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STUDIES IN SPLINE FUNCTIONS AND
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AN EXTREMAL PROPERTY OF MULTIPLE GAUSSIAN NODES

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1. FORMULATION AND STATEMENT OF RESULTS

In the previous paper [1], we established the existence and uniqueness of a Gaussian type quadrature formula having multiple nodes of prescribed odd multiplicities for an Extended Complete Chebyshev (ECT) system. Our result refined and extended work of Turán [3] and Popoviciu [2], where the method of proof is very different. Turán and Popoviciu exploited the connection between quadrature formulae and an associated minimum problem. In this paper we analyze more deeply this relationship.

Specifically, Turán [3] proved the following theorem:

Theorem A. *Let* k *and* ℓ *be positive integers,* ℓ *odd. Then there exists a unique set of nodes* $\{t_i^*\}_1^k$ *and coefficients* $\{a_{ij}^*\}$ *such that*

$$(1.1)\qquad \int_a^b f(x)\, dx = \sum_{i=1}^{k} \sum_{j=0}^{\ell-1} a_{ij}^* f^{(j)}(t_i^*)$$

holds for all $f \in \mathcal{P}_{(\ell+1)k-1}$, *where* $\mathcal{P}_m$ *consists of the set of all polynomials of degree* $\leq m$.

The deduction of (1.1) emanates from a study of

$$(1.2)\qquad \min \int_a^b [p(x)]^{\ell+1}\, dx \quad ,$$

where the minimum is extended over all monic polynomials of degree k. (See the introduction of [1] for more details on this procedure.)

Popoviciu [2] generalized the Turán result as follows.

Theorem B. *For any set of given positive odd integers* $\{\mu_i\}_1^k$, *there exist distinct nodes* $\{t_i^*\}_1^k$ *located in* (a,b) *and real numbers* $\{a_{ij}^*\}_{i=1}^{k} {}_{j=0}^{\mu_i-1}$ *with the property that*

$$\int_a^b f(x)\,dx = \sum_{i=1}^{k} \sum_{j=0}^{\mu_i-1} a_{ij}^* f^{(j)}(t_i^*) \quad , \tag{1.3}$$

holds for all $f \in \mathcal{P}_{n-1}$, *where* $n = \sum_{i=1}^{k} \mu_i + k$.

The equation (1.3) derives from the analysis of the extremal problem

$$\inf \int_a^b \prod_{i=1}^{k} (x-t_i)^{\mu_i+1}\,dx \quad , \tag{1.4}$$

where the infimum is evaluated with respect to all real $\{t_i\}_1^k$. The resulting quadrature formula of Theorem B is non-unique partly attributable to the fact that the class of admissible polynomials in (1.4) is non-convex.

It is not immediately evident that (1.2) and (1.4) entail (1.1) and (1.3), respectively. The intrinsic connection follows with the help of

Proposition 1.1 . *Let* $F(t_1,\ldots,t_k) = \int_a^b \prod_{i=1}^{k} (x-t_i)^{\mu_i+1}\,dx$, *where* $\{\mu_i\}_1^k$ *are all fixed odd positive integers. Let the domain* D *of* $F(t_1,\ldots,t_k)$ *consist of all sets of* k *distinct* t 's , $a \leq t \leq b$. *If* $F(t_1,\ldots,t_k)$ *admits a critical point* $\{t_i^*\}_1^k$ *in* D , *then* (1.3) *holds with nodes* $\{t_i^*\}_1^k$.

Proof. The critical point conditions applied to $\{t_i^*\}_1^k$ are

$$\frac{\partial}{\partial t_i} F(t_1^*,\ldots,t_k^*) = c_i \int_a^b \left[\prod_{j=1}^{k} (x-t_j^*)^{\mu_j}\right] \left[\prod_{\substack{j=1\\ j\neq i}}^{k} (x-t_j^*)\right] dx = 0 ,$$

$c_i = -(\mu_i+1)$, $i = 1,\ldots,k$.

Because $t_i^* \neq t_j^*$ for $i \neq j$, we readily deduce

$$\int_a^b \left[\prod_{j=1}^{k} (x-t_j^*)^{\mu_j}\right] q(x)\, dx = 0 \qquad (1.5)$$

valid for all $q \in \mathcal{P}_{k-1}$.

Next we construct the unique quadrature formula of the form (1.3), having nodes $\{t_i^*\}_1^k$, exact with respect to all $f \in \mathcal{P}_{n-1-k}$, $n = \sum_{i=1}^{k} \mu_i + k$. More specifically, the coefficients $\{a_{ij}^*\}$ are determined maintaining (1.3) for all $f \in \mathcal{P}_{n-1-k}$. This is manifestly feasible since the determinant of the associated linear system is non-zero. Set $p^*(x) = \prod_{i=1}^{k} (x-t_i^*)^{\mu_i}$. Any $g \in \mathcal{P}_{n-1}$ can be factored to $g(x) = p^*(x)q(x)+q_1(x)$, where $q(x) \in \mathcal{P}_{k-1}$, and $q_1(x) \in \mathcal{P}_{n-1-k}$. By construction, $R(f) = \int_a^b f(x)\,dx - \sum_{i=1}^{k} \sum_{j=0}^{\mu_i-1} a_{ij}^* f^{(j)}(t_i^*) = 0$ on $\mathcal{P}_{n-1-k}$. Therefore, $R(g) = R(p^*q+q_1) = R(p^*\cdot q)$. However, (1.5) entails $R(p^*\cdot q) = 0$, and consequently $R(g) = 0$ for all $g \in \mathcal{P}_{n-1}$.

Q.E.D.

Proposition 1.1 has an analogue for an ECT-system $\{u_i(x)\}_1^n$ defined on $[a,b]$. However, since an ECT-system in general lacks the factoring property of the polynomials, the analysis becomes more arduous.

