STRICTLY HERMITIAN POSITIVE DEFINITE FUNCTIONS

By

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Abstract. Let H be any complex inner product space with inner product $< \cdot, \cdot >$. We say that $f : \mathbb{C} \to \mathbb{C}$ is Hermitian positive definite on H if the matrix

$$(*) \qquad (f(< \mathbf{z}^r, \mathbf{z}^s >))_{r,s=1}^n$$

is Hermitian positive definite for all choice of $\mathbf{z}^1, \ldots, \mathbf{z}^n$ in H for all n. It is strictly Hermitian positive definite if the matrix $(*)$ is also non-singular for any choice of distinct $\mathbf{z}^1, \ldots, \mathbf{z}^n$ in H. In this article, we prove that if $\dim H \geq 3$, then f is Hermitian positive definite on H if and only if

$$(**) \qquad f(z) = \sum_{k,\ell=0}^{\infty} b_{k,\ell} z^k \overline{z}^\ell,$$

where $b_{k,\ell} \geq 0$ for all k, ℓ in $\mathbb{Z}_+$, and the series converges for all z in $\mathbb{C}$. We also prove that f of the form $(**)$ is strictly Hermitian positive definite on any H if and only if the set

$$J = \{(k, \ell) : b_{k,\ell} > 0\}$$

is such that $(0, 0) \in J$, and every arithmetic sequence in $\mathbb{Z}$ intersects the values $\{k - \ell : (k, \ell) \in J\}$ an infinite number of times.

1 Introduction

A function (or kernel) K mapping the product space $Z \times Z$ into $\mathbb{C}$ is termed *Hermitian positive definite* if

$$(1.1) \qquad \sum_{r,s=1}^{n} c_r K(\mathbf{z}^r, \mathbf{z}^s) \overline{c}_s \geq 0,$$

for every choice of $\mathbf{z}^1, \ldots, \mathbf{z}^n \in Z$, $c_1, \ldots, c_n \in \mathbb{C}$, and all $n \in \mathbb{N}$. We say that the function K is *positive definite* if $K : X \times X \to \mathbb{R}$ and

$$(1.2) \qquad \sum_{r,s=1}^{n} c_r K(\mathbf{x}^r, \mathbf{x}^s) c_s \geq 0,$$

for every choice of $\mathbf{x}^1, \ldots, \mathbf{x}^n \in X$, $c_1, \ldots, c_n \in \mathbb{R}$, and all $n \in \mathbb{N}$. These functions seem to have been first considered by Mercer [20] in connection with integral

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equations. We use the term *strict* if strict inequalities occur in (1.1) and (1.2) for every choice of distinct $\mathbf{z}^1, \ldots, \mathbf{z}^n \in Z$ or $\mathbf{x}^1, \ldots, \mathbf{x}^n \in X$, as appropriate, and nonzero $c_1, \ldots, c_n$.

Hermitian positive definite and positive definite functions have been much studied in various contexts and guises. One of the first characterizations of sets of Hermitian positive definite functions is Bochner's Theorem; see, e.g., [4], [9, Sect. 6.5] and [16, Chap. 10]. The function

$$K(x, y) = f(x - y),$$

where $X = \mathbb{R}$ and f is continuous at 0, is Hermitian positive definite if and only if f is of the form

$$f(t) = \int_{-\infty}^{\infty} e^{itx} d\mu(x),$$

where $d\mu$ is a finite, nonnegative measure on $\mathbb{R}$. (Essentially the same result holds for $X = \mathbb{R}^n$; see Bochner [4].)

In the theory of radial basis functions, there has been much interest in characterizing positive definite functions of the form

$$K(\mathbf{x}, \mathbf{y}) = f(\|\mathbf{x} - \mathbf{y}\|),$$

where $\|\cdot\|$ is some norm. There is a large literature connected with such problems. Schoenberg [26] characterized such functions where $\|\cdot\|$ is the Euclidean norm on $X = \mathbb{R}^n$. For the analogous problem with the ℓ_p norm $\|\cdot\|_p$ on $\mathbb{R}^n$, see [12] and the references therein. Schoenberg [27] also characterized positive definite functions of the form

$$K(\mathbf{x}, \mathbf{y}) = f(\|\mathbf{x} - \mathbf{y}\|),$$

where $\|\cdot\|$ is again the Euclidean norm on $\mathbb{R}^n$ but $X = S^{n-1}$ is the unit sphere in $\mathbb{R}^n$, $n \geq 2$. As there is a simple 1-1 correspondence between $\|\mathbf{x} - \mathbf{y}\|$ and the standard inner product $< \mathbf{x}, \mathbf{y} >$ for all $\mathbf{x}, \mathbf{y} \in S^{n-1}$, it is more convenient to consider

$$K(\mathbf{x}, \mathbf{y}) = g(< \mathbf{x}, \mathbf{y} >).$$

Schoenberg proved that for $g \in C[-1, 1]$, the kernel K is positive definite if and only if g is of the form

$$g(t) = \sum_{r=0}^{\infty} a_r P_r^{\lambda}(t), \tag{1.3}$$

where $a_r \geq 0$ for all r, $\sum_{r=0}^{\infty} a_r P_r^{\lambda}(1) < \infty$ and the P_r^{λ} are the Gegenbauer (ultraspherical) polynomials with $\lambda = (n-2)/2$. As these kernels are often used for interpolation, there has been much effort put into determining exact conditions for when $g(< \cdot, \cdot >)$ is strictly positive definite. It was recently proven by Chen, Menegatto and Sun [7] that for $n \geq 3$, a function of the form (1.3) is strictly positive definite if and only if the set

$$\{r : a_r > 0\}$$

contains an infinite number of even and an infinite number of odd integers.

Hermitian positive definite and positive definite functions also arise in the study of reproducing kernels; see, e.g., [1], [21] and [10, Chap. X]. Each reproducing kernel is a (Hermitian) positive definite function and vice versa. Strict (Hermitian) positive definiteness is also desired when considering the reproducing kernel, since it is equivalent to the linear independence of point functionals in the associated reproducing kernel space.

A corollary (quite literally, a footnote) in [27] is the result that if for every positive definite matrix $(a_{rs})_{r,s=1}^n$, all $n \in \mathbb{N}$, the function $f : \mathbb{R} \to \mathbb{R}$ is such that

$$(f(a_{rs}))_{r,s=1}^n$$

is also positive definite, then f is necessarily of the form

$$f(t) = \sum_{k=0}^{\infty} b_k t^k, \tag{1.4}$$

where the $b_k \geq 0$ for all k and the series converges for all $t \in \mathbb{R}$. That is, f is real entire and absolutely monotone on $\mathbb{R}_+$. The converse direction is a simple consequence of the fact that the class of positive definite functions is a positive cone and the Schur Product Theorem and is an exercise in Pólya–Szegö [24, p. 101]. For other approaches and generalizations, see [25], [8], [2, p. 59] and [11]. The same problem for matrices of a fixed size, i.e., all $n \times n$ matrices for a fixed n, seems much more difficult; see, e.g., [14].

If H is a real inner product space of dimension m, then the set of matrices $(< \mathbf{x}^r, \mathbf{x}^s >)_{r,s=1}^n$, all n, obtained by choosing arbitrary $\mathbf{x}^r$ in H is exactly the set of all positive definite matrices of rank at most m. Thus, if $\dim H = \infty$, (1.4) provides the exact characterization of all functions f such that

$$f(< \mathbf{x}, \mathbf{y} >) \tag{1.5}$$

is positive definite. In [19], this result was significantly improved. They proved that if $\dim H \geq 2$, then $f : \mathbb{R} \to \mathbb{R}$ is positive definite on H (in the sense of (1.5)) if and only if f is of the form (1.4). If $\dim H = 1$ then this result is not valid. In

[23] it was shown that assuming that f has the form (1.4), f (as given in (1.5)) is strictly positive definite on H if and only if the set

$$\{k \,:\, b_k > 0\}$$

contains the index 0 plus an infinite number of even integers and an infinite number of odd integers.

