SIAM J. NUMER. ANAL.
Vol. 17, No. 3, June 1980

0036-1429/80/1703-0005 $01.00/0

ON THE ERDÖS CONJECTURE CONCERNING MINIMAL NORM INTERPOLATION ON THE UNIT CIRCLE*

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Abstract. The purpose of this note is to complete the proof of the Erdös conjecture concerning optimal points of polynomial interpolation on the unit circle when the number of these points is even.

1. Introduction. Let $\Gamma=\{z: |z|=1\}$, $D=\{z: |z|\leqq 1\}$ and $C(D)$ denote the space of continuous functions in D supplied with the supnorm $\|f\|=\max_{z\in D}|f(z)|$. Let $z=\{z_1, z_2, \cdots, z_{n+1}\}$ be $(n+1)$ distinct points on Γ, and P_z be the Lagrange interpolating operator based on this point. Obviously, P_z is a projection from $C(D)$ onto π_n, the space of polynomials of degree $\leqq n$. The norm of P_z is given by

$$\|P_z\|\equiv\|\Lambda_z\|=\max_{z\in D}\Lambda_z(z)==\max_{z\in\Gamma}\Lambda_z(z),$$

where $\Lambda_z(z)\equiv\sum_{k=1}^{n+1}\prod_{\substack{j=1\\ j\neq k}}^{n+1}|z-z_j|/|z_k-z_j|$ is the so-called Lebesgue function.

It has been conjectured by Erdös [3] that the norm $\|\Lambda_z\|$ is minimal when the nodes $\{z_k\}$ are equally distributed on Γ. For the case when the number of interpolation nodes is odd, this conjecture has been proved in [2]. The proof is based on the equivalence of the above problem to a corresponding minimum norm problem for trigonometric polynomials on $[0, 2\pi)$. This latter problem has been solved by de Boor and Pinkus in [1], where they extended techniques primarily developed by Kilgore [4] in order to characterize the optimal points of interpolation by algebraic polynomials on a finite real interval. Here we are completing the proof of the Erdös conjecture for the case when the number of interpolation points is even. The proof also consists in reducing the minimum norm problem on the unit circle to the corresponding trigonometric problem and using the techniques of [4] and [1], but instead of usual trigonometric polynomials the "trigonometric" polynomials of the half-angles will be used.

2. Results. The main result we are going to prove is the following theorem.

THEOREM 1. *Among all polynomial projections from $C(D)$ onto π_n induced by interpolation at $(n+1)$ distinct points $z_1, \cdots, z_{n+1}$ on the unit circle, the projections corresponding to the equally distributed points, i.e., $z_k^*=e^{2k\pi/(n+1)+\alpha}$ $(k=1,\cdots,n+1;$ some $\alpha)$ are the only ones of minimal norm.*

Proof. Since Theorem 1 has been proven for $n=2m$ in [2], we consider only the case when the degree of the polynomial is odd, $n=2m-1$. As has been noted above, the proof consists in reducing this minimum norm problem to the equivalent "trigonometric" interpolation problem. For this purpose let us define the following $2m$-dimensional subspace

$$H_m=\operatorname{span}\left\{\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, \cdots, \sin\frac{2m-1}{2}\theta, \cos\frac{2m-1}{2}\right\},$$

and let $T_{\underline{\theta}}: \tilde{C}[0, 2\pi)\to H_m$ be the interpolating operator at the points $\underline{\theta}=(\theta_1,\cdots,\theta_{2m})$, where $\tilde{C}[0, 2\pi)$ denotes the class of continuous functions satisfying $f(0)=-f(2\pi)$. The operator $T_{\underline{\theta}}$ is well-defined since H_m is a Chebyshev system of order $2m$ on $[0, 2\pi)$ (see, for example, [5]).

* Received by the editors October 10, 1979.

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Now if $\Lambda'_{\underline{\theta}}(\theta)$ denotes the associated Lebesgue function then the following identification may be easily verified (see [2]).

$$\Lambda'_{\underline{\theta}}(\theta)=\Lambda_{\underline{z}}(e^{i\theta}), \qquad 0\leqq\theta<2\pi, \quad \underline{z}=\{e^{i\theta_k}\}_{k=1}^{2m}; \tag{1}$$
$$0\leqq\theta_1<\cdots<\theta_{2m}<2\pi.$$

Equation (1) shows that the proof of Theorem 1 is equivalent to the proof of the following theorem.

THEOREM 2. $\min_{\underline{\theta}\in\Theta}\max_{0\leqq\theta<2\pi}\Lambda'_{\underline{\theta}}(\theta)$ *is attained only if* $\theta_k=(k-1)\pi/m+\alpha$; $k=1,\cdots,2m$, *some* α.