Theorem B will follow from Proposition 1.1 once we establish the existence of an attained minimum of (1.4) in D . Theorem C pertains to this objective.

<u>Theorem C.</u> <u>Let</u> $\{\mu_i\}_1^k$ <u>be positive odd integers. Consider</u>

$$\min_{\underline{t}\in\Gamma} \int_a^b \prod_{i=1}^k (x-t_i)^{\mu_i+1}\,dx \quad , \tag{1.6}$$

<u>where</u> $\Gamma = \{\underline{t} = (t_1,\ldots,t_k) : a \leq t_1 \leq \ldots \leq t_k \leq b\}$. <u>Any attained minimum</u> $\{t_i^*\}_1^k$ <u>of</u> (1.6) <u>satisfies</u>

$a < t_1^* < \ldots < t_k^* < b$.

<u>Proof.</u> Since Γ is closed, the existence of a minimum $\{t_i^*\}_1^k$ of (1.6) follows. We embody the proof of Theorem C in the next two lemmas.

<u>Lemma 1.1 .</u> $a < t_1^*$ <u>and</u> $t_k^* < b$.

<u>Proof.</u> Assume, without loss of generality, $a = t_1^* < t_2^*$ and consider

$$F(t_1) = \int_a^b \left[\prod_{i=2}^k (x-t_i^*)^{\mu_i+1}\right]\left[(x-t_1)^{\mu_1+1}\right] dx \quad .$$

The assumption $a = t_1^*$ gives $F(a+\varepsilon) \geq F(a)$ implying $F'(a) \geq 0$. A direct calculation produces

$$F'(a) = -(\mu_1+1) \int_a^b \left[\prod_{i=2}^k (x-t_i^*)^{\mu_i+1}\right] (x-a)^{\mu_1}\,dx$$

which is manifestly negative since $\prod_{i=2}^k (x-t_i^*)^{\mu_i+1} (x-a)^{\mu_1} \geq 0$ for all $x \in [a,b]$ as μ_i are all odd. The foregoing contradiction can be averted provided $t_1^* > a$ prevails. A parallel analysis proves that $t_k^* < b$.

<u>Lemma 1.2 .</u> $t_1^* < t_2^* < \ldots < t_k^*$.

<u>Proof.</u> We assume, without loss of generality, that

$t_{j-1}^* < t_j^* = t_{j+1}^* = t^* < t_{j+2}^*$ for some $j(1 \leq j \leq k-1)$ where $t_0^* = a$, $t_{k+1}^* = b$. Set $F(s_1,s_2) =$

$$F(t_j,t_{j+1}) = \int_a^b \left[\prod_{\substack{i=1 \\ i\neq j,j+1}}^{k} (x-t_i^*)^{\mu_i+1} \right] (x-t_j)^{\mu_j+1} (x-t_{j+1})^{\mu_{j+1}+1} dx.$$

A Taylor expansion gives $F(t^*-\delta,t^*+\varepsilon) = F(t^*,t^*)-\delta F_1(t^*,t^*) + \varepsilon F_2(t^*,t^*) + \frac{\delta^2}{2} F_{11}(t^*,t^*) + \frac{\varepsilon^2}{2} F_{22}(t^*,t^*) - \varepsilon\delta F_{12}(t^*,t^*) + O(|\varepsilon|^3 + |\delta|^3)$, where $F_i(t^*,t^*) = \frac{\partial}{\partial s_i} F(t^*,t^*)$, $i = 1,2$ and $F_{ij}(t^*,t^*) = \frac{\partial^2 F(t^*,t^*)}{\partial s_i \partial s_j}$. By assumption, $F(t^*-\delta,t^*+\varepsilon) \geq F(t^*,t^*)$ for ε , δ small and positive. Since $F(t^*,t^*)$ is a critical point of $F(s_1,s_2)$ it follows that $F_1(t^*,t^*) = F_2(t^*,t^*) = 0$, and consequently $\delta^2 F_{11}(t^*,t^*) + \varepsilon^2 F_{22}(t^*,t^*) - 2\varepsilon\delta F_{12}(t^*,t^*) \geq 0$. Let $\delta = c\varepsilon$, $c > 0$, so that the preceding inequality reduces to

(1.7) $c^2 F_{11}(t^*,t^*) - 2c F_{12}(t^*,t^*) + F_{22}(t^*,t^*) \geq 0$

for all $c > 0$. Computing F_{11} , F_{12} and F_{22} , and substituting into (1.7), we obtain

(1.8) $c^2(\mu_j+1)\mu_j - 2c(\mu_j+1)(\mu_{j+1}+1) + (\mu_{j+1}+1)\mu_{j+1} \geq 0$ for all $c > 0$.

However, this quadratic has two distinct positive real roots, a fact incompatible with (1.8). Thus $t_j^* = t_{j+1}^*$ is impossible. Lemma 1.2 is proved and therefore also Theorem C.

Let $F(t_1,\ldots,t_k) = \int_a^b \left[\prod_{i=1}^{k} (x-t_i)^{\mu_i+1} \right] dx$. From Theorem C , $F(t_1^*,\ldots,t_k^*)$ presents a strict local minimum and thus by Proposition 1.1, Theorem B is confirmed. If the $\{\mu_i\}_1^k$

are not all equal, then by a rearrangement of the $\{t_i\}_1^k$ we can exhibit a number of quadrature formulae fulfilling (1.3). By contrast we proved in [1] uniqueness of the quadrature formula (1.3) subject to a specific ordering of the $\{t_i\}_1^k$. In particular, there exists a unique point $\underline{t}^* \in \Gamma$ arranged in the order $a < t_1^* < \ldots < t_k^* < b$, which is a critical point of $F(t_1,\ldots,t_k)$.