In this paper, we generalize both of these results to $\mathbb{C}$. In [13], see also [2, p. 171] and [11], it was proven that if $f : \mathbb{C} \to \mathbb{C}$ and for all Hermitian positive definite matrices $(a_{rs})_{r,s=1}^n$ (and all n) the matrix $(f(a_{rs}))_{r,s=1}^n$ is Hermitian positive definite, then f is necessarily of the form

$$f(z) = \sum_{k,\ell=0}^{\infty} b_{k,\ell} z^k \overline{z}^\ell \tag{1.6}$$

where the $b_{k,\ell} \geq 0$ for all $k, \ell \in \mathbb{Z}_+$, and the series converges for all $z \in \mathbb{C}$. If H is a complex inner product space of dimension m, then the set of matrices $(< \mathbf{z}^r, \mathbf{z}^s >)_{r,s=1}^n$, all n, obtained by choosing arbitrary $\mathbf{z}^r$ in H is exactly the set of all positive definite matrices of rank at most m.

In Section 2, we prove that this characterization (1.6) remains valid for every complex inner product space H of dimension at least 3. That is, if dim $H \geq 3$ and the matrix

$$(f(< \mathbf{z}^r, \mathbf{z}^s >))_{r,s=1}^n$$

is Hermitian positive definite for all choices of $\mathbf{z}^1, \ldots, \mathbf{z}^n$ in H, all $n \in \mathbb{N}$, then f is necessarily of the form (1.6). For $m = 1$, this result is not valid. We do not know what happens when $m = 2$. Our proof of this result uses a generalization of the method of proof in [11] and also uses the result of [19] and an extension theorem for separately real analytic functions.

In Section 3, we assume f is of the form (1.6) and prove the following appealing characterization of strictly Hermitian positive definite functions. Let

$$J = \{(k, \ell) \,:\, b_{k,\ell} > 0\}.$$

We show that $f(< \cdot, \cdot >)$ is a strictly Hermitian positive definite function if and only if $(0, 0) \in J$ and

$$|\{k - \ell :\, (k, \ell) \in J\,,\, k - \ell = q \,(\mathrm{mod}\, p)\}| = \infty$$

for all choices of $p \in \mathbb{N}$ and $q \in \{0, 1, \ldots, p - 1\}$. The latter condition simply says that every arithmetic sequence in $\mathbb{Z}$ intersects the values $\{k - \ell \,:\, (k, \ell) \in J\}$ an infinite number of times. Our proof of this result utilizes the Skolem–Mahler–Lech Theorem from number theory and a generalization thereof due to M. Laurent.

The paper [30] contains some sufficient conditions for when a function of the form (1.6) generates a strictly Hermitian positive definite function of a fixed order n on the complex unit sphere S^∞ of ℓ^2.

2 Hermitian positive definite functions

In this section, we prove the following.

Theorem 2.1. *Let H be a complex inner product space with* $\dim H \geq 3$. *Then for every choice of $\mathbf{z}^1, \ldots, \mathbf{z}^n \in H$, the matrix*

$$(f(<\mathbf{z}^r, \mathbf{z}^s>))_{r,s=1}^n$$

is Hermitian positive definite if and only if f is of the form

$$f(z) = \sum_{k,\ell=0}^{\infty} b_{k,\ell} z^k \overline{z}^\ell,$$

where $b_{k,\ell} \geq 0$, all $k, \ell \in \mathbb{Z}_+$, and the series converges for all $z \in \mathbb{C}$.

Let $\mathcal{H}_m(\mathbb{C})$ denote the set of all Hermitian positive definite matrices of rank at most m, and $\mathcal{H}_m(\mathbb{R})$ the set of all (real) positive definite matrices of rank at most m. If H is a complex inner product space of dimension m, then for any $\mathbf{z}^1, \ldots, \mathbf{z}^n \in H$, the matrix

$$(<\mathbf{z}^r, \mathbf{z}^s>)_{r,s=1}^n$$

is in $\mathcal{H}_m(\mathbb{C})$. The converse direction also holds. That is, for every n and any Hermitian positive definite matrix $A = (a_{rs})_{r,s=1}^n$ of rank at most m, there exist $\mathbf{z}^1, \ldots, \mathbf{z}^n \in H$ such that

$$a_{rs} = <\mathbf{z}^r, \mathbf{z}^s>, \qquad r, s = 1, \ldots, n.$$

Thus characterizing all Hermitian positive definite functions on H is equivalent to characterizing those $f : \mathbb{C} \to \mathbb{C}$ for which

$$(f(a_{rs}))_{r,s=1}^n$$

is Hermitian positive definite for all possible Hermitian positive definite matrices $A = (a_{rs})_{r,s=1}^n$ of rank at most $\dim H$. This same result holds over the reals. We denote by $\mathcal{F}_m(\mathbb{C})$ the class of all Hermitian positive definite functions on Hermitian positive definite matrices of rank at most m, and by $\mathcal{F}_m(\mathbb{R})$ the analogous set with respect to the reals.

For $m = 1$, the claim of Theorem 2.1 is not valid. For example, it is readily verified that

$$f(z) = \begin{cases} |z|^\alpha, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is in $\mathcal{F}_1(\mathbb{C})$ for every $\alpha \in \mathbb{R}$. It is possible that Theorem 2.1 is valid for $m = 2$. This case remains open.

It is convenient to divide the proof of Theorem 2.1 into a series of steps. Sufficiency is simple and known. We include a proof thereof for completeness. This and the next lemma are also to be found in [11].

Lemma 2.2. *Assume $f, g \in \mathcal{F}_m(\mathbb{C})$ and $a, b \geq 0$. Then*

(i) $af + bg \in \mathcal{F}_m(\mathbb{C})$;

(ii) $f \cdot g \in \mathcal{F}_m(\mathbb{C})$;

(iii) *the three functions* $1, z, \overline{z}$ *are in* $\mathcal{F}_m(\mathbb{C})$;

(iv) $\overline{f} \in \mathcal{F}_m(\mathbb{C})$;

(v) $\mathcal{F}_m(\mathbb{C})$ *is closed under pointwise convergence.*

Proof. (i) follows from the fact that $\mathcal{H}_\infty(\mathbb{C})$, the set of all Hermitian positive definite matrices, is a positive cone. (v) is a consequence of the closure of $\mathcal{H}_\infty(\mathbb{C})$. If the $n \times n$ matrix A is in $\mathcal{H}_m(\mathbb{C})$, then the $n \times n$ matrices $f(A)$ and $g(A)$ ($f(A) = (f(a_{rs}))_{r,s=1}^n$) are in $\mathcal{H}_\infty(\mathbb{C})$. (ii) is a consequence of the Schur Product Theorem; see, e.g., [15, p. 309]. That is, for $A \in \mathcal{H}_m(\mathbb{C})$ the matrix $f(A)g(A) = (f(a_{rs})g(a_{rs}))_{r,s=1}^n$ is in $\mathcal{H}_\infty(\mathbb{C})$. (iii) and (iv) essentially follow by definition. □

The sufficiency part of Theorem 2.1 immediately follows from Lemma 2.2. We now consider the necessity.

Lemma 2.3. *Let $f \in \mathcal{F}_m(\mathbb{C})$, and set*

$$f(z) = u(z) + iv(z)$$

where $u(z) = \operatorname{Re} f(z)$ and $v(z) = \operatorname{Im} f(z)$. Then for $z \in \mathbb{C}$,

(i) $f(\overline{z}) = \overline{f(z)}$;

(ii) $u(\overline{z}) = u(z)$;

(iii) $v(\overline{z}) = -v(z)$;

(iv) $u \in \mathcal{F}_m(\mathbb{C})$;

(v) $f|_{\mathbb{R}} = u|_{\mathbb{R}} \in \mathcal{F}_m(\mathbb{R})$.

