Functional Inequalities for Spectral Radii of Nonnegative Matrices

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ABSTRACT

Let P_n^+ denote the set of all square $n \times n$ nonnegative matrices. For $A_k = (a_{ij}^k)_{i,j=1}^n$, $k = 1,\dots,m$ (k not a power), we set

$$f(A_1,\dots,A_m) = \left(f\left(a_{ij}^1,\dots,a_{ij}^m\right)\right)_{i,j=1}^n.$$

For each $A \in P_n^+$, we let $\rho(A)$ denote its spectral radius. This paper is concerned with the characterization of those functions $f:\mathbb{R}_+^m \to \mathbb{R}_+$ satisfying either

$$\rho(f(A_1,\dots,A_m)) \leqslant f(\rho(A_1),\dots,\rho(A_m)) \tag{1}$$

or

$$f(\rho(A_1),\dots,\rho(A_m)) \leqslant \rho(f(A_1,\dots,A_m)) \tag{2}$$

for all $A_1,\dots,A_m \in P_n^+$ and every $n \in \mathbb{N}$. We totally characterize all functions satisfying (1). We delineate various classes of functions which satisfy (2). If $f(\mathbf{0}) = 0$, and f is bounded above in some neighborhood of any $\boldsymbol{\alpha} \in \operatorname{int} \mathbb{R}_+^m$, then we totally characterize all f satisfying (2).

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1. INTRODUCTION

For each $n \in \mathbb{N}$, let P_n^+ denote the set of $n \times n$ nonnegative matrices, and $P^+ = \bigcup_{n \geqslant 1} P_n^+$. Given $A_k \in P_n^+$, $A_k = (a_{ij}^k)_{i,j=1}^n$, $k = 1,\dots,m$ (k not a power), and $f: \mathbb{R}_+^m \to \mathbb{R}_+$, we let $f(A_1,\dots,A_m) = (f(a_{ij}^1,\dots,a_{ij}^m))_{i,j=1}^n$. In this paper we consider the set of all such functions satisfying

$$\rho(f(A_1,\dots,A_m)) \leqslant f(\rho(A_1),\dots,\rho(A_m)) \tag{1.1}$$

for $A_k \in P_n^+$, $k = 1,\dots,m$, and every $n \in \mathbb{N}$, as well as the converse inequality

$$f(\rho(A_1),\dots,\rho(A_m)) \leqslant \rho(f(A_1,\dots,A_m)), \tag{1.2}$$

where $\rho(\cdot)$ is the spectral radius.

In Section 2 we totally characterize all functions $f: \mathbb{R}_+^m \to \mathbb{R}_+$ satisfying (1.1) (Theorem 2.1). This class is exactly the set of functions satisfying the functional inequalities

(i) $$f(\mathbf{a}) + f(\mathbf{b}) \leqslant f(\mathbf{a}+\mathbf{b}),$$

(ii) $$\prod_{k=1}^{s} f^{1/s}(\mathbf{a}_k) \leqslant f\left((\mathbf{a}_1 \circ \cdots \circ \mathbf{a}_s)^{1/s}\right), \qquad s = 2,3,\dots,$$

for all vectors in $\mathbb{R}_+^m$; here and throughout this paper, if $\mathbf{a} = (a_1,\dots,a_m)$, $\mathbf{b} = (b_1,\dots,b_m)$, then we define

$$\mathbf{a} \circ \mathbf{b} = (a_1 b_1,\dots,a_m b_m)$$

to be the Hadamard product of $\mathbf{a}$ and $\mathbf{b}$, and

$$\mathbf{a}^t = \left(a_1^t,\dots,a_m^t\right).$$

We also present various properties and examples for this set of functions.

In Section 3 we study (1.2). We prove that $f:\mathbb{R}_+^m \to \mathbb{R}_+$ satisfies (1.2) if it is in any one of the following three classes:

(a) f is bounded above at some point $\alpha \in \operatorname{int} \mathbb{R}_+^m$, $f(x_1,\dots,x_m) = \max_{k=1,\dots,m} f_k(x_k)$, where $f_k:\mathbb{R}_+ \to \mathbb{R}_+$, and each f_k satisfies

(i) $$f_k(a+b) \leqslant f_k(a) + f_k(b),$$

(ii) $$f_k(\sqrt{ab}) \leqslant \sqrt{f_k(a) f_k(b)}$$

for all $a, b > 0$.

(b) f is componentwise decreasing.

(c) There exists a $c > 0$ such that $c \leqslant f(\mathrm{x}) \leqslant 2c$ for all $\mathrm{x} \in \mathbb{R}_+^m$.

Moreover, if $f(\mathbf{0}) = 0$ and f is bounded above at some point $\alpha \in \operatorname{int} \mathbb{R}_+^m$, then we prove that f satisfies (1.2) if and only if it is in (a). However, there do exist functions not in (a), (b), or (c) which satisfy (1.2). Thus this study is incomplete. Again we present various properties and examples.

This work was partially motivated by the following result (see Karlin and Ost [6] and Elsner, Johnson, and Dias da Silva [2]). Let $A_1,\dots,A_m \in P_n^+$ and $\alpha_k > 0$, $k = 1,\dots,m$, with $\sum_{k=1}^m \alpha_k \geqslant 1$. Set $C = (c_{ij})$, where

$$c_{ij} = \left(a_{ij}^1\right)^{\alpha_1} \cdots \left(a_{ij}^m\right)^{\alpha_m}, \qquad i, j = 1,\dots,n.$$

Then

$$\rho(C) \leqslant \rho(A_1)^{\alpha_1} \cdots \rho(A_m)^{\alpha_m}. \tag{1.3}$$

Setting $f(x_1,\dots,x_m) = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, we may rewrite this result as

$$\rho(f(A_1,\dots,A_m)) \leqslant f(\rho(A_1),\dots,\rho(A_m)).$$

If we let $x^0 = 1$ for *all* $x \geqslant 0$, then it is easily checked that (1.3) holds if $\alpha_k \geqslant 0$, $\sum_{k=1}^m \alpha_k \geqslant 1$. We will need this result in Section 2. Further motivation was provided by a talk given by the first author at the Fourth Haifa Matrix Conference (January 1988), in which the inequality (1.2) was proven for the specific case of $m = 1$ and $f(x) = x + \operatorname{sgn}(x)$. This result appears in [1], and is an immediate consequence of our results (Propositions 3.3 and 3.7).