In this paper we are concerned with the extension of Theorem C to ECT-systems. Let $\{u_i(x)\}_1^{n+1}$ be an ECT-system on $[a,b]$. We determine

$$(1.9) \qquad u(x;t_1,\ldots,t_k) = \sum_{i=1}^{n+1} a_i u_i(x)$$

as the unique "polynomial" (real linear combination of $\{u_i(x)\}$) vanishing at t_i with multiplicity μ_i+1, $n = \sum_{i=1}^{k} \mu_i+k$, and $a_{n+1} = 1$. We shall prove the following theorem.

Theorem D. *Let $\{u_i(x)\}_1^{n+1}$ and $u(x;t_1,\ldots,t_k)$ be delimited as above. Assume $\{\mu_i\}_1^k$ are prescribed odd positive integers, and $\Gamma = \{(t_1,\ldots,t_k)\colon a \leq t_1 \leq \ldots \leq t_k \leq b\}$. Then*

$$(1.10) \qquad \min_{\underline{t}\in\Gamma} \int_a^b u(x;t_1,\ldots,t_k)\,dx$$

is uniquely attained for $\{t_i^*\}_1^k$, $a < t_1^* < \ldots < t_k^* < b$, and the $\{t_i^*\}_1^k$ are independent of $u_{n+1}(x)$.

For any ECT-system, the fact that the solution to (1.10) is unique and independent of $u_{n+1}(x)$ is perhaps striking in view of the non-convex domain attendant to the minimum problem.

The uniqueness assertion pertaining to $\{t_i^*\}_1^k$ and the fact that $\{t_i^*\}_1^k$ is independent of $u_{n+1}(x)$ follows on the basis of Proposition 1.1 and the uniqueness result of the quadrature formula affirmed in [1]. Therefore, we need only concern ourselves with proving that any minimum of (1.10) must satisfy $a < t_1^* < \dots < t_k^* < b$.

2. PRELIMINARIES AND SOME DETERMINANTAL IDENTITIES

For ease of exposition, we fix some notation and terminology.

1.

$$U\begin{pmatrix}1, & \dots & ,\ell\\ t_1,&\dots,&t_\ell\end{pmatrix} = \begin{vmatrix} u_1(t_1) & \dots & u_1(t_\ell)\\ \vdots & & \vdots\\ u_\ell(t_1) & \dots & u_\ell(t_\ell)\end{vmatrix}$$

while $U^*\begin{pmatrix}1, & \dots & \dots & \dots & ,\ell\\ \underbrace{t_1,\dots,t_1}_{\nu_1}, & \underbrace{t_2,\dots,t_2}_{\nu_2}, & \dots, & \underbrace{t_p,\dots,t_p}_{\nu_p}\end{pmatrix}$, $\ell = \sum_{i=1}^{p} \nu_i$,

shall denote the determinant of the matrix with columns

$\{u_1^{(j)}(t_i),\dots,u_\ell^{(j)}(t_i)\}$, $j = 0,1,\dots,\nu_i-1$; $i = 1,\dots,p$.

2. In the four quantities defined next, $\{t_i^*\}_2^k$ and $\{\mu_i\}_2^k$ are held fixed.

a) $W_0(\ell;t;\nu) = U^*\begin{pmatrix}1, & \dots & \dots & \dots & ,\ell\\ \underbrace{t,\dots,t}_{\nu}, & \underbrace{t_2^*,\dots,t_2^*}_{\mu_2+1}, & \dots, & \underbrace{t_k^*,\dots,t_k^*}_{\mu_k+1}\end{pmatrix}$,

$\ell = \nu + \sum_{i=2}^{k} \mu_i+k-1$.

b) $W_0(m;t;\nu;x) = U^*\begin{pmatrix} 1, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & ,m \\ \underbrace{t,\ldots,t}_{\nu}, & \underbrace{t_2^*,\ldots,t_2^*}_{\mu_2+1}, & \ldots, & \underbrace{t_k^*,\ldots,t_k^*}_{\mu_k+1}, & x \end{pmatrix}$

where $m = \nu + \sum_{i=2}^{k} \mu_i + k$.

c) $\hat{W}_0(\ell;t;\nu)$ stands for the determinant of the matrix associated with $W_0(\ell+1;t;\nu+1)$ with the last row and ν^{th} column deleted. In display

$$\hat{W}_0(\ell;t;\nu) = \begin{vmatrix} u_1(t),\ldots,u_1^{(\nu-2)}(t),u_1^{(\nu)}(t),u_1(t_2^*),\ldots,u_1^{(\mu_k)}(t_k^*) \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ u_\ell(t),\ldots,u_\ell^{(\nu-2)}(t),u_\ell^{(\nu)}(t),u_\ell(t_2^*),\ldots,u_\ell^{(\mu_k)}(t_k^*) \end{vmatrix} .$$

d) $\hat{W}_0(\ell;t;\nu;x)$ is defined to be the determinant of the matrix associated with $W_0(\ell+1;t;\nu+1;x)$ with the last row and ν^{th} column deleted.

Note that

$$\frac{d}{dt} W_0(\ell;t;\nu) = \hat{W}_0(\ell;t;\nu)$$

and $$\frac{d}{dt} W_0(\ell;t;\nu;x) = \hat{W}_0(\ell;t;\nu;x) .$$

3. In the following six determinantal expressions $t_1^*,\ldots,t_{i-1}^*,t_{i+2}^*,\ldots,t_k^*$ and $\mu_1,\ldots,\mu_{i-1},\mu_{i+2},\ldots,\mu_k$ are held fixed.