Proof. (i) If $A \in \mathcal{H}_m(\mathbb{C})$, then $\overline{a}_{rs} = a_{sr}$. As $f(A) \in \mathcal{H}_\infty(\mathbb{C})$, we also have $f(a_{sr}) = \overline{f(a_{rs})}$. Thus

$$f(\overline{a}_{rs}) = f(a_{sr}) = \overline{f(a_{rs})}$$

and $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$. (ii) and (iii) are consequences of (i). To prove (iv), we note that from Lemma 2.2 (iv), we have $\overline{f} \in \mathcal{F}_m(\mathbb{C})$. Thus from Lemma 2.2 (i),

$$u = \frac{f + \overline{f}}{2} \in \mathcal{F}_m(\mathbb{C}).$$

To prove (v), note that $f|_{\mathbb{R}} = u|_{\mathbb{R}}$ as a consequence of (iii). Now $u|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ and $u \in \mathcal{F}_m(\mathbb{C})$. Thus $u|_{\mathbb{R}} \in \mathcal{F}_m(\mathbb{R})$. □

Lemma 2.4. *Let $f \in \mathcal{F}_m(\mathbb{C})$, $m \geq 2$. Then for any $a, b \in \mathbb{C}$,*

$$g(z) = f(z+1) + f(|a|^2 z + |b|^2) \pm \left[f(az+b) + f(\overline{a}z + \overline{b})\right]$$

is in $\mathcal{F}_{m-1}(\mathbb{C})$.

Proof. Assume A is an $n \times n$ matrix in $\mathcal{H}_{m-1}(\mathbb{C})$. Then for any $a \in \mathbb{C}$, the $2n \times 2n$ matrix

$$\begin{pmatrix} A & aA \\ \overline{a}A & |a|^2 A \end{pmatrix}$$

is also in $\mathcal{H}_{m-1}(\mathbb{C})$.

Let J denote the $n \times n$ matrix all of whose entries are one. Then $J \in \mathcal{H}_1(\mathbb{C})$, and the $2n \times 2n$ matrix

$$\begin{pmatrix} J & bJ \\ \overline{b}J & |b|^2 J \end{pmatrix}$$

is also in $\mathcal{H}_1(\mathbb{C})$ for any $b \in \mathbb{C}$. Thus

$$\begin{pmatrix} A+J & aA+bJ \\ \overline{a}A+\overline{b}J & |a|^2 A + |b|^2 J \end{pmatrix}$$

is in $\mathcal{H}_m(\mathbb{C})$. As $f \in \mathcal{F}_m(\mathbb{C})$, it follows that

$$\begin{pmatrix} f(A+J) & f(aA+bJ) \\ f(\overline{a}A+\overline{b}J) & f(|a|^2 A + |b|^2 J) \end{pmatrix}$$

is in $\mathcal{H}_\infty(\mathbb{C})$.

For any $\mathbf{c}^T = (c_1, \ldots, c_n) \in \mathbb{C}^n$, set

$$(\mathbf{c}^T, \pm\mathbf{c}^T) = (c_1, \ldots, c_n, \pm c_1, \ldots, \pm c_n) \in \mathbb{C}^{2n}.$$

Thus

$$(\mathbf{c}^T, \pm\mathbf{c}^T) \begin{pmatrix} f(A+J) & f(aA+bJ) \\ f(\bar{a}A+\bar{b}J) & f(|a|^2A+|b|^2J) \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \pm\mathbf{c} \end{pmatrix} \geq 0.$$

That is,

$$\mathbf{c}^T \left[f(A+J) + f(|a|^2A+|b|^2J) \pm \left[f(aA+bJ) + f(\bar{a}A+\bar{b}J)\right]\right] \mathbf{c} \geq 0,$$

which implies that

$$g(z) = f(z+1) + f(|a|^2z+|b|^2) \pm \left[f(az+b) + f(\bar{a}z+\bar{b})\right]$$

is in $\mathcal{F}_{m-1}(\mathbb{C})$. □

Setting $z = t \in \mathbb{R}$, we obtain from Lemma 2.4 and Lemma 2.3 (v) that

$$h(t) = u(t+1) + u(|a|^2t+|b|^2) \pm \left[u(at+b) + u(\bar{a}t+\bar{b})\right]$$

is in $\mathcal{F}_{m-1}(\mathbb{R})$. Furthermore, from Lemma 2.3 (ii),

$$u(\bar{a}t+\bar{b}) = u(at+b).$$

Thus

$$h(t) = u(t+1) + u(|a|^2t+|b|^2) \pm 2u(at+b)$$

is in $\mathcal{F}_{m-1}(\mathbb{R})$ for any $a, b \in \mathbb{C}$.

We require the following result.

Theorem 2.5 ([19]). *Assume $f \in \mathcal{F}_m(\mathbb{R})$, $m \geq 2$. Then*

$$f(t) = \sum_{k=0}^{\infty} c_k t^k$$

with $c_k \geq 0$ for all k, and the series converges for all $t \in \mathbb{R}$.

There is a slight oversight in the proof in [19]. It is also necessary in their proof that f be bounded in a neighborhood of the origin. It may be easily shown that this holds for $f \in \mathcal{F}_m(\mathbb{R})$ when $m \geq 2$.

Thus, for $m \geq 3$ and for all $a, b \in \mathbb{C}$,

$$h(t) = u(t+1) + u(|a|^2t+|b|^2) \pm 2u(at+b) \tag{2.1}$$

is in $\mathcal{F}_2(\mathbb{R})$ and thus has a power series expansion, with nonnegative coefficients, which converges for all t. Since $u(t)$ has this property, so does $u(t+1)$ and $u(|a|^2t+|b|^2)$. Thus

$$u(at+b)$$

has a power series expansion in t, which converges for all $t \in \mathbb{R}$, for each fixed $a, b \in \mathbb{C}$. In this power series expansion, the coefficients need not be nonnegative.

It is convenient to consider u as a map from $\mathbb{R}^2$ to $\mathbb{R}$, rather than from $\mathbb{C}$ to $\mathbb{R}$. Thus, if $w = w_1 + iw_2 \in \mathbb{C}$, we set $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$ and write

$$U(\mathbf{w}) = u(w)\,.$$

However, we also write

$$U(\mathbf{a}t + \mathbf{b}) = u(at+b)$$

as we consider $t \in \mathbb{R}$ as a parameter.

Proposition 2.6. *Let U be as above and $m \geq 3$.*

(i) *For each $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, we have*

$$U(\mathbf{a}t+\mathbf{b}) = \sum_{k=0}^{\infty} c_k(\mathbf{a},\mathbf{b})t^k, \tag{2.2}$$

where the power series converges for all $t \in \mathbb{R}$.

(ii) $U \in C(\mathbb{R}^2)$.

(iii) *For each $M > 0$, there exists a sequence of positive numbers $(b_k(M))_{k=0}^{\infty}$ such that for each k, and all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ satisfying $|\mathbf{a}|, |\mathbf{b}| \leq M$, we have*

$$|c_k(\mathbf{a},\mathbf{b})| \leq b_k(M) \quad \textit{and} \quad \limsup_{k\to\infty} \sqrt[k]{b_k(M)} = 0\,.$$

Proof. (i) is just a restatement of the consequence of Theorem 2.5. We prove (ii) as follows. Let h be as in (2.1). Since $h \in \mathcal{F}_2(\mathbb{R})$, it follows that

$$h(t) = \sum_{k=0}^{\infty} c_k t^k,$$

where $c_k \geq 0$ and the series converges for all $t \in \mathbb{R}$. Thus h is also increasing and nonnegative on $\mathbb{R}_+$. We therefore have $h(t) - h(0) \geq 0$ for all $t \in \mathbb{R}_+$, which can be rewritten as

$$u(t+1) - u(1) + u(|a|^2t + |b|^2) - u(|b|^2) \geq 2|u(at+b) - u(b)|\,.$$

Fix $b \in \mathbb{C}$. As $u|_{\mathbb{R}} \in C(\mathbb{R})$, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $t \in \mathbb{R}$ satisfies $|t| < \delta$ and $a \in \mathbb{C}$, $|a| \le 1$, then

$$|u(t+1) - u(1)| < \varepsilon$$

and

$$|u(|a|^2 t + |b|^2) - u(|b|^2)| \le |u(t + |b|^2) - u(|b|^2)| < \varepsilon .$$

Thus,

$$|u(at+b) - u(b)| < \varepsilon$$

for all $a \in \mathbb{C}$ satisfying $|a| \le 1$ and $t \in [0, \delta)$. This proves (ii).