Proof. The proof follows the pattern of the proof given in [1]. To ease notation let us make a change of variable so that our domain is $[0,\pi)$ rather than $[0,2\pi)$. Therefore set

$$S_m=\operatorname{span}\{\sin\theta,\cos\theta,\sin 3\theta,\cos 3\theta,\cdots,\sin(2m-1)\theta,\cos(2m-1)\theta\}.$$

Now set

$$\Theta=\{\underline{\theta}:\underline{\theta}=(\theta_1,\cdots,\theta_{2m}),\quad 0\leqq\theta_1<\cdots<\theta_{2m}<\pi\},$$

and for each $\underline{\theta}$ let $S_k\in S_m$ denote the unique "polynomial" satisfying $S_k(\theta_j)=\delta_{kj}$, $k,j=1,\cdots,2m$. It may be verified that $\Lambda_{\underline{\theta}}(\theta)=\sum_{k=1}^{2m}|S_k(\theta)|$, where

$$S_k(\theta)=\prod_{\substack{j=1\\ j\neq k}}^{2m}\sin(\theta-\theta_j)/\sin(\theta_k-\theta_j).$$

Note that $\Lambda_{\underline{\theta}}(\theta)$, on each interval (θ_k,θ_{k+1}), $k=1,\cdots,2m$ $(\theta_{2m+1}\equiv\pi+\theta_1)$, is a "polynomial" in S_m, which will be denoted by F_k. Let τ_k denote the unique point in (θ_k,θ_{k+1}) at which $\Lambda_{\underline{\theta}}(\theta)$ attains its maximum value, and let $\lambda_k(\underline{\theta})$ denote this maximum value, $k=1,2,\cdots,2m$. Without loss of generality let $\theta_1=0$. It remains to solve the problem of finding the minimum of $\|\Lambda_{\underline{\theta}}\|$ over $\underline{\theta}\in\Theta$.

We begin with a proof of the claim that

$$J_k=\det\left\|\frac{\partial\lambda_i(\underline{\theta})}{\partial\theta_j}\right\|_{\substack{i=1\\ i\neq k}\ j=2}^{2m\ \ 2m}\neq 0\quad\text{for all }\underline{\theta}\in\Theta\text{ and }k=1,\cdots,2m. \tag{2}$$

In view of a formula in [4, p. 275],

$$\frac{\partial\lambda_i(\underline{\theta})}{\partial\theta_j}=-F_i'(\theta_j)S_j(\tau_i)=\prod_{k=1}^{2m}\sin(\tau_i-\theta_k)\frac{F_i'(\theta_j)}{\sin(\theta_j-\tau_i)}\Big/\prod_{\substack{k=1\\ k\neq j}}^{2m}\sin(\theta_j-\theta_k),$$

so that $J_k\neq 0$ for all $\underline{\theta}\in\Theta$ is equivalent to proving that

$$\det\left\|q_i(\theta_j)\right\|_{\substack{i=1\\ i\neq k}\ j=2}^{2m\ \ 2m}\neq 0 \tag{3}$$

for all $\underline{\theta}\in\Theta$, where $q_i(\theta)=F_i'(\theta)/\sin(\theta-\tau_i)$, $i=1,\cdots,2m$. In order to prove (3), suppose to the contrary that $J_k=0$ for some $k=1,\cdots,2m$ and some $\underline{\theta}\in\Theta$. Then there exists a set of nontrivial coefficients, $a_1,\cdots,a_{k-1},a_{k+1},\cdots,a_{2m}$ for which $\sum_{i=1,i\neq k}^{2m}a_iq_i(\theta)$ vanishes at $\theta_2\theta_3,\cdots,\theta_{2m}$. We now show that this implies $\sum_{i=1,i\neq k}^{2m}a_iq_i(\theta)\equiv 0$. Indeed $q_i(\theta)$ is a polynomial in T_{2m-2}, the trigonometric polynomials of degree $\leqq 2m-2$, such that $q_i(\theta+\pi)=q_i(\theta)$, $\theta\in[0,\pi)$. Thus on $[0,2\pi)$, the combination $\sum_{i=1,i\neq k}^{2m}a_iq_i(\theta)$, which is also a polynomial in T_{2m-2}, vanishes at $\theta_2,\theta_3,\cdots,\theta_{2m},\theta_2+\pi,\cdots,\theta_{2m}+\pi$, i.e., $2(2m-1)$ times.

However every nontrivial function in T_{2m-2} has at most $2(2m-2)$ zeros on $[0, 2\pi)$. Thus $\sum_{i=1, i\neq k}^{2m} a_i q_i(\theta) \equiv 0$.