a) $W_i(\ell;t;\nu) =$

$$U^*\begin{pmatrix} 1, & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & ,\ell \\ \underbrace{t_1^*,\ldots,t_1^*}_{\mu_1+1},\ldots, & \underbrace{t_{i-1}^*,\ldots,t_{i-1}^*}_{\mu_{i-1}+1}, & \underbrace{t,\ldots,t}_{\nu}, & \underbrace{t_{i+2}^*,\ldots,t_{i+2}^*}_{\mu_{i+2}+1},\ldots, & \underbrace{t_k^*,\ldots,t_k^*}_{\mu_k+1} \end{pmatrix}$$

b) $W_i(\ell;t;\nu;x) =$

$$U^*\begin{pmatrix} 1, & \cdots & & & & & & & , \ell \\ \underbrace{t_1^*,\dots,t_1^*}_{\mu_1+1},\dots,\underbrace{t_{i-1}^*,\dots,t_{i-1}^*}_{\mu_{i-1}+1},\underbrace{t,\dots,t}_{\nu},\underbrace{t_{i+2}^*,\dots,t_{i+2}^*}_{\mu_{i+2}+1},\dots,\underbrace{t_k^*,\dots,t_k^*}_{\mu_k+1},x \end{pmatrix}.$$

c) $\hat{W}_i(\ell;t;\nu)$ and

d) $\hat{W}_i(\ell;t;\nu;x)$ are defined in the analogous manner.

e) $V_i(\ell;t;\varepsilon;\alpha;\beta)$

$$= U^*\begin{pmatrix} 1, & \cdots & & & & & & , \ell \\ \underbrace{t_1^*,\dots,t_1^*}_{\mu_1+1},\dots,\underbrace{t_{i-1}^*,\dots,t_{i-1}^*}_{\mu_{i-1}+1},\underbrace{t,\dots,t}_{\alpha},\underbrace{t+\varepsilon,\dots,t+\varepsilon}_{\beta},\underbrace{t_{i+2}^*,\dots,t_{i+2}^*}_{\mu_{i+2}+1},\dots,\underbrace{t_k^*,\dots,t_k^*}_{\mu_k+1} \end{pmatrix}$$

f) $V_i(\ell;t;\varepsilon;\alpha;\beta;x)$

$$= U^*\begin{pmatrix} 1, & \cdots & & & & & & , \ell \\ \underbrace{t_1^*,\dots,t_1^*}_{\mu_1+1},\dots,\underbrace{t_{i-1}^*,\dots,t_{i-1}^*}_{\mu_{i-1}+1},\underbrace{t,\dots,t}_{\alpha},\underbrace{t+\varepsilon,\dots,t+\varepsilon}_{\beta},\underbrace{t_{i+2}^*,\dots,t_{i+2}^*}_{\mu_{i+2}+1},\dots,\underbrace{t_k^*,\dots,t_k^*}_{\mu_k+1} \end{pmatrix}.$$

When $k = 2$, we drop the subscript i in the definition of the W's and V's.

4. We shall exploit Sylvester's determinant identity recorded here for easy reference. Let A be a fixed $m \times m$ matrix. Specify two sets of p tuples of indices, $1 \leq \gamma_1 < \dots < \gamma_p \leq m$, $1 \leq \delta_1 < \dots < \delta_p \leq m$ to be held fixed. For each index i $(1 \leq i \leq m)$ not contained in the set $\underline{\gamma} = (\gamma_1,\dots,\gamma_p)$, and index j $(1 \leq j \leq m)$ not contained in the set $\underline{\delta} = (\delta_1,\dots,\delta_p)$, we form

$$b_{ij} = A\begin{pmatrix} k_1,\dots,k_{p+1} \\ \ell_1,\dots,\ell_{p+1} \end{pmatrix}$$

where $\{k_1,\dots,k_{p+1}\}$ comprises the set of indices

$\{i,\gamma_1,\dots,\gamma_p\}$ arranged in increasing order, and $\{\ell_1,\dots,\ell_{p+1}\}$ consists of the set of indices $\{j,\delta_1,\dots,\delta_p\}$ also appearing in increasing order. For any selection of indices $i_1 < \dots < i_q$, $i_\nu \not\in \underline{\gamma}$, $\nu = 1,\dots,q$, and $j_1 < \dots < j_q$, $j_\mu \not\in \underline{\delta}$, $\mu = 1,\dots,q$, $q \le m-p$, we have the identity (known as the Sylvester determinant identity)

$$(2.1)\quad B\begin{pmatrix} i_1,\dots,i_q \\ j_1,\dots,j_q \end{pmatrix} = \left[A\begin{pmatrix} \gamma_1,\dots,\gamma_p \\ \delta_1,\dots,\delta_p \end{pmatrix}\right]^{q-1} A\begin{pmatrix} \alpha_1,\dots,\alpha_{p+q} \\ \beta_1,\dots,\beta_{p+q} \end{pmatrix}$$

where $\{\alpha_1,\dots,\alpha_{p+q}\} = \{i_1,\dots,i_q,\gamma_1,\dots,\gamma_p\}$ and $\{\beta_1,\dots,\beta_{p+q}\} = \{j_1,\dots,j_q,\delta_1,\dots,\delta_p\}$ are each arranged in increasing order. The submatrix of A, composed of the rows of indices $\gamma_1,\dots,\gamma_p$ and columns of indices $\delta_1,\dots,\delta_p$ is called the pivot block in the application of Sylvester's determinant identity.

5. Proposition 2.1. $W_0(n+1;t;\mu_1+1;x)\,\hat{W}_0(n;t;\mu_1+1) - \hat{W}_0(n+1;t;\mu_1+1;x)\,W_0(n;t;\mu_1+1) = W_0(n;t;\mu_1;x)\,W_0(n+1;t;\mu_1+2)$.

Proof. Consider the $n+2 \times n+2$ matrix

$$C = \begin{bmatrix} u_1(t),\dots,u_1^{(\mu_1)}(t), & u_1^{(\mu_1+1)}(t), & u_1(t_2^*),\dots,u_1^{(\mu_k)}(t_k^*), & u_1(x) \\ \vdots & \vdots & \vdots & \vdots \\ u_{n+1}(t),\dots,u_{n+1}^{(\mu_1)}(t), & u_{n+1}^{(\mu_1+1)}(t), & u_{n+1}(t_2^*),\dots,u_{n+1}^{(\mu_k)}(t_k^*), & u_{n+1}(x) \\ 0 \ \dots \ 0 & 0 & 0 \ \dots \ 0 & 1 \end{bmatrix}.$$

(Recall that $n = \sum_{i=1}^{k} \mu_i + k$.) Proposition 2.1 results by application of Sylvester's determinant identity applied to the above matrix, where the pivot block is composed of the first n rows of C, and all columns of C aside from the $\mu_1 + 1^{st}$ and $\mu_1 + 2^{nd}$.