From the above,

$$u(t+1) + u(|a|^2 t + |b|^2) \pm 2u(at+b)$$

has a power expansion with nonnegative coefficients, as do $u(t+1)$ and $u(|a|^2t+|b|^2)$. Furthermore, if

$$u(|a|^2 t + |b|^2) = \sum_{k=0}^{\infty} d_k(|a|^2, |b|^2) t^k$$

and $|a|, |b| \le M$, then as is easily verified

$$0 \le d_k(|a|^2, |b|^2) \le d_k(M^2, M^2) .$$

Thus

$$|c_k(\mathbf{a}, \mathbf{b})| \le d_k(1,1) + d_k(M^2, M^2)$$

for each k. As

$$\sum_{k=0}^{\infty} d_k(M^2, M^2) t^k$$

is entire, we have

$$\limsup_{k \to \infty} \sqrt[k]{d_k(M^2, M^2)} = 0$$

for each M. This proves (iii). □

As a consequence of Proposition 2.6, we have the following result.

Theorem 2.7. *Assume U satisfies the conditions of Proposition 2.6. Then U is the restriction to $\mathbb{R}^2$ of an entire function $\widetilde{U}$ on $\mathbb{C}^2$.*

To verify Theorem 2.7, we need less than we have shown in Proposition 2.6. In (i) of Proposition 2.6, we proved that $U(\mathrm{w})$ has a convergent power series expansion on every straight line in $\mathbb{R}^2$. It suffices, in the proof of Theorem 2.7, for this property to hold only on lines parallel to the axes. From that and the other results proved in Proposition 2.6, Theorem 2.7 follows as a consequence of a result in Bernstein [3, p. 101]. It also follows from results in Browder [5], Cameron–Storvick [6] and Siciak [29]; see the review article by Nguyen [22]. Theorem 2.7 implies that the series expansion for U converges appropriately. That is,

$$U((x,y)) = u(x+iy) = \sum_{k,\ell=0}^{\infty} a_{k,\ell} x^k y^\ell,$$

where the power series converges absolutely for all $(x,y) \in \mathbb{R}^2$.

We now continue as in [11].

Proposition 2.8. *Let $f \in \mathcal{F}_m(\mathbb{C})$, $m \geq 3$, and $v \doteq \operatorname{Im} f$. Then*

$$v(x+iy) = \sum_{k,\ell=0}^{\infty} c_{k,\ell} x^k y^\ell,$$

where the power series converges absolutely for all $(x,y) \in \mathbb{R}^2$.

Proof. From Lemma 2.2 (ii) and (iii), we have $zf(z) \in \mathcal{F}_m(\mathbb{C})$. Thus from Theorem 2.7, its real part

$$xu(x+iy) - yv(x+iy)$$

has a power series expansion, which in turn implies that

$$yv(x+iy) = \sum_{k,\ell=0}^{\infty} d_{k,\ell} x^k y^\ell,$$

with the power series converging absolutely. Setting $y = 0$, we obtain

$$0 = \sum_{k,\ell=0}^{\infty} d_{k,0} x^k .$$

Thus $d_{k,0} = 0$ for all k and

$$v(x+iy) = \sum_{k,\ell=0}^{\infty} d_{k,\ell+1} x^k y^\ell ,$$

where the power series converges absolutely. □

We can now finally prove Theorem 2.1.

Proof of Theorem 2.1. As $f \in \mathcal{F}_m(\mathbb{C})$, $m \geq 3$, we have from Theorem 2.7 and Proposition 2.8 that

$$f(x+iy) = \sum_{k,\ell=0}^{\infty} e_{k,\ell} x^k y^\ell$$

and the power series converges absolutely for all $(x,y) \in \mathbb{R}^2$. Substituting $x = (z+\overline{z})/2$ and $y = (z-\overline{z})/2$, we obtain

$$f(z) = \sum_{k,\ell=0}^{\infty} b_{k,\ell} z^k \overline{z}^\ell;$$

and this power series also converges absolutely for all $z \in \mathbb{C}$. It remains to prove that $b_{k,\ell} \geq 0$ for all $(k,\ell) \in \mathbb{Z}_+^2$. For $\varepsilon > 0$ and $z_1, \ldots, z_n \in \mathbb{C}$, the matrix

$$A = (\varepsilon z_r \overline{z}_s)_{r,s=1}^n$$

is Hermitian positive definite of rank 1. Thus

$$\sum_{r,s=1}^{n} c_r f(\varepsilon z_r \overline{z}_s) \overline{c}_s \geq 0$$

for all $c_1, \ldots, c_n \in \mathbb{C}$. Substituting for f, we obtain

$$\sum_{k,\ell=0}^{\infty} b_{k,\ell} \varepsilon^{k+\ell} \sum_{r,s=1}^{n} c_r z_r^k \overline{z}_s^k \overline{z}_r^\ell z_s^\ell \overline{c}_s = \sum_{k,\ell=0}^{\infty} b_{k,\ell} \varepsilon^{k+\ell} \left| \sum_{r=1}^{n} c_r z_r^k \overline{z}_r^\ell \right|^2 \geq 0 \, .$$

This inequality must hold for all choices of $n \in \mathbb{N}$, $\varepsilon > 0$, and $z_1, \ldots, z_n, c_1, \ldots, c_n \in \mathbb{C}$. Given $(i,j) \in \mathbb{Z}_+^2$, it is possible to choose n and $z_1, \ldots, z_n, c_1, \ldots, c_n \in \mathbb{C}$ such that

$$\sum_{r=1}^{n} c_r z_r^k \overline{z}_r^\ell = 0$$

for all $(k,\ell) \in \mathbb{Z}_+^2$ satisfying $k + \ell \leq i + j$ except that

$$\sum_{r=1}^{n} c_r z_r^i \overline{z}_r^j \neq 0 \, .$$

Letting $\varepsilon \downarrow 0$ then proves $b_{i,j} \geq 0$. □

3 Strictly Hermitian positive definite functions

In this section, we always assume that f is of the form

$$f(z) = \sum_{k,\ell=0}^{\infty} b_{k,\ell} z^k \overline{z}^\ell, \tag{3.1}$$

where the $b_{k,\ell} \geq 0$ for all k, ℓ in $\mathbb{Z}_+$, and the series converges for all $z \in \mathbb{C}$. We prove the following result.

Theorem 3.1. *Assume f is of the form* (3.1). *Set*

$$J = \{(k,\ell) \,:\, b_{k,\ell} > 0\}\,.$$

Then f is strictly Hermitian positive definite on H if and only if $(0,0) \in J$ and for each $p \in \mathbb{N}$ and $q \in \{0,1,\dots,p-1\}$,

$$|\{k-\ell :\, (k,\ell) \in J\,,\, k-\ell = q\,(\mathrm{mod}\, p)\}| = \infty\,.$$

The conditions of Theorem 3.1 are independent of H. We first show that it suffices in the proof of Theorem 3.1 to consider only the standard inner product on $\mathbb{C}$. In this, we follow the analysis in [23].