We end this introduction by introducing an ancillary function on P^+ which we will need in the subsequent analysis.

For $A \in P_n^+$, set

$$\mu(A) = \max\left(\prod_{j=1}^{s} a_{i_j i_{j+1}}\right)^{1/s},$$

where $i_1, \ldots, i_s$ are distinct indices in $\{1, \ldots, n\}$ and $i_{s+1} = i_1$. This functional μ was introduced in Engel and Schneider [3], where it was proven that

$$\mu(A) = \inf\left\{\max_{i,j}(D^{-1}AD)_{ij} \colon D \text{ any positive diagonal matrix}\right\}.$$

Subsequently Friedland [4] proved that

$$\mu(A) = \lim_{r \to \infty} \rho(A^{[r]})^{1/r},$$

where $A^{[r]} = (a_{ij}^r)_{i,j=1}^n$ (r a power). While we will use neither of the above characterizations, we will use the inequality provided by this fourth characterization, to be found in Elsner, Johnson, and Dias da Silva [2]:

$$\mu(A) = \max\{\rho(A \circ B) \colon B \in P_n^+,\ \rho(B) \leqslant 1\}. \tag{1.4}$$

Paralleling the previous notation, $A \circ B$ is the Hadamard product of A and B, i.e.,

$$A \circ B = (a_{ij}b_{ij})_{i,j=1}^n.$$

Note that it easily follows from the above that $0 \leqslant \mu(A) \leqslant \rho(A) \leqslant n\mu(A)$ for all $A \in P_n^+$. Thus in particular, $\mu(A) = 0$ if and only if $\rho(A) = 0$, for any $A \in P^+$.

2. THE INEQUALITY $\rho(f(A_1, \ldots, A_m)) \leqslant f(\rho(A_1), \ldots, \rho(A_m))$

The main result of this section is the following:

THEOREM 2.1. *Let $f \colon \mathbb{R}_+^m \to \mathbb{R}_+$. Then f satisfies*

$$\rho(f(A_1, \ldots, A_m)) \leqslant f(\rho(A_1), \ldots, \rho(A_m)) \tag{2.1}$$

for all $A_1,\dots, A_m \in P_n^+$ *and all* $n \in \mathbb{N}$ *if and only if*

$$f(\mathbf{a})+f(\mathbf{b}) \leqslant f(\mathbf{a}+\mathbf{b}), \qquad \textit{all} \quad \mathbf{a},\mathbf{b} \in \mathbb{R}_+^m \tag{2.2a}$$

$$\prod_{k=1}^{s} f^{1/s}(\mathbf{a}_k) \leqslant f\big((\mathbf{a}_1 \circ \cdots \circ \mathbf{a}_s)^{1/s}\big), \qquad \textit{all} \quad \mathbf{a}_1,\dots,\mathbf{a}_s \in \mathbb{R}_+^m, \quad s=2,3,\dots . \tag{2.2b}$$

Furthermore if f is continuous, or $m=1$, *then it suffices to take only* $s=2$ *in* (2.2b).

The proof of the "if" part of this theorem is long, detailed, and arduous. Therefore we first separate out a series of results needed in its proof.

LEMMA 2.2. *Assume* $f:\mathbb{R}_+^m \to \mathbb{R}_+$ *satisfies* (2.2). *Then*:

(i) $f(\mathbf{0})=0$.
(ii) f *is componentwise increasing.*
(iii) *If* $f(a_1,\dots,a_m)>0$ *and* $a_r=0$, *then*

$$f(a_1,\dots,a_{r-1},0,a_{r+1},\dots,a_m)=f(a_1,\dots,a_{r-1},x,a_{r+1},\dots,a_m)$$

for all $x>0$.
(iv) *For all* $\lambda_j>0$, $\lambda_j \in \mathbb{Q}$, $\sum_{j=1}^r \lambda_j=1$,

$$\prod_{j=1}^{r} f^{\lambda_j}(\mathbf{a}_j) \leqslant f\big(\mathbf{a}_1^{\lambda_1} \circ \cdots \circ \mathbf{a}_r^{\lambda_r}\big).$$

Proof. Both (i) and (ii) are immediate consequences of (2.2a). To prove (iii), assume that $f(a_1,\dots,a_m)>0$ with $a_r=0$. From (2.2b) with $s=2$,

$$f^{1/2}(a_1,\dots,a_{r-1},0,a_{r+1},\dots,a_m)f^{1/2}(a_1,\dots,a_{r-1},x,a_{r+1},\dots,a_m)$$
$$\leqslant f(a_1,\dots,a_{r-1},0,a_{r+1},\dots,a_m).$$

Thus

$$f(a_1,\dots,a_{r-1},x,a_{r+1},\dots,a_m) \leqslant f(a_1,\dots,a_{r-1},0,a_{r+1},\dots,a_m)$$

for every $x \geqslant 0$. Since f is componentwise increasing, equality holds.

To prove (iv), let $\lambda_j = p_j/q_j$, where $p_j, q_j \in \mathbb{N}$. Apply (2.2b) with $s = \prod_{j=1}^{r} q_j$, and repeat $\mathbf{a}_j$ therein $\lambda_j s$ times. ■

LEMMA 2.3. *Assume $f: \mathbb{R}_+^m \to \mathbb{R}_+$ satisfies* (2.2). *Set*

$$D = \{\mathbf{a} : f(\mathbf{a}) > 0\}.$$

(i) *If $\mathbf{a}_1, \dots, \mathbf{a}_s \in D$, then $(\mathbf{a}_1 \circ \cdots \circ \mathbf{a}_s)^{1/s} \in D$.*
(ii) *If $\mathbf{a} \in \operatorname{int} D$, $\mathbf{b} \in \mathbb{R}_+^m$, then $\mathbf{a} + \mathbf{b} \in \operatorname{int} D$.*
(iii) *If $\mathbf{a} \in D$, $\mathbf{b} > \mathbf{0}$ (i.e., $b_i > 0$, $i = 1, \dots, m$), then $\mathbf{a} + \mathbf{b} \in \operatorname{int} D$.*
(iv) *If $\mathbf{a} \in D$, $\lambda > 1$, then $\lambda \mathbf{a} \in \operatorname{int} D$.*