Q.E.D.

6. Proposition 2.2. If $J_{nm} = \frac{1}{(n-m)!}$ for $n \geq m$; 0 for $n < m$, then for any non-negative integers b and a,

$$1)\quad J\begin{pmatrix} b,b+1,\ldots,b+a \\ 0,1,\ldots,a \end{pmatrix} = \frac{a!(a-1)!\ldots 1!}{(b+a)!\ldots(b+1)!b!},$$

and

$$2)\quad J\begin{pmatrix} b,b+1,\ldots,b+i-1,b+i+1,\ldots,b+a+1 \\ 0,1, \quad \ldots \quad ,a \end{pmatrix} =$$

$$= \frac{a!(a-1)!\ldots 1!}{(b+a+1)!\ldots(b+i+1)!(b+i-1)!\ldots b!}\binom{a+1}{i},$$

$$i = 0,1,\ldots,a+1 .$$

Proof. To prove 1), note that $\binom{n}{m} = \frac{n!}{m!} J_{nm}$, and letting $k_{nm} = \binom{n}{m}$, 1) is equivalent to showing

$K\begin{pmatrix} b,b+1,\ldots,b+a \\ 0,1 \;,\ldots, \; a \end{pmatrix} \equiv 1$. This latter identity is readily validated invoking repeatedly the identity $\binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1}$.

The proof of 2) reduces to showing

$K\begin{pmatrix} b,b+1,\ldots,b+i-1,b+i+1,\ldots,b+a+1 \\ 0,1 \;, \quad \ldots \quad ,a \end{pmatrix} = \binom{a+1}{i}$. We proceed by induction on a. Note that

$$\binom{a+1}{i} = \binom{a}{i} + \binom{a}{i-1} = K\begin{pmatrix} b,b+1,\ldots,b+i-1,b+i+1,\ldots,b+a \\ 0,1 \quad , \quad \ldots \quad ,a-1 \end{pmatrix} +$$

$$+ K\begin{pmatrix} b,b+1,\ldots,b+i-2,b+i,\ldots,b+a \\ 0,1 \quad , \quad \ldots \quad ,a-1 \end{pmatrix}$$

by the induction hypothesis. Subtract from each row of $K\begin{pmatrix} b,b+1,\ldots,b+i-1,b+i+1,\ldots,b+a+1 \\ 0,1 \quad , \quad \ldots \quad ,a \end{pmatrix}$ the previous row and use the binomial identities $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ and $\binom{n+1}{k} - \binom{n-1}{k} = \binom{n}{k-1} + \binom{n-1}{k-1}$.

Q.E.D.

3. PROOF OF EXTREMAL PROPERTY (THEOREM D)

Let $F(t_1,\ldots,t_k) = \int_a^b u(x;t_1,\ldots,t_k)dx$, where $\underline{t} \in \Gamma$.

As we stated in the introduction, we need to prove that any $\{t_i^*\}_1^k$ minimizing $F(t_1,\ldots,t_k)$ over Γ satisfies $a < t_1^* < \ldots < t_k^* < b$. The proof of Theorem C is instructive. In the present context due to the fact that a general ECT-system lacks the factoring properties of polynomial systems, the analysis is more intricate.

Lemma 3.1 . $a < t_1^*$ and $t_k^* < b$.

Proof. Assume, without loss of generality, that $a = t_1^* < t_2^* < \ldots < t_k^* < b$. Since $F(a,t_2^*,\ldots,t_k^*)$ is a minimum for $\underline{t} \in \Gamma$, $\left.\frac{\partial}{\partial t} F(t,t_2^*,\ldots,t_k^*)\right|_{t=a} \geq 0$. Now, for $a \leq t_1 < \ldots < t_k \leq b$,

$$(3.1)\qquad u(x;t_1,\dots,t_k) = \frac{U^*\begin{pmatrix}1, & \cdot & \cdot & \cdot & ,n+1\\ t_1,\dots,t_1,t_2,\dots,t_2,\dots,t_k,\dots,t_k,x\end{pmatrix}}{U^*\begin{pmatrix}1, & \cdot & \cdot & \cdot & ,n\\ t_1,\dots,t_1,t_2,\dots,t_2,\dots,t_k,\dots,t_k\end{pmatrix}}$$

(t_i occurring μ_i+1 times, $i = 1,2,\dots,k$) . Therefore,

$$(3.2)\qquad \frac{\partial}{\partial t_1} F(t_1,t_2^*,\dots,t_k^*)\Big|_{t_1=a} = \frac{\partial}{\partial t_1}\int_a^b \frac{W_0(n+1;t_1;\mu_1+1;x)}{W_0(n;t_1;\mu_1+1)}\,dx\Big|_{t_1=a}$$

$$= \int_a^b \frac{\hat{W}_0(n+1;a;\mu_1+1;x)\,W_0(n;a;\mu_1+1) - W_0(n+1;a;\mu_1+1;x)\,\hat{W}_0(n;a;\mu_1+1)}{[W_0(n,a,\mu_1+1)]^2}\,dx .$$

Simplifying in accordance with Proposition 2.1, we obtain,

$$\frac{\partial}{\partial t_1} F(t_1,t_2^*,\dots,t_k^*)\Big|_{t_1=a} = -\int_a^b \frac{W_0(n;a;\mu_1;x)\,W_0(n+1,a;\mu_1+2)}{[W_0(n;a;\mu_1+1)]^2}\,dx .$$

Because $\{u_i(x)\}_1^{n+1}$ is an ECT-system on $[a,b]$, the determinants $W_0(n;a;\mu_1+1)$ and $W_0(n+1;a;\mu_1+2)$ are positive. Moreover, since all μ_i are odd we have $W_0(n;a;\mu_1;x) \geq 0$ for x in $[a,b]$ and vanishing only at $x = a,t_2^*,\dots,t_k^*$. Therefore $\frac{\partial}{\partial t_1} F(t_1,t_2^*,\dots,t_k^*)\Big|_{t_1=a} < 0$, contradicting a previous fact. Thus, we necessarily have $t_1^* > a$. A similar analysis validates the inequality $t_k^* < b$.

