space which contains the unit vector e^j for all $j \in J$ and contains $|x|$ if it contains x, and let $A: S \to l_\infty(I)$ be bounded and sr. If Ax uniformly alternates for some $x \in S$, then $Ac_0 \subseteq c_0$, i.e., the columns of A are already in c_0.*

Proof. We have to prove that

$$K := \{i \in I : |A(i,k)| \geq \alpha\}$$

is finite for all $k \in J$ and all $\alpha > 0$. By considering $-A$ and/or inverting the order of the rows of A if necessary, we can insure that A is tp_2. Then, for $j \leq k$, $i' \leq i$, we have

$$A(i,j)A(i',k) \leq A(i',j)A(i,k)$$

or, for $i' \in K$,

$$A(i,j) \leq \frac{A(i,k)}{\alpha} A(i',j).$$

Therefore,

$$\sum_{j \leq k} |x(j)| \sup_{i \geq i'} A(i,j) \leq \frac{\|A(\cdot,k)\|}{\alpha} \sum_{j \leq k} A(i',j)|x(j)| \leq \frac{\|A(\cdot,k)\|}{\alpha} \|A|x|\|$$

hence

$$\sum_{j \leq k} |x(j)| \, \|A(\cdot,j)_{[K]}\| < \infty,$$

with

$$[K] := [\inf K, \sup K].$$

By inverting the order of rows and of columns of A (which preserves tp_2), we obtain the same inequality for the sum over $j \geqslant k$; hence altogether

$$\sum_{j \in J} \| A(\cdot, j)_{[K]} \| \, |x(j)| < \infty.$$

Since $\Sigma_{j \in J} A(\cdot, j)x(j)$ uniformly alternates, this implies that we can find some *finite* index set L such that $A_{[K], L}$ already maps x_L to a uniformly alternating sequence, and this, by a standard sr result (see, e.g., [K, Chapter 5, §1] or the beginning of the proof of Theorem 1 in [BFP]), implies that $|[K]| \leqslant |L|$, and thus $|K| < \infty$. ∎

This theorem makes it possible to give a shorter proof of

THEOREM 2 of [BFP]. *If* $A \in \mathbb{R}^{I \times J}$ *is sr and* l_∞*-invertible, then* $D^J A^{-1} D^I$ *is also sr.*

by mimicking the proof of Proposition 1 of [BFP], since the assumptions allow the conclusion that A maps $c_0(J)$ to $c_0(I)$. Hence Corollary 2 to Theorem 1 of [BFP] shows that A^{-1} *is the pointwise limit of certain matrices* E^L *as the index interval* L *converges to* I. Explicitly, E^L equals ${A_{K,L}}^{-1}$ on $K \times L$ and vanishes off $K \times L$, with K some index set depending on L. This implies that E^L is sr, and hence so is its pointwise limit A^{-1}.

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