The proof of this lemma is an immediate and simple consequence of (2.2). We omit it. It is to be understood that by $\operatorname{int} D$ we mean the interior of D *relative to* $\mathbb{R}_+^m$. Thus $\operatorname{int} D$ may include boundary points of $\mathbb{R}_+^m$. We always assume that $f \not\equiv 0$ and thus $\operatorname{int} D \neq \varnothing$. In the next series of results we use the following notation. If $\mathbf{b}, \mathbf{c} \in \mathbb{R}_+^m$, we let $\mathbf{b}^{\mathbf{c}} = b_1^{c_1} \cdots b_m^{c_m}$.

PROPOSITION 2.4. *Assume $f: \mathbb{R}_+^m \to \mathbb{R}_+$ satisfies* (2.2). *Let $\boldsymbol{\rho} = (\rho_1, \dots, \rho_m) \in \operatorname{int} D$. Then there exists an $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$, $\alpha_i \geqslant 0$, $\sum_{i=1}^m \alpha_i \geqslant 1$, such that*

$$f(\mathbf{a}) \leqslant f(\boldsymbol{\rho}) \boldsymbol{\rho}^{-\boldsymbol{\alpha}} \mathbf{a}^{\boldsymbol{\alpha}} \tag{2.3}$$

for all $\mathbf{a} \in \mathbb{R}_+^m$, where it is understood that $x^0 \equiv 1$ for all $x \geqslant 0$. Furthermore $\alpha_r = 0$ if $\rho_r = 0$.

Proof. Let

$$F = \{\mathbf{a} : \mathbf{a} \in \operatorname{int} D,\ a_i > 0,\ i = 1, \dots, m\}.$$

We first prove (2.3) under the assumption that $\boldsymbol{\rho} \in F$. To this end, set

$$E = \{\mathbf{b} : e^{\mathbf{b}} \in F\}.$$

(Observe that for each $\mathbf{a} \in F$ there exists a $\mathbf{b} \in E$ such that $\mathbf{a} = e^{\mathbf{b}}$.) Note that E is open. For each $\mathbf{b} \in E$, set

$$g(\mathbf{b}) = \ln f(e^{\mathbf{b}}),$$

i.e., $g(\mathbf{b}) = \ln f(e^{b_1},\dots,e^{b_m})$. From (2.2b) with $s=2$, we obtain

$$\frac{g(\mathbf{b})+g(\mathbf{c})}{2} \leqslant g\left(\frac{\mathbf{b}+\mathbf{c}}{2}\right)$$

for all $\mathbf{b},\mathbf{c} \in E$. Thus g is *midconcave* on E. From Lemma 2.2(ii) we also have that g is componentwise increasing. Thus g is concave and continuous on E; see e.g. Jensen [5] or Roberts and Varberg [7]. In particular, at each $\mathbf{b} \in E$ a *subgradient* to g exists. That is, corresponding to $\mathbf{b} \in E$ there exists an $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)$ such that

$$g(\mathbf{c}) - g(\mathbf{b}) \leqslant (\boldsymbol{\alpha}, \mathbf{c}-\mathbf{b}) \tag{2.4}$$

for all $\mathbf{c} \in E$ [where $(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{i=1}^m \alpha_i \beta_i$]. Let $e^{\mathbf{b}} = \boldsymbol{\rho}$. By assumption $\mathbf{b} \in E$. Translating (2.4) back to f, we obtain (2.3) for all $\mathbf{a} \in F$. To extend (2.3) to all $\mathbf{a} \in \mathbb{R}_+^m$ we first need some additional facts.

Let $\mathbf{a} \in F$. Then $\mathbf{a}_x = (a_1,\dots,a_{r-1},x,a_{r+1},\dots,a_m) \in \operatorname{int} D$ for all $x \geqslant a_r$, by Lemma 2.3(ii). Since f is componentwise increasing, $f(\mathbf{a}) \leqslant f(\mathbf{a}_x)$ for all $x \geqslant a_r$. Therefore

$$0 < f(\mathbf{a}) \leqslant f(\mathbf{a}_x) \leqslant f(\boldsymbol{\rho})\boldsymbol{\rho}^{-\boldsymbol{\alpha}}\mathbf{a}_x^{\mathbf{a}}$$

for all $x \geqslant a_r$. If $\alpha_r < 0$, then letting $x \uparrow \infty$, a contradiction ensues. Thus $\alpha_r \geqslant 0$ for each $r \in \{1,\dots,m\}$.

From (2.2a), we have $2f(\boldsymbol{\rho}) \leqslant f(2\boldsymbol{\rho})$. Since $2\boldsymbol{\rho} \in F$ [Lemma 2.3(iv)], we have from (2.3)

$$2f(\boldsymbol{\rho}) \leqslant f(2\boldsymbol{\rho}) \leqslant f(\boldsymbol{\rho})\boldsymbol{\rho}^{-\boldsymbol{\alpha}}(2\boldsymbol{\rho})^{\boldsymbol{\alpha}} = f(\boldsymbol{\rho})2^{\sum_{i=1}^m \alpha_i}.$$

Thus $\sum_{i=1}^m \alpha_i \geqslant 1$.