<u>Lemma 3.2 .</u> $t_1^* < t_2^* < \dots < t_k^*$.

<u>Proof.</u> We shall assume, without loss of generality, that $a < t_1^* < \dots < t_{i-1}^* < t_i^* = t_{i+1}^*$ $(= t^*)$ $< t_{i+2}^* < \dots < t_k^* < b$, for some i $(1 \leq i \leq k-1)$. Set for convenience $t_0^* = a$ and $t_{k+1}^* = b$. Fix $t_j = t_j^*$, $j \neq i$, $i+1$, and regard

$F(t_1,\dots,t_k)$ as a function of t_i and t_{i+1}, denoted more compactly by $G(t_i,t_{i+1})$. A Taylor expansion gives

$$G(t^*-\delta,t^*+\varepsilon) = G(t^*,t^*)-\delta\, G_1(t^*,t^*)+\varepsilon G_2(t^*,t^*) + \frac{\delta^2}{2} G_{11}(t^*,t^*) +\frac{\varepsilon^2}{2} G_{22}(t^*,t^*)-\varepsilon\delta\, G_{12}(t^*,t^*)+0(|\varepsilon|^3+|\delta|^3),$$

where G_i and G_{ij} are appropriate first and second order partials, respectively.

We shall prove later (Lemma 3.4) that $G_1(t^*,t^*) = G_2(t^*,t^*) = 0$. When this is done, then since $G(t^*,t^*)$ is a local minimum of $G(t_i,t_{i+1})$, we may conclude that

$$\delta^2 G_{11}(t^*,t^*) + \varepsilon^2 G_{22}(t^*,t^*) - 2\varepsilon\delta\, G_{12}(t^*,t^*) \geq 0$$

for all ε,δ small and positive. In particular, with $\delta = c\varepsilon$, $c \geq 0$, it follows that

$$c^2 G_{11}(t^*,t^*)-2c\, G_{12}(t^*,t^*)+G_{22}(t^*,t^*) \geq 0 \quad \text{for all} \quad c > 0 .$$

We shall ultimately contradict this statement thereby compelling $t_1^* < t_2^* < \dots < t_k^*$.

The proof of Lemma 3.2 is divided into a series of steps.

<u>Lemma 3.3.</u>

$$V_i(\ell;t;\varepsilon;\alpha;\beta) = \varepsilon^{\alpha\beta}\Big[W_i(\ell;t;\alpha+\beta)\, J\begin{pmatrix}\alpha,\alpha+1,\dots,\alpha+\beta-1\\ 0,1\ ,\dots,\ \beta-1\end{pmatrix} + \varepsilon\hat{W}_i(\ell;t;\alpha+\beta)\, J\begin{pmatrix}\alpha,\alpha+1,\dots,\alpha+\beta-2,\alpha+\beta\\ 0,1\ ,\dots,\ \beta-1\end{pmatrix} + 0(\varepsilon^2)\Big],$$

<u>where</u> $J\begin{pmatrix}\alpha_1,\dots,\alpha_s\\ \beta_1,\dots,\beta_s\end{pmatrix}$ <u>is determined as in Proposition</u> 2.2.

<u>Proof.</u> To ease the notational complexity we carry out the proof assuming $k=2$, such that $\alpha+\beta = \ell$, and

$$V(\ell;t;\varepsilon;\alpha;\beta) = U^*\begin{pmatrix} 1, & \cdots & ,\alpha+\beta \\ \underbrace{t,\ldots,t}_{\alpha}, & \underbrace{t+\varepsilon,\ldots,t+\varepsilon}_{\beta} \end{pmatrix}$$ (a little reflection

reveals that the proof is the same for any k apart from more tedious notational inconveniences).

Now, each $u_i^{(j)}(t+\varepsilon)$ permits a Taylor series expansion
$u_i^{(j)}(t+\varepsilon) = \sum_{k=0}^{\ell-j} \frac{\varepsilon^k}{k!} u_i^{(j+k)}(t) + O(\varepsilon^{\ell-j+1})$, $i = 1,\ldots,\alpha+\beta$; $j = 0,1,\ldots,\beta-1$. Let $y_{ij} = u_i^{(j)}(t)$, $j = 0,1,\ldots,s$; $i = 1,\ldots,\alpha+\beta$, where $s \geq \ell = \alpha+\beta$, and

$$M_{ij} = \begin{cases} \delta_{ij} & i = 1,\ldots,s \ ; \ j = 1,\ldots,\alpha \\ \dfrac{\varepsilon^{i-j+\alpha}}{(i-j+\alpha)!} & i = 1,\ldots,s \ ; \ j = \alpha+1,\ldots,\alpha+\beta \end{cases} .$$

(If $i-j+\alpha < 0$, we define $M_{ij} = 0$.)