Proposition 3.2. *Theorem 3.1 is valid if and only if it holds for the standard inner product on $\mathbb{C}$, namely*

$$< z, w >= z\overline{w}\,,$$

for $z, w \in \mathbb{C}$.

The main tool used in the proof of Proposition 3.2 is Proposition 3.3, which appears in [23] in the real case, and is essentially based on an exercise in [24, p. 287].

Proposition 3.3. *Let H be a complex inner product space, and $\mathbf{z}^1,\dots,\mathbf{z}^n$ any n distinct points in H. There then exist distinct $z_1,\dots,z_n \in \mathbb{C}$ and a Hermitian positive definite matrix $(m_{rs})_{r,s=1}^n$ such that*

$$< \mathbf{z}^r, \mathbf{z}^s >= z_r\overline{z}_s + m_{rs}\,.$$

Proof. Set

$$a_{rs} =< \mathbf{z}^r, \mathbf{z}^s >\,, \qquad r,s = 1,\dots,n\,.$$

Since the $\mathbf{z}^1,\dots,\mathbf{z}^n$ are distinct points in H, the Hermitian positive definite matrix

$$A = (a_{rs})_{r,s=1}^n$$

has no two identical rows (or columns). For assume there are two identical rows indexed by i and j. Then

$$a_{ii} = a_{ij} = a_{ji} = a_{jj}\,,$$

which implies that

$$< \mathbf{z}^i, \mathbf{z}^i >=< \mathbf{z}^i, \mathbf{z}^j >=< \mathbf{z}^j, \mathbf{z}^j >\,.$$

But then

$$\| < \mathbf{z}^j, \mathbf{z}^j > \mathbf{z}^i - < \mathbf{z}^i, \mathbf{z}^i > \mathbf{z}^j \| = 0$$

and thus $\mathbf{z}^i = \mathbf{z}^j$, contradicting our assumption.

As A is an $n \times n$ Hermitian positive definite matrix, it may be decomposed as

$$A = C^T \overline{C},$$

where $C = (c_{kr})_{k=1}^{m}{}_{r=1}^{n}$ is an $m \times n$ matrix, $\overline{C} = (\bar{c}_{kr})_{k=1}^{m}{}_{r=1}^{n}$ and rank$A = m$. Thus

$$a_{rs} = \sum_{k=1}^{m} c_{kr}\bar{c}_{ks}\,, \qquad r,s = 1,\dots,n\,.$$

Since no two rows of A are identical, it follows that no two columns of C are identical. Hence there exists a $\mathbf{v} \in \mathbb{C}^m$, $\|\mathbf{v}\| = 1$, for which

$$\mathbf{v}C = \mathbf{z}$$

with $\mathbf{z} = (z_1,\dots,z_n)$, where the $z_1,\dots,z_n$ are all distinct.

Let V be any $m \times m$ unitary matrix ($V^T\overline{V} = I$) whose first row is $\mathbf{v}$. Let U be the $m \times m$ matrix whose first row is $\mathbf{v}$ and all of whose other entries are zero. Then

$$A = C^T\overline{C} = C^T V^T \overline{V}\overline{C} = C^T U^T \overline{U}\overline{C} + C^T(V-U)^T(\overline{V}-\overline{U})\overline{C},$$

since $U^T(\overline{V}-\overline{U}) = (V-U)^T\overline{U} = 0$. From the above, it follows that $(C^TU^T\overline{U}\overline{C})_{rs} = z_r\bar{z}_s$ and $M = C^T(V-U)^T(\overline{V}-\overline{U})\overline{C}$ is Hermitian positive definite. This proves the proposition. □

We now prove Proposition 3.2.

Proof of Proposition 3.2. Let H be any complex inner product space, and let $\mathbf{z} \in H$, $\|\mathbf{z}\| = 1$. Then for $z, w \in \mathbb{C}$,

$$f(< z\mathbf{z}, w\mathbf{z} >) = f(z\overline{w})\,.$$

This immediately implies that if f is not a strictly Hermitian positive definite function on $\mathbb{C}$, then it is not a strictly Hermitian positive definite function on any complex inner product space H.

The converse direction is a consequence of Proposition 3.3. Assume f is a strictly Hermitian positive definite function on $\mathbb{C}$. Let $\mathbf{z}^1,\dots,\mathbf{z}^n$ be distinct points in H. By Proposition 3.3, there exist distinct $z_1,\dots,z_n \in \mathbb{C}$ and a Hermitian positive definite matrix $(m_{rs})_{r,s=1}^n$ such that

$$< \mathbf{z}^r, \mathbf{z}^s > = z_r\bar{z}_s + m_{rs}\,.$$

As $(m_{rs})_{r,s=1}^n$ is a Hermitian positive definite matrix, for any nonzero $c_1,\dots,c_n$

$$\sum_{r,s=1}^n c_r < \mathbf{z}^r, \mathbf{z}^s > \bar{c}_s = \sum_{r,s=1}^n c_r z_r \bar{z}_s \bar{c}_s + \sum_{r,s=1}^n c_r m_{rs}\bar{c}_s \geq \sum_{r,s=1}^n c_r z_r \bar{z}_s \bar{c}_s\,.$$

Similarly, it is well-known that

$$\sum_{r,s=1}^{n} c_r < \mathbf{z}^r, \mathbf{z}^s >^k \overline{< \mathbf{z}^r, \mathbf{z}^s >}^\ell \overline{c}_s \geq \sum_{r,s=1}^{n} c_r (z_r \overline{z}_s)^k (\overline{z}_r z_s)^\ell \overline{c}_s \,.$$

This is a consequence of inequalities between the Schur (Hadamard) product of two Hermitian positive definite matrices; see, e.g., [15, p. 310]. Thus, for f of the form (3.1),

$$\sum_{r,s=1}^{n} c_r f(< \mathbf{z}^r, \mathbf{z}^s >) \overline{c}_s = \sum_{k,\ell=0}^{\infty} b_{k,\ell} \sum_{r,s=1}^{n} c_r < \mathbf{z}^r, \mathbf{z}^s >^k < \mathbf{z}^s, \mathbf{z}^r >^\ell \overline{c}_j$$

$$\geq \sum_{k,\ell=0}^{\infty} b_{k,\ell} \sum_{r,s=1}^{n} c_r (z_r \overline{z}_s)^k (\overline{z}_r z_s)^\ell \overline{c}_s = \sum_{r,s=1}^{n} c_r f(z_r \overline{z}_s) \overline{c}_s \,.$$

As the $z_1, \ldots, z_n$ are distinct, the $c_1, \ldots, c_n$ are nonzero and f is a strictly Hermitian positive definite function on $\mathbb{C}$, it follows that the above quantity is strictly positive. Thus f is a strictly Hermitian positive definite function on H. □

We therefore assume in what follows that $H = \mathbb{C}$. Let $z_1, \ldots, z_n$ be n distinct points in $\mathbb{C}$ and $c_1, \ldots, c_n \in \mathbb{C} \backslash \{0\}$. We want conditions implying

$$\sum_{r,s=1}^{n} c_r f(z_r \overline{z}_s) \overline{c}_s > 0 \,.$$

From (3.1) we have

$$\sum_{r,s=1}^{n} c_r f(z_r \overline{z}_s) \overline{c}_s = \sum_{r,s=1}^{n} c_r \left(\sum_{k,\ell=0}^{\infty} b_{k,\ell} (z_r \overline{z}_s)^k (\overline{z}_r z_s)^\ell \right) \overline{c}_s = \sum_{k,\ell=0}^{\infty} b_{k,\ell} \left| \sum_{r=1}^{n} c_r z_r^k \overline{z}_r^\ell \right|^2 .$$

Now $b_{k,\ell} \geq 0$ and $|\sum_{r=1}^n c_r z_r^k \overline{z}_r^\ell|^2 \geq 0$. Thus f is a strictly Hermitian positive definite function if and only if for all choices of n, distinct points $z_1, \ldots, z_n$ in $\mathbb{C}$ and nonzero values $c_1, \ldots, c_n$, we always have

$$\sum_{r=1}^{n} c_r z_r^k \overline{z}_r^\ell \neq 0 \tag{3.2}$$

for some $(k, \ell) \in J$.

