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# The Closed Linear Span of $\{x^k - c_k\}_1^\infty$

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Let $\{c_k\}_1^\infty$ be a given real sequence. We wish to determine, in the first instance, easily verified conditions on $\{c_k\}_1^\infty$ which imply that the sequence of functions $\{x^k - c_k\}_1^\infty$ is *total* in $C[0, 1]$; that is, that the closed linear span $\bar{V}\{x^k - c_k\}$ is all of $C[0, 1]$ or, in other words, that every real function, continuous in $[0, 1]$, is the limit, in the uniform norm, of a sequence of finite linear combinations of the $x^k - c_k$. When this happens we refer to $\{c_k\}_1^\infty$ as an *approximating sequence*. Since the sequence $\{x^k\}_0^\infty$ is total in $C[0, 1]$, our problem is equivalent to demanding that the function $f(x) \equiv 1$ belong to $\bar{V}\{x^k - c_k\}$. In this case we wish, in the second instance, to find an effective approximation to $f(x) \equiv 1$ in the uniform norm on $[0, 1]$ by finite linear combinations of the $x^k - c_k$.

An equivalent formulation of our problem, which is sometimes useful, is afforded by the following proposition. This proposition is an elementary application of the Hahn–Banach theorem.

PROPOSITION 1. *The sequence $\{c_k\}_1^\infty$ is an approximating sequence if and*



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*only if it is not a Hausdorff moment sequence, i.e., if and only if, with* $c_0 = 1$, *there is no real function* $\mu$ *of bounded variation on* $[0, 1]$ *satisfying*

$$c_k = \int_0^1 x^k \, d\mu(x), \qquad k = 0, 1, 2, 3, \ldots . \tag{1}$$

To determine whether or not $\{c_k\}_1^\infty$ is an approximating sequence, one might try to apply the following well-known criterion [1, p. 99].

PROPOSITION 2. *A necessary and sufficient condition for the existence of a real function* $\mu$ *of bounded variation, satisfying* (1), *is the convergence, as* $n \uparrow \infty$, *of*

$$\sum_{\nu=0}^{n} |\lambda_{n\nu}|, \tag{2}$$

*where*

$$\lambda_{n\nu} = \binom{n}{\nu} \Delta^{n-\nu} c_\nu = \binom{n}{\nu} \sum_{k=0}^{n-\nu} (-1)^k \binom{n-\nu}{k} c_{\nu+k}, \qquad \nu = 0, 1, \ldots, n.$$

Note that the value of $c_0$ is immaterial in the above theorem since we may add point masses at $x = 0$. The convergence of (2) is rather difficult to verify and so we seek simpler sufficient conditions for a sequence $\{c_k\}_1^\infty$ to be an approximating sequence. We now present some such conditions.

1. If $\{c_k\}_1^\infty$ is a Hausdorff moment sequence, then $|c_k| \leqslant \|\mu\|$ for all $k$, where $\|\mu\|$ denotes the total variation of $\mu$. In addition, it follows from the dominated convergence theorem that the sequence $\{c_k\}_1^\infty$ must converge, i.e., $\lim_{k \to \infty} c_k$ exists. Thus if either $\limsup_{k \to \infty} |c_k| = \infty$ or $\lim_{k \to \infty} c_k$ does not exist, then $\{c_k\}_1^\infty$ is an approximating sequence. In the first case the function 1 can be approximated by $f_k(x) \equiv (c_{n_k})^{-1}(x^{n_k} - c_{n_k})$ for a suitable subsequence $\{c_{n_k}\}_1^\infty$.

2. If $\{c_k\}_1^\infty$ is *not* an approximating sequence and if we change $c_k$ for a finite number of $k$'s, however slightly, then the resulting sequence $\{c'_k\}_1^\infty$ is an approximating sequence. For if

$$F(z) = \int_0^1 x^z \, d\mu(x), \tag{3}$$

then $F(z)$ is holomorphic and bounded in $\operatorname{Re} z > \delta > 0$ for every $\delta > 0$, with $F(k) = c_k$. But such an $F(z)$ is uniquely determined by its values at all but a finite number of the $k$'s. In particular, if $c_k = c$ for all $k$, then $\{c_k\}$ is not an approximating sequence, so that if $c_k = c$ for all but a finite number of $k$, and there exists a $j$ for which $c_j \neq c$, then $\{c_k\}$ is an approximating sequence.