Then,

$$U^*\begin{pmatrix} 1, & \cdots & ,\alpha+\beta \\ \underbrace{t,\ldots,t}_{\alpha}, & \underbrace{t+\varepsilon,\ldots,t+\varepsilon}_{\beta} \end{pmatrix} = \sum_{1\leq i_1<\cdots<i_{\alpha+\beta}\leq s}$$

$$\times\ Y\begin{pmatrix} 1\ ,\ldots,\alpha+\beta \\ i_1,\ldots,i_{\alpha+\beta} \end{pmatrix} M\begin{pmatrix} i_1,\ldots,i_{\alpha+\beta} \\ 1\ ,\ldots,\alpha+\beta \end{pmatrix} + O(\varepsilon^{\alpha\beta+2}) .$$

In light of the structure of M , we find that

$$U^*\begin{pmatrix}1, & \cdots & ,\alpha+\beta \\ \underbrace{t,\ldots,t}_{\alpha},\underbrace{t+\varepsilon,\ldots,t+\varepsilon}_{\beta}\end{pmatrix} = \sum_{\alpha+1\le i_{\alpha+1}<\ldots<i_{\alpha+\beta}\le s}$$

$$\times\, Y\begin{pmatrix}1, & \cdots & \alpha+\beta \\ 1,\ldots,\alpha,i_{\alpha+1},\ldots,i_{\alpha+\beta}\end{pmatrix} M\begin{pmatrix}i_{\alpha+1},\ldots,i_{\alpha+\beta} \\ \alpha+1\ ,\ldots,\ \alpha+\beta\end{pmatrix} + O(\varepsilon^{\alpha\beta+2}).$$

Consider $M\begin{pmatrix}i_{\alpha+1},\ldots,i_{\alpha+\beta} \\ \alpha+1\ ,\ldots,\ \alpha+\beta\end{pmatrix}$. Expanding this determinant we find that the smallest power in ε is $\varepsilon^{\alpha\beta}$ and its term is

$$M\begin{pmatrix}\alpha+1,\ldots,\alpha+\beta \\ \alpha+1,\ldots,\alpha+\beta\end{pmatrix} \cdot Y\begin{pmatrix}1,\ldots,\alpha+\beta \\ 1,\ldots,\alpha+\beta\end{pmatrix}$$

$$= \varepsilon^{\alpha\beta}\ J\begin{pmatrix}\alpha,\alpha+1,\ldots,\alpha+\beta-1 \\ 0,1\ ,\ldots,\ \beta-1\end{pmatrix} \cdot W(\alpha+\beta;t;\alpha+\beta)\ .$$

Also, the only term of the power $\varepsilon^{\alpha\beta+1}$ is

$$M\begin{pmatrix}\alpha+1,\ldots,\alpha+\beta-1,\alpha+\beta+1 \\ \alpha+1, & \cdots & ,\alpha+\beta\end{pmatrix}\ Y\begin{pmatrix}1, & \cdots & ,\alpha+\beta \\ 1,\ldots,\alpha+\beta-1,\alpha+\beta+1\end{pmatrix}\ .$$

The identity of the lemma is hereby established. Q.E.D.

Since $G(t^*,t^*)$ is a minimum of $G(t,t)$ over $t^*_{i-1} < t < t^*_{i+2}$, then viewing t^* as a node of multiplicity $\mu_i + \mu_{i+1} + 2$, with t_i and t_{i+1} coalesced, it follows that

$$\frac{\partial G}{\partial t}\,(t^*,t^*) = 0\ .$$

Now,

$$(3.3)\qquad G(t,t) = \int_a^b \frac{W_i(n+1;t;\mu+\nu;x)}{W_i(n;t;\mu+\nu)}\,dx\quad ,$$

where $\mu_i+1 = \mu$, $\mu_{i+1}+1 = \nu$. Differentiating (3.3) with respect to t , and adapting the analysis of Lemma 3.1, see following (3.2), we infer the equation

$$(3.4) \qquad \int_a^b W_i(n;t^*;\mu+\nu-1;x)\,dx = 0 .$$

Lemma 3.4 . $G_1(t^*,t^*) = G_2(t^*,t^*) = 0$.

Proof. We shall prove that $G_2(t^*,t^*) = 0$; the proof of $G_1(t^*,t^*) = 0$ is similar. By definition

$$(3.5) \qquad G_2(t^*,t^*) = \lim_{\varepsilon\to 0} \frac{G(t^*,t^*+\varepsilon)-G(t^*,t^*)}{\varepsilon} .$$

Now, $G(t^*,t^*+\varepsilon) = \int_a^b \frac{V_i(n+1;t^*;\varepsilon;\mu;\nu;x)}{V_i(n;t^*;\varepsilon;\mu;\nu)}\,dx$ and

$$G(t^*,t^*) = \int_a^b \frac{W_i(n+1;t^*;\mu+\nu;x)}{W_i(n;t^*;\mu+\nu)}\,dx .$$

The computation of (3.5) with the aid of Proposition 2.1 with application of Lemma 3.3 to $V_i(n+1;t^*;\varepsilon;\mu;\nu;x)$ and $V_i(n;t^*;\varepsilon;\mu;\nu)$ leads to the representation

$$G_2(t^*,t^*) = -\frac{J\begin{pmatrix}\mu,\mu+1,\ldots,\mu+\nu-2,\mu+\nu\\ 0,1\ \ ,\ldots,\ \ \nu-1\end{pmatrix}}{J\begin{pmatrix}\mu,\mu+1,\ldots,\mu+\nu-1\\ 0,1\ \ ,\ldots,\ \ \nu-1\end{pmatrix}} \frac{W_i(n+1;t^*;\mu+\nu+1)}{[W_i(n;t^*;\mu+\nu)]^2}$$

$$\times \int_a^b W_i(n;t^*;\mu+\nu-1;x)\,dx .$$

The assertion of the lemma is now manifest by virtue of (3.4).

Lemma 3.5. *If* $\int_a^b W_i(n;t^*;\mu+\nu-1;x)\,dx = 0$, *then* $\int_a^b \hat{W}_i(n;t^*;\mu+\nu-1;x)\,dx < 0$.