One direction in the proof of Theorem 3.1 is elementary.

Proposition 3.4. *If f is a strictly Hermitian positive definite function, then $(0,0) \in J$ and for each $p \in \mathbb{N}$, $q \in \{0, 1, \ldots, p-1\}$, we have*

$$|\{k - \ell \,:\, (k, \ell) \in J, k - \ell = q \,(\mathrm{mod}\, p)\}| = \infty \,.$$

Proof. To prove that $(0,0) \in J$, we simply take $n = 1$, $z_1 = 0$ and $c_1 \neq 0$ in (3.2). As f is a strictly Hermitian positive definite function,

$$c_1 z_1^k \bar{z}_1^\ell \neq 0$$

for some $(k,\ell) \in J$. But

$$c_1 z_1^k \bar{z}_1^\ell = 0$$

for all $(k,\ell) \in \mathbb{Z}_+^2 \backslash \{(0,0)\}$, which implies that we must have $(0,0) \in J$.

Let us now assume that there exists a $p \in \mathbb{N}$ and $q \in \{0,1,\ldots,p-1\}$ for which

$$|\{k-\ell \,:\, (k,\ell) \in J, k-\ell = q \,(\mathrm{mod}\, p)\}| = N < \infty .$$

Thus if $(k,\ell) \in J$ and $k - \ell = q \,(\mathrm{mod}\, p)$, then

$$k - \ell = q + a_m p, \qquad m = 1,\ldots,N,$$

for some $a_1,\ldots,a_N \in \mathbb{Z}$.

Choose $\theta_1,\ldots,\theta_{N+1}$ in $[0,2\pi)$ such that the $p(N+1)$ points

$$z_{r,t} = e^{i(\theta_r + 2\pi t/p)}, \qquad r = 1,\ldots,N+1;\, t = 0,1,\ldots,p-1,$$

are all distinct.

There exist $c_1,\ldots,c_{N+1}$, not all zero, such that

$$\sum_{r=1}^{N+1} c_r e^{i\theta_r(q+a_m p)} = 0, \qquad m = 1,\ldots,N. \tag{3.3}$$

Since $e^{i\frac{2\pi t}{p}}$, $t = 0,1,\ldots,p-1$, are distinct, the $p \times p$ matrix

$$\left(e^{i\frac{2\pi ts}{p}}\right)_{t,s=0}^{p-1}$$

is nonsingular and there exist $d_0,\ldots,d_{p-1}$, not all zero, such that

$$\sum_{t=0}^{p-1} d_t e^{i\frac{2\pi ts}{p}} = 0, \qquad s = 0,1,\ldots,p-1\,;\, s \neq q. \tag{3.4}$$

We claim that

$$\sum_{t=0}^{p-1}\sum_{r=1}^{N+1} d_t c_r z_{r,t}^k \bar{z}_{r,t}^\ell = 0$$

for all $(k,\ell) \in J$. As the $z_{r,t}$ are distinct and the $d_t c_r$ are not all zero, this would imply that f is not a strictly Hermitian positive definite function.

Now
$$\sum_{t=0}^{p-1}\sum_{r=1}^{N+1} d_t c_r z_{r,t}^k \overline{z}_{r,t}^\ell = \sum_{t=0}^{p-1}\sum_{r=1}^{N+1} d_t c_r e^{i(\theta_r+\frac{2\pi t}{p})k} e^{-i(\theta_r+\frac{2\pi t}{p})\ell}$$
$$= \left(\sum_{t=0}^{p-1} d_t e^{i\frac{2\pi t}{p}(k-\ell)}\right)\left(\sum_{r=1}^{N+1} c_r e^{i\theta_r(k-\ell)}\right).$$
If $(k,\ell) \in J$ and $k-\ell = q\,(\operatorname{mod} p)$, then $k-\ell = q + a_m p$; and from (3.3), the right-hand factor is zero. If $(k,\ell) \in J$ and $k-\ell \neq q\,(\operatorname{mod} p)$, then $k-\ell = s\,(\operatorname{mod} p)$, $s \in \{0,1,\ldots,p-1\}\backslash\{q\}$. Thus
$$k-\ell = s + mp$$
for some $s \in \{0,1,\ldots,p-1\}\backslash\{q\}$ and $m \in \mathbb{Z}$, and
$$e^{i\frac{2\pi t}{p}(k-\ell)} = e^{i\frac{2\pi ts}{p}} e^{i2\pi tm} = e^{i\frac{2\pi ts}{p}}.$$
From (3.4), the left-hand factor is zero. This proves the proposition. □

It is the converse direction which is less elementary. We prove the following result, which completes the proof of Theorem 3.1.

Theorem 3.5. *Let $J \subseteq \mathbb{Z}_+^2$ be such that for each $p \in \mathbb{N}$ and $q \in \{0,1,\ldots,p-1\}$, we have*
$$|\{k-\ell : (k,\ell) \in J,\ k-\ell = q\,(\operatorname{mod} p)\}| = \infty.$$
Then for all n, distinct nonzero points $z_1,\ldots,z_n$ in $\mathbb{C}$ and nonzero values $c_1,\ldots,c_n$ in $\mathbb{C}$, we have
$$\sum_{r=1}^{n} c_r z_r^k \overline{z}_r^\ell \neq 0$$
for some $(k,\ell) \in J$.

Note that we have here assumed that each of the $z_1,\ldots,z_n$ is nonzero and we have dropped the condition $(0,0) \in J$. It is easily shown that we can make this assumption.

A *linear recurrence relation* is a series of equations of the form

$$a_{s+m} = a_{s+m-1}w_1 + \cdots + a_s w_m\,, \qquad s \in \mathbb{Z}, \tag{3.5}$$

satisfied by the *recurrence sequence* $\{a_s\}_{s\in\mathbb{Z}}$ for some given $w_1,\ldots,w_m$ (m finite). We assume that the w_j and a_s are in $\mathbb{C}$. Associated with each such recurrence sequence is a *generalized power sum*

$$a_s = \sum_{j=1}^{r} P_j(s) u_j^s, \tag{3.6}$$

where the P_j are polynomials in s of degree ∂P_j with $\sum_{j=1}^{r} \partial P_j + 1 \le m$, and vice versa. That is, each generalized power sum of the form (3.6) gives rise to a linear recurrence relation of the form (3.5).

We are interested in the form (3.6). A delightful theorem in number theory which concerns recurrence sequences is the Skolem–Mahler–Lech Theorem; see, e.g., [28, p. 38].

Theorem 3.6 (Skolem–Mahler–Lech). *Assume that $\{a_s\}_{s\in\mathbb{Z}}$ is a recurrence sequence, i.e., satisfies* (3.5) *or* (3.6). *Set*

$$\mathcal{A} = \{s : a_s = 0\}.$$

Then $\mathcal{A}$ is the union of a finite number of points and a finite number of full arithmetic sequences.

What is the connection between this result and our problem? Recall that we are concerned with the equations

$$\sum_{r=1}^{n} c_r z_r^k \overline{z}_r^\ell .$$

Assume for the moment that $|z_r| = 1$ for all $r = 1, \ldots, n$. We can then rewrite the above as

$$\sum_{r=1}^{n} c_r z_r^{k-\ell} .$$

Set

$$a_s = \sum_{r=1}^{n} c_r z_r^s , \qquad s \in \mathbb{Z} .$$

These are equations of the form (3.6) and thus $\{a_s\}_{s\in\mathbb{Z}}$ is a recurrence sequence. Let us rewrite and prove Theorem 3.5 in this particular case. (The same holds if we assume $|z_r| = \lambda$ for some $\lambda \in \mathbb{R}_+$ and all $r = 1, \ldots, n$.)