3. Suppose that the sequence $\{c_k\}_1^\infty$ is such that for all $k \geqslant M$,

$$\varepsilon(-1)^k(c_k - c) \geqslant 0, \tag{4}$$

where $c \in \mathbb{R}$ and $\varepsilon \in \{-1, 1\}$, fixed. If $c_k - c \not\equiv 0$, then $\{c_k\}_1^\infty$ is an approximating sequence. By subtracting a point mass at 1, if necessary, we may assume that $c = 0$ and that $c_r \neq 0$ for some $r \geqslant M$. Then for all $n \geqslant r$,

$$\sum_{\nu=0}^{n} |\lambda_{n\nu}| \geqslant |\lambda_{nr}| = \left| \binom{n}{r} \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} c_{r+k} \right|$$

$$\geqslant \binom{n}{r} |c_r| \to \infty \qquad \text{as} \quad n \uparrow \infty,$$

establishing the result.

As a slightly more general example we assume that

$$(-1)^k(c_{n_k} - c) \geqslant 0, \qquad k = 1, 2, \ldots, \tag{5}$$

where $\{n_k\}_1^\infty$ is a subsequence satisfying the Müntz condition $\sum_{k=1}^\infty (n_k)^{-1} = \infty$. If $c_k \neq c$ for some $k \geqslant 1$, then $\{c_k\}_1^\infty$ is an approximating sequence. We may again assume that $c = 0$ and consider the function $F(z)$ of (3). The condition (5) implies that $F(t_k) = 0$ for some $t_k$ with $n_k \leqslant t_k \leqslant n_{k+1}$. Since $\sum_{k=1}^\infty (t_k)^{-1} = \infty$, we see, by the uniqueness theorem for functions holomorphic and bounded in a half-plane, that $F(z) \equiv 0$, contradicting the fact that $c_k \not\equiv 0$.

4. If (4) holds and $c_k \neq c$ for infinitely many $k$, we can explicitly construct a good approximation to $f(x) \equiv 1$ on $[0, 1]$. For $n = 1, 2, 3, \ldots$, let $T_n(x) = \sum_{k=0}^n b_k^{(n)} x^k$ denote the $n$th Chebyshev polynomial of the first kind on $[0, 1]$, normalized so that $\|T_n\|_\infty = b_0^{(n)} = 1$. We choose $\{a_k^{(n)}\}_1^n$ and $A(n)$, $k = 1, 2, 3, \ldots, n$; $n = 1, 2, \ldots$, to satisfy

$$\sum_{k=1}^{n} a_k^{(n)}(x^k - c_k) - 1 = A(n)\, T_n(x).$$

For this we require that $a_k^{(n)} = A(n)\, b_k^{(n)}$ and that

$$\sum_{k=1}^{n} a_k^{(n)} c_k + 1 = -A(n)\, b_0^{(n)} = -A(n).$$

Thus

$$A(n) \left( \sum_{k=1}^{n} b_k^{(n)} c_k \right) + 1 = -A(n)$$

and

$$A(n) = -\left( \sum_{k=1}^{n} b_k^{(n)} c_k + 1 \right)^{-1}.$$

(See below where we show that $A(n)$ is well defined for all sufficiently large $n$.) We may assume that (4) holds with $\varepsilon = 1$ and $c = 0$. Since, for every $n$,

$$\left\| \sum_{k=1}^{n} a_k^{(n)}(x^k - c_k) - 1 \right\|_\infty = |A(n)| \, \|T_n\|_\infty = |A(n)|,$$

it suffices to show that $A(n) \to 0$ as $n \to \infty$ or that

$$\left| \sum_{k=1}^{n} b_k^{(n)} c_k \right| \to \infty \qquad \text{as} \quad n \to \infty. \tag{6}$$

It is known that

$$(-1)^k \, b_k^{(n)} = \frac{2^k}{k!} \frac{n^2(n^2 - 1^2) \cdots (n^2 - (k-1)^2)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)}.$$

Let $k_0$ be the smallest integer $k \geqslant M$ for which $c_k \neq 0$. Then for $n \geqslant M$ we have

$$\left| \sum_{k=1}^{n} b_k^{(n)} c_k \right| \geqslant \left| \sum_{k=M}^{n} b_k^{(n)} c_k \right| - \left| \sum_{k=1}^{M-1} b_k^{(n)} c_k \right| \geqslant |b_{k_0}^{(n)} c_{k_0}| - \left| \sum_{k=1}^{M-1} b_k^{(n)} c_k \right|.$$

Note that it is precisely here that we make use of the hypothesis that $(-1)^k c_k \geqslant 0$, $k \geqslant M$. It is readily shown that the first term on the right of the above is $\geqslant \delta n^{2M}$ for some suitable $\delta > 0$, while the second term is $O(n^{2M-2})$ as $n \uparrow \infty$. Thus (6) holds and the functions

$$\sum_{k=1}^{n} a_k^{(n)}(x^k - c_k)$$

converge uniformly to 1 on $[0, 1]$.