Assume, without loss of generality, that $\mathbf{a} \in \operatorname{int} D$, $a_1 = \cdots = a_r = 0$, and $a_{r+1},\dots,a_m > 0$. Then $(x_1,\dots,x_r,a_{r+1},\dots,a_m) \in \operatorname{int} D$ for all $x_i \geqslant 0$, $i = 1,\dots,r$, and from Lemma 2.2(iii),

$$f(x_1,\dots,x_r,a_{r+1},\dots,a_m) = f(0,\dots,0,a_{r+1},\dots,a_m).$$

Now, for $x_i > 0$, $i = 1,\dots,r$,

$$\begin{aligned} 0 < f(0,\dots,0,a_{r+1},\dots,a_m) &= f(x_1,\dots,x_r,a_{r+1},\dots,a_m) \\ &\leqslant f(\boldsymbol{\rho})\boldsymbol{\rho}^{-\boldsymbol{\alpha}} x_1^{\alpha_1}\cdots x_r^{\alpha_r} a_{r+1}^{\alpha_{r+1}} \cdots a_m^{\alpha_m}. \end{aligned}$$

Thus $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ is uniformly bounded below away from zero for all $x_i > 0$, $i = 1, \ldots, r$. Since $\alpha_i \geqslant 0$ for all i, this implies that $\alpha_i = 0$, $i = 1, \ldots, r$.

We now prove that (2.3) holds for all $\mathbf{a} \in \mathbb{R}_+^m$. Assume $\mathbf{a} \in \operatorname{int} D$ and $a_r = 0$ for some $r \in \{1, \ldots, m\}$. Since $\alpha_r = 0$ if $a_r = 0$, and $f(\tilde{\mathbf{a}}) = f(\mathbf{a})$, where $\tilde{a}_i = a_i$ if $a_i > 0$, $\tilde{a}_i \geqslant 0$ if $a_i = 0$, it follows that

$$f(\mathbf{a}) = f(\tilde{\mathbf{a}}) \leqslant f(\boldsymbol{\rho})\boldsymbol{\rho}^{-\boldsymbol{\alpha}}\tilde{\mathbf{a}}^{\boldsymbol{\alpha}} = f(\boldsymbol{\rho})\boldsymbol{\rho}^{-\boldsymbol{\alpha}}\mathbf{a}^{\boldsymbol{\alpha}}$$

for all $\mathbf{a} \in \operatorname{int} D$, where we set $x^0 \equiv 1$ for all $x \geqslant 0$.

Assume $\mathbf{a} \in D \backslash \operatorname{int} D$. From Lemma 2.3(iv) $\lambda \mathbf{a} \in \operatorname{int} D$ for all $\lambda > 1$, and from Lemma 2.2(ii) $f(\mathbf{a}) \leqslant f(\lambda \mathbf{a})$. Thus

$$f(\mathbf{a}) \leqslant f(\lambda \mathbf{a}) \leqslant f(\boldsymbol{\rho})\boldsymbol{\rho}^{-\boldsymbol{\alpha}}(\lambda \mathbf{a})^{\boldsymbol{\alpha}}$$

for all $\lambda > 1$. By continuity we obtain

$$f(\mathbf{a}) \leqslant f(\boldsymbol{\rho})\boldsymbol{\rho}^{-\boldsymbol{\alpha}}\mathbf{a}^{\boldsymbol{\alpha}}.$$

Since $f(\mathbf{a}) = 0$ if $\mathbf{a} \notin D$, we have proven the validity of (2.3) for all $\mathbf{a} \in \mathbb{R}_+^m$ and $\boldsymbol{\rho} \in F$.

Finally, if $\boldsymbol{\rho} \in \operatorname{int} D$ but $\rho_r = 0$ for some $r \in \{1, \ldots, m\}$, let $\tilde{\boldsymbol{\rho}} = (\tilde{\rho}_1, \ldots, \tilde{\rho}_m)$, where $\tilde{\rho}_i = \rho_i$ if $\rho_i > 0$, $\tilde{\rho}_i > 0$ if $\rho_i = 0$. Thus $\tilde{\boldsymbol{\rho}} \in F$. There then exists an $\boldsymbol{\alpha}$, as desired, such that

$$f(\mathbf{a}) \leqslant f(\tilde{\boldsymbol{\rho}})\tilde{\boldsymbol{\rho}}^{-\boldsymbol{\alpha}}\mathbf{a}^{\boldsymbol{\alpha}}$$

for all $\mathbf{a} \in \mathbb{R}_+^m$. Since $\boldsymbol{\rho} \in \operatorname{int} D$, it follows from the previous argument that $\alpha_r = 0$ if $\rho_r = 0$. Thus $\tilde{\boldsymbol{\rho}}^{-\boldsymbol{\alpha}} = \boldsymbol{\rho}^{-\boldsymbol{\alpha}}$. Furthermore $f(\tilde{\boldsymbol{\rho}}) = f(\boldsymbol{\rho})$. Substituting $\boldsymbol{\rho}$ for $\tilde{\boldsymbol{\rho}}$ in the above gives us the full result. ∎

PROPOSITION 2.5. *Assume* $f: \mathbb{R}_+^m \to \mathbb{R}_+$ *satisfies* (2.2). *Let* $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m) \in D \backslash \operatorname{int} D$. *There then exists an* $r \in \{1, \ldots, m\}$ *with* $\rho_r > 0$ *such that*

$$f(x_1, \ldots, x_m) = 0 \qquad \text{if} \quad x_r = 0, \tag{2.5}$$

$$f(\tilde{\rho}_1, \ldots, \tilde{\rho}_m) = 0 \qquad \text{if} \quad \tilde{\rho}_i \leqslant \rho_i, \ i = 1, \ldots, m, \ \text{and} \ \tilde{\rho}_r < \rho_r. \tag{2.6}$$

Proof. For ease of notation we assume that $\rho_1,\dots,\rho_j > 0$ and $\rho_{j+1} = \cdots = \rho_m = 0$ ($j \geqslant 1$). Set

$$S = \left\{ k : 1 \leqslant k \leqslant j,\ f(x_{1k},\dots,x_{k-1,k},0,x_{k+1,k},\dots,x_{mk}) > 0 \text{ for some } \{x_{ik}\}_{\substack{i=1\\ i\neq k}}^{m} \right\}.$$

$$T = \left\{ k : 1 \leqslant k \leqslant j,\ f\big(\rho_1,\dots,\rho_{k-1},\mu_k,\rho_{k+1},\dots,\rho_j,0,\dots,0\big) > 0 \text{ for some } \mu_k < \rho_k \right\}.$$

If $S \cup T \neq \{1,\dots,j\}$, then the proposition holds (since f is componentwise increasing).