Proposition 3.7. *Let $J^* \subseteq \mathbb{Z}$ be such that for each $p \in \mathbb{N}$ and $q \in \{0, 1, \ldots, p-1\}$, we have*

$$|\{s : s \in J^*,\ s = q\,(\mathrm{mod}\,p)\}| = \infty .$$

Then for all n, distinct nonzero points $z_1, \ldots, z_n$ in $\mathbb{C}$ and nonzero values $c_1, \ldots, c_n$ in $\mathbb{C}$, we have

$$\sum_{r=1}^{n} c_r z_r^s \neq 0$$

for some $s \in J^*$.

Proof. Set

$$a_s = \sum_{r=1}^{n} c_r z_r^s, \qquad s \in \mathbb{Z},$$

and let

$$\mathcal{A} = \{s : a_s = 0\}.$$

Assume $a_s = 0$ for all $s \in J^*$, i.e., $J^* \subseteq \mathcal{A}$. The Skolem–Mahler–Lech Theorem implies that J^* is contained in the set $\mathcal{A}$ which is the union of a finite number of points and a finite number of full arithmetic sequences. As the a_s cannot all be zero, it is readily verified that there must exist an arithmetic sequence disjoint from $\mathcal{A}$. Thus there exists a $p \in \mathbb{N}$ and $q \in \{0, 1, \ldots, p-1\}$ for which

$$\{s : s \in J^*, \, s = q \,(\operatorname{mod} p)\} = \emptyset.$$

This contradiction proves the proposition. □

Remark 1. The above is equivalent to the following. Let $J^* \subseteq \mathbb{Z}$ and

$$\Pi_{J^*} = \operatorname{span}\{z^k : k \in J^*\}.$$

Then for every finite point set E in $\mathbb{C}$,

$$\dim \Pi_{J^*}|_E = |E|$$

if and only if $0 \in J^*$ and J^* satisfies the criteria of Proposition 3.7.

Remark 2. By a *discrete measure* $d\mu$ on $[0, 2\pi)$ we mean a measure of the form

$$d\mu = \sum_{r=1}^{n} c_r \delta_{\theta_r}$$

for some finite n, where δ_θ is the Dirac-Delta point measure at θ. From Propositions 3.4 and 3.7 the $J^* \subseteq \mathbb{Z}$, as above, characterize the sets of uniqueness for the Fourier coefficients of a discrete measure. Set

$$\widehat{\mu}(s) = \int_0^{2\pi} e^{i\theta s} d\mu(\theta), \qquad s \in \mathbb{Z}.$$

If $d\mu_1$ and $d\mu_2$ are two discrete measures and

$$\widehat{\mu}_1(s) = \widehat{\mu}_2(s)$$

for all $s \in J^*$, then $d\mu_1 = d\mu_2$.

Unfortunately, there seems to be no a priori reason to assume that the $\{z_r\}$ are all of equal modulus. We nonetheless prove that such is effectively the case. We prove this fact via the following generalization of the Skolem–Mahler–Lech Theorem.

Theorem 3.8 (Laurent [17], [18, p. 26]). *Let*

$$a_{k,\ell} = \sum_{r=1}^{n} c_r z_r^k w_r^\ell$$

for some $(z_r, w_r) \in \mathbb{C}^2$ *and* $c_r \in \mathbb{C}$, $r = 1, \ldots, n$. *Let*

$$\mathcal{B} = \{(k,\ell) : a_{k,\ell} = 0\}\,.$$

Then $\mathcal{B}$ *is the union of a finite number of translates of subgroups of* $\mathbb{Z}^2$.

Note that this is not a full generalization of the Skolem–Mahler–Lech Theorem in that the polynomial parts of the generalized power sum, see (3.6), are here assumed to be constants.

In our problem, we are given

$$a_{k,\ell} = \sum_{r=1}^{n} c_r z_r^k \overline{z}_r^\ell,$$

i.e., $\overline{z}_r = w_r$. Additionally, and importantly, we only consider $(k,\ell) \in \mathbb{Z}_+^2$ although, since $z_r \neq 0$ for all n, we can and do define $a_{k,\ell}$ for all $(k,\ell) \in \mathbb{Z}^2$.

What are the subgroups of $\mathbb{Z}^2$? Each subgroup is given as the set of (k,ℓ) satisfying

$$\begin{pmatrix} k \\ \ell \end{pmatrix} = p \begin{pmatrix} a \\ b \end{pmatrix} + q \begin{pmatrix} c \\ d \end{pmatrix},$$

where p, q vary over all $\mathbb{Z}$, and a, b, c, d are given integers with $ad - bc \notin \{-1, 1\}$. If $ad - bc \in \{-1, 1\}$, then the set of solutions is all of $\mathbb{Z}^2$. If $ad - bc = 0$, then the solution set can be rewritten as

$$(k,\ell) = (js, jt)\,, \qquad j \in \mathbb{Z}$$

for some $(s,t) \in \mathbb{Z}^2$. This includes the case $(s,t) = (0,0)$. For $|ad - bc| \geq 2$, the subgroup is a lattice.

Let us denote a translate of a subgroup of $\mathbb{Z}^2$ by L.

Proposition 3.9. *Assume* $|L \cap \mathbb{Z}_+^2| = \infty$, *and*

$$\sum_{r=1}^{n} c_r z_r^k \overline{z}_r^\ell = 0$$

for all $(k,\ell) \in L$. *Then*

$$\sum_{\{r:|z_r|=\lambda\}} c_r z_r^k \bar{z}_r^\ell = 0$$

for all $(k,\ell) \in L$ *and each* $\lambda \in \mathbb{R}_+$.

Of course, we only consider values λ equal to one of the $|z_1|,\ldots,|z_n|$. For other values of λ, the set $\{r : |z_r| = \lambda\}$ is empty.

Before proving Proposition 3.9 we first prove an ancillary result.

Proposition 3.10. *Assume*

$$\sum_{r=1}^{n} d_r w_r^j = 0$$

for all $j \in \mathbb{Z}$ *with* $d_r \neq 0$ *and* $w_r \in \mathbb{C}\backslash\{0\}$, $r = 1,\ldots,n$. *Then*

$$\sum_{\{r:w_r=\mu\}} d_r w_r^j = 0$$

for each $\mu \in \mathbb{C}$ *and all* $j \in \mathbb{Z}$.

Proof. Let $\mu_1,\ldots,\mu_m$ denote the distinct values of the $w_1,\ldots,w_n$, and assume that ℓ_s of the w_j's equal μ_s, $s = 1,\ldots,m$. Thus $\sum_{s=1}^m \ell_s = n$.

For any k,

$$\det\left(w_r^{j+k}\right)_{r=1\, j=0}^{n\quad n-1} = w_1^k \cdots w_n^k \prod_{1\le t<s\le n} (w_s - w_t)\,.$$

Thus

$$\operatorname{rank}\left(w_r^j\right)_{r=1,\, j\in\mathbb{Z}}^n = m\,.$$

The linear subspace

$$D = \{\mathbf{d} = (d_1,\ldots,d_n)\ :\ \sum_{r=1}^n d_r w_r^j = 0\,,\ \text{all } j \in \mathbb{Z}\}$$

is therefore of dimension $n - m$.