Proof. On the basis of Proposition 2.1, we obtain $W_i(n-1;t^*;\mu+\nu-2;x)\; W_i(n;t^*;\mu+\nu) = W_i(n;t^*;\mu+\nu-1;x) \times \hat{W}_i(n-1;t^*;\mu+\nu-1) - \hat{W}_i(n;t^*;\mu+\nu-1;x)\; W_i(n-1;t^*;\mu+\nu-1)$. Now, $W_i(n;t^*;\mu+\nu)$ and $W_i(n-1;t^*;\mu+\nu-1)$ are positive and $\int_a^b W_i(n;t^*;\mu+\nu-1;x)\,dx = 0$ by assumption. Thus

$$\operatorname{sgn}\left[\int_a^b W_i(n-1;t^*;\mu+\nu-2;x)\,dx\right] = -\operatorname{sgn}\left[\int_a^b \hat{W}_i(n;t^*;\mu+\nu-1;x)\,dx\right].$$

Since $\mu+\nu-2$ is even, $W_i(n-1;t^*;\mu+\nu-2;x) \geq 0$ and $\not\equiv 0$ for some $x \in [a,b]$. Therefore

$$\int_a^b \hat{W}_i(n;t^*;\mu+\nu-1;x)\,dx < 0 .$$

Q.E.D.

Lemma 3.6. $G_{22}(t^*,t^*) =$

$$- \frac{J\begin{pmatrix}\mu,\mu+1,\ldots,\mu+\nu-3,\mu+\nu-1\\ 0,1\;\;,\ldots,\;\;\nu-2\end{pmatrix} J\begin{pmatrix}\mu,\mu+1,\ldots,\mu+\nu\\ 0,1\;\;,\ldots,\;\;\nu\end{pmatrix}}{\left[J\begin{pmatrix}\mu,\mu+1,\ldots,\mu+\nu-1\\ 0,1\;\;,\ldots,\;\;\nu-1\end{pmatrix}\right]^2}$$

$$\times \frac{W_i(n+1;t^*;\mu+\nu+1)}{[W_i(n;t^*;\mu+\nu)]^2} \int_a^b \hat{W}_i(n;t^*;\mu+\nu-1;x)\,dx .$$

Proof. The verification of this identity uses the facts of Lemma 3.3 and equation (3.4) and the representation

$$G_2(t^*,t^*+\varepsilon) = -\int_a^b \frac{V_i(n+1;t^*;\varepsilon;\mu;\nu+1)V_i(n;t^*;\varepsilon;\mu;\nu-1;x)}{[V_i(n;t^*;\varepsilon;\mu;\nu)]^2} \, dx \; .$$

Q.E.D.

Set $$M_i = -\frac{W_i(n+1;t^*;\mu+\nu+1)}{[W_i(n;t^*;\mu+\nu)]^2} \int_a^b \hat{W}_i(n;t^*;\mu+\nu-1;x) \, dx \; .$$

On the basis of Lemma 3.5 we find that $M_i > 0$. Moreover, an application of Proposition 2.2 yields

$$G_{22}(t^*,t^*) = \frac{\nu(\nu-1)}{(\mu+\nu)(\mu+\nu-1)} M_i \quad .$$

By symmetry considerations, we have

$$G_{11}(t^*,t^*) = \frac{\mu(\mu-1)}{(\mu+\nu)(\mu+\nu-1)} M_i \quad .$$

It is possible to calculate $G_{12}(t^*,t^*)$ following the above techniques. However, we can also ascertain $G_{12}(t^*,t^*)$ from the equation

$$\frac{\partial^2 G(t^*,t^*)}{\partial t^2} = G_{11}(t^*,t^*) + G_{22}(t^*,t^*) + 2\, G_{12}(t^*,t^*)$$

coupled to the fact of

Lemma 3.7 . $$\frac{\partial^2 G(t^*,t^*)}{\partial t^2} = M_i \quad .$$

Proof. Paralleling the discussion of (3.2) and (3.3), we get

$$\frac{\partial G(t^*,t^*)}{\partial t} = -\frac{W_i(n+1;t^*;\mu+\nu+1)}{[W_i(n;t^*;\mu+\nu)]^2} \int_a^b W_i(n;t^*;\mu+\nu-1;x) \, dx \quad .$$

Since, by (3.4), $\int_a^b W_i(n;t^*;\mu+\nu-1;x) \, dx = 0$,

$$\frac{\partial^2 G(t^*,t^*)}{\partial t^2} = -\frac{W_i(n+1;t^*;\mu+\nu+1)}{[W_i(n;t^*;\mu+\nu)]^2}\int_a^b \frac{\partial}{\partial t} W_i(n;t^*;\mu+\nu-1;x)\,dx$$

$$= -\frac{W_i(n+1;t^*;\mu+\nu+1)}{[W_i(n;t^*;\mu+\nu)]^2}\int_a^b \hat{W}_i(n;t^*;\mu+\nu-1;x)\,dx$$

$$= M_i \ .$$

Q.E.D.

Combining as indicated earlier leads to

$$G_{12}(t^*,t^*) = \frac{\mu\nu}{(\mu+\nu)(\mu+\nu-1)} M_i \ .$$

Substituting in the values of G_{11}, G_{12} and G_{22} in $c^2G_{11}(t^*,t^*)-2c\,G_{12}(t^*,t^*) + G_{22}(t^*,t^*) \geq 0$, we obtain

$$\text{(3.6)} \qquad \frac{M_i}{(\mu+\nu)(\mu+\nu-1)} [c^2\mu(\mu-1) - 2c\mu\nu + \nu(\nu-1)] \geq 0$$

for all $c > 0$, where $\mu,\nu \geq 2$. However, it is easy to see that the quadratic changes sign (twice in fact) for c positive. This contradiction as argued earlier implies $t_1^* < \dots < t_k^*$.

Q.E.D.

The proof of Theorem D is complete.

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