The proof is by negation. Assume $S \cup T = \{1,\dots,j\}$. Now $S \neq \{1,\dots,j\}$. For if $S = \{1,\dots,j\}$ then from (2.2b),

$$0 < \left[\prod_{k=1}^{j} f(x_{1k},\dots,x_{k-1,k},0,x_{k+1,k},\dots,x_{mk})\right] f\big(\rho_1,\dots,\rho_j,0,\dots,0\big)$$
$$\leqslant f^{j+1}(0,\dots,0) = 0,$$

a contradiction. Let $\tilde{S} = \{1,\dots,j\}\backslash T$. Then $\tilde{S} \cup T = \{1,\dots,j\}$, $\tilde{S} \cap T = \varnothing$. For each $\lambda_k > 0$, $\lambda_k \in \mathbb{Q}$, $k = 1,\dots,j$, $\sum_{k=1}^{j}\lambda_k = 1$, we have from (2.2b)

$$0 < \left[\prod_{k\in\tilde{S}} f^{\lambda_k}(x_{1k},\dots,x_{k-1,k},0,x_{k+1,k},\dots,x_{mk})\right]$$
$$\times\left[\prod_{k\in T} f^{\lambda_k}\big(\rho_1,\dots,\rho_{k-1},\mu_k,\rho_{k+1},\dots,\rho_j,0,\dots,0\big)\right]$$
$$\leqslant f(\sigma_1,\dots,\sigma_m),$$

where $\sigma_i = 0$ for $i = j+1,\dots,m$ (since $T \neq \varnothing$) and $i \in \tilde{S}$. For $i \in T$,

$$\sigma_i = \left[\prod_{k\in\tilde{S}} x_{ik}^{\lambda_k}\right]\left[\prod_{\substack{k\in T\\ k\neq i}} \rho_i^{\lambda_k}\right]\mu_i^{\lambda_i}.$$

Since $\mu_i < \rho_i$, it is possible to choose $\lambda_k > 0$, $\lambda_k \in \mathbb{Q}$, $k = 1,\dots,j$, $\sum_{k=1}^{j}\lambda_k = 1$, so that $\sigma_i < \rho_i$, $i \in T$. Thus

$$f\big(\sigma_1,\dots,\sigma_j,0,\dots,0\big) > 0,$$

where $\sigma_i < \rho_i$, $i = 1,\dots,j$. This contradicts the fact that $\rho \notin \operatorname{int} D$ [see Lemma 2.3(iv)] and proves the proposition. ∎

With these preliminaries we can now proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. $\Rightarrow$: Assume f satisfies (2.1). Let $\mathbf{a},\mathbf{b} \in \mathbb{R}^m_+$, $\mathbf{a} = (a_1,\dots,a_m)$, $\mathbf{b} = (b_1,\dots,b_m)$, and set

$$A_k = \begin{pmatrix} a_k & b_k \\ a_k & b_k \end{pmatrix}, \qquad k = 1,\dots,m.$$

Then $\rho(A_k) = a_k + b_k$, and $f(\rho(A_1),\dots,\rho(A_m)) = f(\mathbf{a}+\mathbf{b})$. Furthermore

$$\rho(f(A_1,\dots,A_m)) = \rho\begin{pmatrix} f(a_1,\dots,a_m) & f(b_1,\dots,b_m) \\ f(a_1,\dots,a_m) & f(b_1,\dots,b_m) \end{pmatrix}$$
$$= f(\mathbf{a}) + f(\mathbf{b}).$$

This proves (2.2a). Note that this also implies that $f(\mathbf{0}) = 0$.

Now set

$$A_k = \begin{pmatrix} 0 & a_{k1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{k,s-1} \\ a_{ks} & 0 & \cdots & 0 \end{pmatrix}, \qquad k = 1,\dots,m,$$

where each A_k is an $s \times s$ matrix. Then $\rho(A_k) = [\prod_{j=1}^s a_{kj}]^{1/s}$. Set

$$\mathbf{a}_j = (a_{1j},\dots,a_{mj}), \qquad j = 1,\dots,s.$$

Then $f(\rho(A_1),\dots,\rho(A_m)) = f((\mathbf{a}_1 \circ \cdots \circ \mathbf{a}_s)^{1/s})$. Since $f(\mathbf{0}) = 0$,

$$\rho(f(A_1,\dots,A_m)) = \rho\begin{pmatrix} 0 & f(\mathbf{a}_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\mathbf{a}_{s-1}) \\ f(\mathbf{a}_s) & 0 & \cdots & 0 \end{pmatrix}$$
$$= \prod_{k=1}^{s} f^{1/s}(\mathbf{a}_k).$$

This proves (2.2b).

$\Leftarrow$: Assume $f:\mathbb{R}_+^m \to \mathbb{R}_+$ satisfies (2.2). For given $A_1,\dots,A_m \in P_n^+$, let $\rho_k = \rho(A_k)$, $k=1,\dots,m$. We divide the proof of this part of the theorem into three cases.

Case 1. $\boldsymbol{\rho} = (\rho_1,\dots,\rho_m) \notin D$, *i.e.*, $f(\rho_1,\dots,\rho_m) = 0$. We must prove that $\rho(f(A_1,\dots,A_m)) = 0$. As noted in Section 1, this is equivalent to proving that $\mu(f(A_1,\dots,A_m)) = 0$.