Now if, for example, $w_1 = \cdots = w_{\ell_1} = \mu_1$ (and $w_j \neq \mu_1$ for $j > \ell_1$), then

$$\operatorname{rank}\left(w_t^j\right)_{t=1,\, j\in\mathbb{Z}}^{\ell_1} = 1$$

and there are therefore $\ell_1 - 1$ linearly independent vectors $\mathbf{d}^1,\ldots,\mathbf{d}^{\ell_1}$ in D satisfying

$$d_t^1 = \cdots = d_t^{\ell_1} = 0$$

for all $t = \ell_1 + 1, \ldots, n$. In this manner, we obtain

$$\sum_{s=1}^{m} (\ell_s - 1) = n - m$$

linearly independent vectors in D. As

$$\dim D = n - m,$$

we have therefore found a basis for D. The proposition now easily follows. $\square$

Proof of Proposition 3.9. A translate of a subgroup L of $\mathbb{Z}^2$ is given by the formula

$$\begin{pmatrix} k \\ \ell \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + p \begin{pmatrix} a \\ b \end{pmatrix} + q \begin{pmatrix} c \\ d \end{pmatrix},$$

where p, q vary over all $\mathbb{Z}$. We divide the proof of this proposition into two cases.

Case 1. $ad - bc = 0$. In this case,

$$(k, \ell) = (\alpha + js, \beta + jt).$$

If $st < 0$ or if $s = t = 0$, then $|L \cap \mathbb{Z}_+^2| < \infty$. Thus we may assume that $s, t \geq 0$ and $s + t > 0$. Now for $(k, \ell) \in L$,

$$0 = \sum_{r=1}^{n} c_r z_r^k \bar{z}_r^\ell = \sum_{r=1}^{n} c_r z_r^{\alpha+js} \bar{z}_r^{\beta+jt} = \sum_{r=1}^{n} \left(c_r z_r^\alpha \bar{z}_r^\beta\right) \left(z_r^s \bar{z}_r^t\right)^j = \sum_{r=1}^{n} d_r w_r^j$$

for all $j \in \mathbb{Z}$, where

$$d_r = c_r z_r^\alpha \bar{z}_r^\beta \quad \text{and} \quad w_r = z_r^s \bar{z}_r^t.$$

Note that $d_r \neq 0$, since $c_r \neq 0$ and $z_r \neq 0$. Furthermore, as $s + t > 0$ and

$$|w_r| = |z_r|^{s+t},$$

we have

$$\{r : |z_r| = \lambda\} = \{r : |w_r| = \lambda^{s+t}\},$$

i.e., the indices $\{1, \ldots, n\}$ divide in the same way when considering the distinct $|z_r|$ or the distinct $|w_r|$. We apply Proposition 3.10 to obtain our result.

Case 2. $|ad - bc| \geq 2$. In this case,

$$\begin{pmatrix} k \\ \ell \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + p \begin{pmatrix} a \\ b \end{pmatrix} + q \begin{pmatrix} c \\ d \end{pmatrix}$$

and necessarily $|L \cap \mathbb{Z}_+^2| = \infty$. Now

$$0 = \sum_{r=1}^{n} c_r z_r^k \overline{z}_r^\ell = \sum_{r=1}^{n} c_r z_r^{\alpha+pa+qc} \overline{z}_r^{\beta+pb+qd} = \sum_{r=1}^{n} \left(c_r z_r^\alpha \overline{z}_r^\beta\right) \left(z_r^a \overline{z}_r^b\right)^p \left(z_r^c \overline{z}_r^d\right)^q$$

for all $p, q \in \mathbb{Z}$.

We must have $a + b \neq 0$ or $c + d \neq 0$. For otherwise, $ad - bc = 0$. Assume, without loss of generality, that $a + b \neq 0$. Fixing any $q \in \mathbb{Z}$, we have

$$0 = \sum_{r=1}^{n} \left(c_r z_r^\alpha \overline{z}_r^\beta \left(z_r^c \overline{z}_r^d\right)^q\right) \left(z_r^a \overline{z}_r^b\right)^p = \sum_{r=1}^{n} d_r w_r^p$$

for all $p \in \mathbb{Z}$, where

$$d_r = c_r z_r^\alpha \overline{z}_r^\beta (z_r^c \overline{z}_r^d)^q \neq 0 \quad \text{and} \quad w_r = z_r^a \overline{z}_r^b .$$

As $|w_r| = |z_r|^{a+b}$ and $a + b \neq 0$, we have

$$\{r : |z_r| = \lambda\} = \{r : |w_r| = \lambda^{a+b}\} .$$

Applying Proposition 3.10 gives us our result. □

Our proof of Theorem 3.5 is a consequence of Theorem 3.8 and Propositions 3.7 and 3.9.

Proof of Theorem 3.5. Assume to the contrary that there exists $J \subseteq \mathbb{Z}_+^2$ such that for each $p \in \mathbb{N}$ and $q \in \{0, 1, \ldots, p-1\}$ we have

$$|\{k - \ell : (k, \ell) \in J,\ k - \ell = q \,(\mathrm{mod}\, p)\}| = \infty ,$$

and yet there exist distinct nonzero points $z_1, \ldots, z_n$ in $\mathbb{C}$ and nonzero values $c_1, \ldots, c_n$ in $\mathbb{C}$ such that

$$\sum_{r=1}^{n} c_r z_r^k \overline{z}_r^\ell = 0$$

for all $(k, \ell) \in J$. Set

$$\mathcal{B} = \{(k, \ell) : (k, \ell) \in \mathbb{Z}^2, \sum_{r=1}^{n} c_r z_r^k \overline{z}_r^\ell = 0\} .$$

Then $J \subseteq \mathcal{B}$, and from Theorem 3.8 we know that $\mathcal{B}$ is a union of a finite number of translates of subgroups of $\mathbb{Z}^2$.

Let $\mathcal{B}^*$ be the union of those translates of subgroups of $\mathbb{Z}^2$ in $\mathcal{B}$ which have an infinite number of elements in $\mathbb{Z}_+^2$. That is,

$$\mathcal{B}^* = \bigcup_{s=1}^{m} L_s ,$$

where L_s is a translate of a subgroup of $\mathbb{Z}^2$, $L_s \subseteq \mathcal{B}$ and $|L_s \cap \mathbb{Z}_+^2| = \infty$. Note that $\mathcal{B}^*$ contains all of the elements of J, except perhaps for a finite number of indices. Thus

$$|\{k-\ell \,:\, (k,\ell) \in \mathcal{B}^*,\, k-\ell = q\,(\operatorname{mod} p)\}| = \infty\,,$$

for all p and q as above.

Applying Proposition 3.9 to the subgroups L_s of $\mathcal{B}^*$, we obtain

$$\sum_{\{r:|z_r|=\lambda\}} c_r z_r^k \overline{z}_r^\ell = 0$$

for all $(k,\ell) \in \mathcal{B}^*$. This implies that for each $\lambda \in \mathbb{R}_+$, we have

$$\sum_{\{r:|z_r|=\lambda\}} c_r z_r^{k-\ell} = 0$$

for all $(k,\ell) \in \mathcal{B}^*$. Applying Proposition 3.7 to

$$J^* = \{k-\ell \,:\, (k,\ell) \in \mathcal{B}^*\},$$

we obtain a contradiction. □

Remark 3. As in Remark 1, the condition on J given in Theorem 3.1 is necessary and sufficient so that for every finite point set E in $\mathbb{C}$,

$$\dim \Pi_J|_E = |E|$$

where

$$\Pi_J = \operatorname{span}\{\, z^k \overline{z}^\ell \,:\, (k,\ell) \in J\,\}\,.$$

Remark 4. If we restrict ourselves to S, the unit sphere in H, i.e., $S = \{z \,:\, z \in H, \|z\| = 1\}$, then it readily follows from the above analysis that f of the form (3.1) is strictly Hermitian positive definite on S if and only if for each $p \in \mathbb{N}$ and $q \in \{0,1,\ldots,p-1\}$,

$$|\{k-\ell :\, (k,\ell) \in J\,,\, k-\ell = q\,(\operatorname{mod} p)\}| = \infty\,.$$

In other words, the condition is exactly the same as for all of H except that we do not have or need $(0,0) \in J$.

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