Now we shall complete the proof of (3) by showing that if $\sum_{i=1}^{2m} b_i q_i(\theta) \equiv 0$, then either all $b_i = 0$, $i = 1, \cdots, 2m$ or all the coefficients $b_i \neq 0$, $i = 1, \cdots, 2m$, contradicting the above choice of a_i. The proof of this last statement is based on the following lemma, which is analogous to Lemma 6 of [1] and is given here without proof.

LEMMA. *Let* $\tau_1^{(i)}, \cdots, \tau_{2m-1}^{(i)}$ *denote the zeros of* $F_i'(\theta)$, *all simple, cyclically ordered in some interval of length* π *so that* $\tau_k^{(k)} = \tau_k$, $k = 1, \cdots, 2m-1$, *and* $\tau_1^{(2m)} = \tau_{2m}$. *Then*

$$0 \leqq \tau_1^{(i)} < \tau_1^{(i-1)} < \cdots < \tau_1^{(1)} < \tau_2^{(2m)} < \cdots < \tau_{2m-1}^{(1)} < \tau_1^{(2m)} < \cdots \tau_1^{(i+1)} < \pi$$

for some $i = 1, \cdots, 2m-1$.

Assume without loss of generality that $q_i(\tau_1) > 0$, $i = 1, \cdots, 2m$ and $b_1 \geqq 0$. From the Lemma it then follows that

$$\begin{aligned} \operatorname{sgn} q_i(\tau_j) &= (-1)^{j+1} \quad \text{for } i, j = 2, \cdots, 2m, \ i \neq j, \\ \operatorname{sgn} q_i(\tau_i) &= (-1)^i \quad \text{for } i = 2, \cdots, 2m, \\ \operatorname{sgn} q_1(\tau_j) &= (-1)^j \quad \text{for } j = 2, \cdots, 2m. \end{aligned}$$

Since $q_i(\tau_1) > 0$ for $i = 1, \cdots, 2m$, the set $N = \{i: i = 2, \cdots, 2m, b_i < 0\}$ is not empty. Let $P = \{2, \cdots, 2m\} \backslash N$ and set

$$r(\theta) = b_1 q_1(\theta) + \sum_{i \in N} b_i q_i(\theta) = -\sum_{i \in P} b_i q_i(\theta).$$

Now $(-1)^j r(\tau_j) = \sum_{i \in P} b_i (-1)^{j+1} q_i(\tau_j) \geqq 0$ for $j \notin P$, while

$$(-1)^j r(\tau_j) = b_1 (-1)^j q_1(\tau_j) + \sum_{i \in N} (-b_i)(-1)^{j+1} q_i(\tau_j) > 0 \quad \text{for } j \in P.$$

Thus $r(\theta)$, being a linear combination of the q_i, has at least $(2m-1)$ zeros in $[0, \pi)$ and hence by the previous analysis $r(\theta) \equiv 0$. Since $0 \equiv r(\theta)$, but $(-1)^j r(\tau_j) > 0$ for $j \in P$, it follows that $P = \phi$. Hence $b_i < 0$, $i = 2, \cdots, 2m$, which in turn implies by the above reasoning that $b_1 > 0$. Thus $b_i \neq 0$ for all $i = 1, \cdots, 2m$. This completes the proof of (3).

In order to complete the proof of Theorem 2, we note that if we are at any $\underline{\theta}^*$, a minimum of $\|\Lambda_{\underline{\theta}}\|$ we must have $\lambda_1(\underline{\theta}^*) = \cdots = \lambda_{2m}(\underline{\theta}^*)$. For assume to the contrary that there exists a $\underline{\theta}^* \in \Theta$, which is a minimum of $\|\Lambda_{\underline{\theta}}\|$, with at least one of the $\lambda_i(\underline{\theta}^*)$, say $\lambda_1(\underline{\theta}^*)$, strictly less than $\|\Lambda_{\underline{\theta}^*}\|$. Since $J_1 \neq 0$, it follows by the implicit function theorem that in any neighborhood of $\underline{\theta}^*$ we may strictly decrease the $\lambda_i(\underline{\theta})$, $i = 2, \cdots, 2m$, contradicting the minimality of $\underline{\theta}^*$. Since for the choice $\tilde{\theta}_k = (k-1)\pi/(2m)$, $k = 1, \cdots, 2m$ we have $\lambda_1(\tilde{\underline{\theta}}) = \cdots = \lambda_{2m}(\tilde{\underline{\theta}})$, it remains to show that there is a unique point $\underline{\theta}^* \in \Theta$ for which $\lambda_1(\underline{\theta}^*) = \cdots = \lambda_{2m}(\underline{\theta}^*)$. This is dcmonstrated by reasoning totally analogous to that found in the proof of Theorem 1 of [1].

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