Assume $A_k = (a_{ij}^k)_{i,j=1}^n$, $k=1,\dots,m$ (the k is not a power). By definition, there exist $\{i_1,\dots,i_s\} \subseteq \{1,\dots,n\}$, $1 \leqslant s \leqslant n$ $(i_{s+1} = i_1)$, such that

$$\mu(f(A_1,\dots,A_m)) = \prod_{j=1}^{s} f^{1/s}\left(a_{i_j i_{j+1}}^1,\dots,a_{i_j i_{j+1}}^m\right).$$

Applying (2.2b) we obtain

$$\mu(f(A_1,\dots,A_m)) \leqslant f\left(\left(\prod_{j=1}^{s} a_{i_j i_{j+1}}^1\right)^{1/s},\dots,\left(\prod_{j=1}^{s} a_{i_j i_{j+1}}^m\right)^{1/s}\right).$$

By definition,

$$\left(\prod_{j=1}^{s} a_{i_j i_{j+1}}^k\right)^{1/s} \leqslant \mu(A_k) \leqslant \rho(A_k) = \rho_k, \qquad k=1,\dots,m.$$

Since f is componentwise increasing [Lemma 2.2(ii)], we finally obtain

$$\mu(f(A_1,\dots,A_m)) \leqslant f(\rho_1,\dots,\rho_m) = 0.$$

Case 2. $\boldsymbol{\rho} = (\rho_1,\dots,\rho_m) \in \operatorname{int} D$. From Proposition 2.4 there exists an $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)$, $\alpha_i \geqslant 0$, $\sum_{i=1}^m \alpha_i \geqslant 1$, such that

$$f(\mathbf{a}) \leqslant f(\boldsymbol{\rho})\boldsymbol{\rho}^{-\boldsymbol{\alpha}}\mathbf{a}^{\boldsymbol{\alpha}}$$

for all $\mathbf{a} \in \mathbb{R}_+^m$, where by definition $x^0 = 1$ for all $x \geqslant 0$. Thus

$$f(A_1,\dots,A_m) \leqslant f(\boldsymbol{\rho})\boldsymbol{\rho}^{-\boldsymbol{\alpha}} A_1^{\alpha_1} \circ \cdots \circ A_m^{\alpha_m},$$

where the inequality is elementwise. Since $A \leqslant B$ implies $\rho(A) \leqslant \rho(B)$, it follows that

$$\rho(f(A_1,\dots,A_m)) \leqslant f(\boldsymbol{\rho})\boldsymbol{\rho}^{-\boldsymbol{\alpha}}\rho(A_1^{\alpha_1} \circ \cdots \circ A_m^{\alpha_m}).$$

Applying (1.3), we obtain

$$\rho(f(A_1,\dots,A_m)) \leqslant f(\rho)\rho^{-\alpha}\rho_1^{\alpha_1}\cdots\rho_m^{\alpha_m} = f(\rho).$$

Case 3. $\rho = (\rho_1,\dots,\rho_m) \in D\backslash \operatorname{int} D$. Let $r \in \{1,\dots,m\}$ be such that $\rho_r > 0$ and (2.5), (2.6) of Proposition 2.5 hold. Set

$$h(x_1,\dots,x_m) = \begin{cases} \dfrac{f(x_1,\dots,x_m)}{x_r}, & x_r > 0, \\ 0, & x_r = 0. \end{cases}$$

From (2.5)

$$h(x_1,\dots,x_m)x_r = f(x_1,\dots,x_m)$$

for all $\mathbf{x} \in \mathbb{R}_+^m$. Furthermore, using (2.2b) we obtain

$$\prod_{k=1}^{s} h^{1/s}(\mathbf{a}_k) \leqslant h\big((\mathbf{a}_1 \circ \cdots \circ \mathbf{a}_s)^{1/s}\big) \tag{2.7}$$

for all $\mathbf{a}_1,\dots,\mathbf{a}_s \in \mathbb{R}_+^m$. Thus from (1.4)

$$\rho(f(A_1,\dots,A_m)) = \rho(h(A_1,\dots,A_m)\circ A_r) \leqslant \mu(h(A_1,\dots,A_m))\rho_r.$$

By definition there exist $\{i_1,\dots,i_s\} \subseteq \{1,\dots,n\}$, $1 \leqslant s \leqslant n$ $(i_{s+1} = i_1)$, such that

$$\mu(h(A_1,\dots,A_m)) = \prod_{j=1}^{s} h^{1/s}\big(a^1_{i_j i_{j+1}},\dots,a^m_{i_j i_{j+1}}\big).$$

From (2.7),

$$\rho(f(A_1,\dots,A_m)) \leqslant h\left(\left(\prod_{j=1}^{s} a^1_{i_j i_{j+1}}\right)^{1/s},\dots,\left(\prod_{j=1}^{s} a^m_{i_j i_{j+1}}\right)^{1/s}\right)\rho_r.$$

By definition,

$$\left(\prod_{j=1}^{s} a^k_{i_j i_{j+1}}\right)^{1/s} \leqslant \mu(A_k) \leqslant \rho_k, \qquad k = 1,\dots,m.$$

If $(\prod_{j=1}^{s} a^{r}_{i_j i_{j+1}})^{1/s} < \rho_r$, then from (2.6),

$$h\left(\left(\prod_{j=1}^{s} a^{1}_{i_j i_{j+1}}\right)^{1/s},\dots,\left(\prod_{j=1}^{s} a^{m}_{i_j i_{j+1}}\right)^{1/s}\right)=0$$

and therefore

$$\rho(f(A_1,\dots,A_m)) = 0 < f(\rho).$$

If $(\prod_{j=1}^{s} a^{r}_{i_j i_{j+1}})^{1/s} = \rho_r$, then

$$\begin{aligned}\rho(f(A_1,\dots,A_m)) &\leqslant h\left(\left(\prod_{j=1}^{s} a^{1}_{i_j i_{j+1}}\right)^{1/s},\dots,\left(\prod_{j=1}^{s} a^{m}_{i_j i_{j+1}}\right)^{1/s}\right)\rho_r\\ &= f\left(\left(\prod_{j=1}^{s} a^{1}_{i_j i_{j+1}}\right)^{1/s},\dots,\left(\prod_{j=1}^{s} a^{m}_{i_j i_{j+1}}\right)^{1/s}\right)\\ &\leqslant f(\rho_1,\dots,\rho_m),\end{aligned}$$

since f is componentwise increasing.

This completes the proof of the fact that (2.2) implies (2.1).

It remains to prove that it suffices, if f is continuous or $m=1$, to only take $s=2$ in (2.2b). We claim that in both these cases (2.2a) and (2.2b) with $s=2$ implies

$$f^{\lambda}(\mathbf{a})f^{1-\lambda}(\mathbf{b}) \leqslant f(\mathbf{a}^{\lambda}\mathbf{b}^{1-\lambda}) \qquad (2.8)$$

for all $\lambda \in (0,1)$ and $\mathbf{a},\mathbf{b} \in \mathbb{R}^{m}_{+}$, whence we obtain (2.2b) for all s.