5. If $|c_{n_k} - c|^{1/n_k} \to 0$ as $k \to \infty$, where the subsequence $\{n_k\}_1^\infty$ satisfies the Müntz condition $\sum_{k=1}^{\infty} (n_k)^{-1} = \infty$ and $c_k \not\equiv c$, then $\{c_k\}_1^\infty$ is an approximating sequence. We assume that $c = 0$ and that $\{c_k\}_1^\infty$ is the Hausdorff moment sequence of a real function $\mu$ of bounded variation. For each $x \in [0, 1)$ assume, without loss of generality, that $\mu(x) = \mu(x + 0)$. Set

$$\rho = \inf\{s \colon \mu(x) = \mu(1), s \leqslant x \leqslant 1\}.$$

Since $c_{n_k} = O(d^{n_k})$ (as $k \to \infty$) for each $d > 0$, it follows from Theorem 2 of

[2] that $\rho = 0$. Thus $\mu$ is the Dirac measure at 0, which contradicts the fact that $c_k \neq 0$ for some $k \geqslant 1$.

In the case when $|c_k|^{1/k} \to 0$ as $k \to \infty$, we shall again explicitly construct a good approximation to $f(x) \equiv 1$. We are unable to do this in the general case where the subsequence $\{n_k\}$ is more lacunary. Since $|c_k|^{1/k} \to 0$, we can find a positive continuous increasing function $\phi(x)$ defined on $[0, \infty)$ such that

$$\phi(x) \to \infty \qquad \text{as} \quad x \to \infty,$$

$$\frac{\log |c_k|}{k} \leqslant -\phi(k), \qquad k = 1, 2,\ldots .$$

Set $\varepsilon(k) = \max\{(\log k)^{-1/2}, (\phi((\log k)^{1/2}))^{-1/2}\}$, $k = 2, 3,\ldots$, so that $\varepsilon(k)$ is positive and

$$\begin{aligned} \varepsilon(k) &\to 0 \qquad \text{as} \quad k \to \infty, \\ \varepsilon(k)\, \phi(\varepsilon(k) \log k) &\to \infty \qquad \text{as} \quad k \to \infty. \end{aligned} \tag{7}$$

We now construct a sequence of polynomials

$$p_n(x) = \sum_{k=1}^{s_n} \alpha_k^{(n)} x^k, \qquad n = 1, 2,\ldots,$$

with the property that $\|p_n\|_\infty \to 0$ as $n \to \infty$, and

$$\left| \sum_{k=1}^{s_n} \alpha_k^{(n)} c_k \right| \geqslant \delta > 0,$$

where $\delta$ is some suitable constant. Assuming that this may be done, we have

$$\left\| 1 + \left( \sum_{k=1}^{s_n} \alpha_k^{(n)} c_k \right)^{-1} \sum_{k=1}^{s_n} \alpha_k^{(n)} (x^k - c_k) \right\|_\infty \leqslant \delta^{-1} \|p_n\|_\infty \to 0$$

as $n \to \infty$, thus achieving the desired approximation to 1.

It remains to construct $p_n$. Assume for simplicity that $c_1 \neq 0$. Set

$$p_n(x) = x(1 - x^r)^n,$$

where $r = [\varepsilon(n) \log n]$. Thus, in the above notation, $s_n = n[\varepsilon(n) \log n] + 1$. It is an elementary exercise to verify that

$$\|p_n\|_\infty = \exp\{(-1 + o(1))(\varepsilon(n))^{-1}\}, \qquad n \to \infty,$$

so that $\|p_n\|_\infty \to 0$ as $n \to 0$. Now

$$\left|\sum_{k=1}^{s_n} \alpha_k^{(n)} c_k\right| \geqslant |c_1| - \binom{n}{1} |c_{r+1}| - \binom{n}{2} |c_{2r+1}| - \cdots.$$

Furthermore,

$$\frac{\log |c_{r+1}|}{r+1} \leqslant -\phi(r+1) \leqslant -\phi(\varepsilon(n)\log n)$$

and thus from (7),

$$\log |c_{r+1}| \leqslant -(\log n)[\varepsilon(n)\,\phi(\varepsilon(n)\log n)] \leqslant -2\log n$$

for all $n$ sufficiently large. Thus $|c_{r+1}| \leqslant n^{-2}$. Similarly it may be shown that $|c_{2r+1}| \leqslant (n^{-2})^2, \ldots, |c_{jr+1}| \leqslant (n^{-2})^j$, for all $r$ sufficiently large and $j = 1, 2, \ldots$. Hence

$$\left|\sum_{k=1}^{s_n} \alpha_k^{(n)} c_k\right| \geqslant |c_1| - \sum_{k=1}^{n} \binom{n}{k} n^{-2k} \geqslant |c_1| - \frac{1}{n} \sum_{k=1}^{\infty} \frac{1}{k!} \geqslant \delta > 0,$$

as required.

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## References

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