In the proof of Proposition 2.4 we obtained, using only (2.2a) and (2.2b) with $s=2$, that

$$g(\mathbf{c}) = \ln f(e^{\mathbf{c}})$$

is concave and continuous on E. Translating back the inequality

$$\lambda g(\mathbf{c}) + (1-\lambda)g(\mathbf{d}) \leqslant g(\lambda c + (1-\lambda)\mathbf{d})$$

for all $\mathbf{c},\mathbf{d} \in E$ and $\lambda \in (0,1)$, we obtain (2.8) for $\mathbf{a},\mathbf{b} \in \operatorname{int} D$ with $a_i, b_i > 0$,

$i = 1,\dots,m$. If $a \notin D$ or $b \notin D$, then (2.8) trivially holds. The remaining cases are dealt with as follows.

If f is continuous then D is open, i.e., $D \backslash \operatorname{int} D = \varnothing$. Thus in this case it remains to prove (2.8) only for $\mathbf{a}, \mathbf{b} \in D$, where $\prod_{i=1}^m a_i = 0$ and/or $\prod_{i=1}^m b_i = 0$. Let $\tilde{\mathbf{a}}$ satisfy $\tilde{a}_i = a_i$ if $a_i > 0$, $\tilde{a}_i > 0$ if $a_i = 0$, and let $\tilde{\mathbf{b}}$ be similarly defined. Then $f(\mathbf{a}) = f(\tilde{\mathbf{a}})$ and $f(\mathbf{b}) = f(\tilde{\mathbf{b}})$ from Lemma 2.2(iii). Applying (2.8), we obtain

$$f^{\lambda}(\mathbf{a}) f^{1-\lambda}(\mathbf{b}) = f^{\lambda}(\tilde{\mathbf{a}}) f^{1-\lambda}(\tilde{\mathbf{b}}) \leqslant f(\tilde{\mathbf{a}}^{\lambda} \tilde{\mathbf{b}}^{1-\lambda})$$

for any $\lambda \in (0,1)$. Since f is continuous, we can let $\tilde{\mathbf{a}} \to \mathbf{a}$ and $\tilde{\mathbf{b}} \to \mathbf{b}$ to obtain (2.8).

Let $m = 1$. If $D \backslash \operatorname{int} D = \varnothing$, there is nothing to prove. We therefore assume that $D \backslash \operatorname{int} D = \{c\}$. The only case left to prove in (2.8) is

$$f^{\lambda}(a) f^{1-\lambda}(c) \leqslant f(a^{\lambda} c^{1-\lambda})$$

for $\lambda \in (0,1)$ and all $a > c$. From the proof of Proposition 2.4 we have that f is continuous on (c, ∞). Since f is componentwise increasing, we also have

$$f(c) \leqslant \lim_{\varepsilon \to 0^+} f(c + \varepsilon).$$

Substituting $c + \varepsilon$ for c in the remaining case of (2.8), letting $\varepsilon \downarrow 0$, and using the above two facts, we obtain the desired conclusion. ∎

In the proof of the equivalence of (2.1) and (2.2) we use (2.2b) for *all* s. If f is continuous or $m = 1$, it is only necessary that (2.2b) hold for $s = 2$. In general we do need (2.2b) to hold for all s. We illustrate this fact with this next example.

Example. Define $f: \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$f(x,y) = \begin{cases} xy, & xy > 1, \\ 1, & xy = 1, \quad x = e^{k/2^m}, \quad k, m \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases}$$

We claim that f satisfies (2.2a) and (2.2b) with $s = 2$, but does not satisfy (2.1).

We first consider (2.2a). We must prove

$$f(a_1, a_2) + f(b_1, b_2) \leqslant f(a_1 + b_1, a_2 + b_2) \tag{2.9}$$

for all $a_1, a_2, b_1, b_2 \geqslant 0$. Since f is componentwise increasing, (2.9) holds if $f(a_1, a_2) = 0$ or $f(b_1, b_2) = 0$. Assume neither is zero. Then $f(a_1, a_2) = a_1 a_2 \geqslant 1$, $f(b_1, b_2) = b_1 b_2 \geqslant 1$. Now $(a_1 + b_1)(a_2 + b_2) \geqslant 4$. Thus

$$f(a_1, a_2) + f(b_1, b_2) = a_1 a_2 + b_1 b_2 \leqslant (a_1 + b_1)(a_2 + b_2) = f(a_1 + b_1, a_2 + b_2).$$

For (2.2b) to hold with $s = 2$ we must verify

$$f^{1/2}(a_1, a_2) f^{1/2}(b_1, b_2) \leqslant f\big((a_1 b_1)^{1/2}, (a_2 b_2)^{1/2}\big). \tag{2.10}$$

If the left hand side of (2.10) is zero, there is nothing to prove. We therefore assume that $f^{1/2}(a_1, a_2) = (a_1 a_2)^{1/2} \geqslant 1$ and $f^{1/2}(b_1, b_2) = (b_1 b_2)^{1/2} \geqslant 1$. If $(a_1 a_2)^{1/2}(b_1 b_2)^{1/2} > 1$, then equality holds in (2.10). If $(a_1 a_2)^{1/2}(b_1 b_2)^{1/2} = 1$, then $a_1 a_2 = b_1 b_2 = 1$, and by definition

$$a_1 = \exp\left(\frac{k_1}{2^{m_1}}\right), \qquad b_1 = \exp\left(\frac{k_2}{2^{m_2}}\right).$$

Thus

$$(a_1 b_1)^{1/2} = \exp\left(\frac{k_1 2^{m_2} + k_2 2^{m_1}}{2^{m_1 + m_2 + 1}}\right)$$

and $f((a_1 b_1)^{1/2}, (a_2 b_2)^{1/2}) = 1$. Thus (2.10) holds.

To see that (2.1) does not hold, set

$$A = \begin{pmatrix} 0 & e & 0 \\ 0 & 0 & e \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & e^{-1} & 0 \\ 0 & 0 & e^{-1} \\ 1 & 0 & 0 \end{pmatrix}.$$

Then

$$f(A, B) = \begin{pmatrix} 0 & f(e, e^{-1}) & 0 \\ 0 & 0 & f(e, e^{-1}) \\ f(1,1) & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

since $f(0,0) = 0$. Now $\rho(f(A, B)) = 1$, $\rho(A) = e^{2/3}$, and $\rho(B) = e^{-2/3}$. But $f(\rho(A), \rho(B)) = f(e^{2/3}, e^{-2/3}) = 0$. Thus (2.1) does not hold.

We now record some properties of the set of functions satisfying (2.1).

PROPOSITION 2.5. *Let* $f, g: \mathbb{R}_+^m \to \mathbb{R}_+$, $h: \mathbb{R}_+ \to \mathbb{R}_+$, *and* $h_1, \dots, h_m: \mathbb{R}_+^p \to \mathbb{R}_+$ *all satisfy* (2.1). *Then so do*:

(i) $h(f(\mathbf{x}))$.

(ii) $f(h_1(\mathbf{x}), \dots, h_m(\mathbf{x}))$.

(iii) $\min\{f(\mathbf{x}), g(\mathbf{x})\}$.

(iv) $f(c\mathbf{x})$ *and* $cf(\mathbf{x})$, $c > 0$.

(v) $f^\alpha(\mathbf{x})g^\beta(\mathbf{x})$ *for* $\alpha, \beta > 0$, $\alpha + \beta \geqslant 1$.

(vi) *The function*

$$f_M(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \in M, \\ 0, & \mathbf{x} \notin M, \end{cases}$$

where M *is any subset of* $\mathbb{R}_+^m$ *with the following properties*:

(1) $\mathbf{a} \in M$, $\mathbf{b} \in \mathbb{R}_+^m$ *implies* $\mathbf{a} + \mathbf{b} \in M$,

(2) $\mathbf{a}_1, \dots, \mathbf{a}_s \in M$ *implies* $(\mathbf{a}_1 \circ \cdots \circ \mathbf{a}_s)^{1/s} \in M$.

Proof. Statements (i)–(iv) can be proved by verifying either (2.1) or (2.2). In (i) and (ii) we use the fact that h and f are componentwise increasing, respectively. (v) follows from (2.1) and the inequality $\rho(A^\alpha \circ B^\beta) \leqslant \rho^\alpha(A)\rho^\beta(B)$ for all $\alpha, \beta > 0$, $\alpha + \beta \geqslant 1$; see (1.3). Using (2.2), we easily verify (vi). ∎

Note that included in (v) are functions of the form $F(\mathbf{x}, \mathbf{y}) = f^\alpha(\mathbf{x})g^\beta(\mathbf{y})$. A set of functions satisfying (2.1) is obtained from the following:

PROPOSITION 2.6. *Let* $c \geqslant 0$, $t > 0$, *and*

$$f(x) = (x^t - c)_+^{1/t} = \begin{cases} (x^t - c)^{1/t}, & x^t \geqslant c, \\ 0, & x^t < c. \end{cases}$$

Then f *satisfies* (2.1).

Proof. For $f(x) = (x^t - c)_+^{1/t}$ the inequality (2.2a) is

$$(a^t - c)_+^{1/t} + (b^t - c)_+^{1/t} \leqslant \left((a+b)^t - c\right)_+^{1/t}. \tag{2.11}$$

If $a^t \leqslant c$ or $b^t \leqslant c$ or $c = 0$, then (2.11) is easily checked. Assume that $a^t, b^t > c > 0$, and set $a^t = A^t + c$, $b^t = B^t + c$, and $C^t = c$. Then (2.11) is equivalent to

$$\left[(A+B)^t + C^t\right]^{1/t} \leqslant (A^t + C^t)^{1/t} + (B^t + C^t)^{1/t}$$

for all $A, B, C > 0$. Now,

$$\begin{aligned}\left[(A+B)^t + C^t\right]^{1/t} &= \frac{A}{A+B}\left[(A+B)^t + C^t\right]^{1/t} + \frac{B}{A+B}\left[(A+B)^t + C^t\right]^{1/t}\\ &= \left[A^t + \left(\frac{A}{A+B}\right)^t C^t\right]^{1/t} + \left[B^t + \left(\frac{B}{A+B}\right)^t C^t\right]^{1/t}\\ &\leqslant (A^t + C^t)^{1/t} + (B^t + C^t)^{1/t},\end{aligned}$$

proving (2.11).

The inequality (2.2b) with $s = 2$ for $f(x)$ is

$$(a^t - c)_+^{1/2t}(b^t - c)_+^{1/2t} \leqslant \left((ab)^{t/2} - c\right)_+^{1/t}. \tag{2.12}$$

Here again, if $a^t \leqslant c$ or $b^t \leqslant c$, then there is nothing to prove. Assume that $a^t, b^t > c$. Raise both sides of (2.12) to the power $2t$. The desired inequality is then seen to be equivalent to $0 \leqslant c(a^{t/2} - b^{t/2})^2$, whose validity is clear. ∎

From Propositions 2.5, 2.6, and since we know that $f(\mathrm{x}) = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ satisfies (2.1) if $\alpha_i > 0$, $\sum_{i=1}^m \alpha_i \geqslant 1$ [see (1.3)], we get:

COROLLARY 2.7. *The function* $f(x_1, \dots, x_m) = (x_1^{t_1} - c_1)_+^{r_1} \cdots (x_m^{t_m} - c_m)_+^{r_m}$ *satisfies* (2.1) *whenever* $c_i \geqslant 0$, $t_i > 0$, *and* $t_i r_i \geqslant 1$, $i = 1, \dots, m$.

Finally we remark that the set of functions satisfying (2.1) is *not* convex. We know that $f(x) = x^r$ satisfies (2.1) whenever $r \geqslant 1$. Nevertheless we have:

PROPOSITION 2.8. *Let* $r, t \geqslant 1$, $r \neq t$. *Then* $f(x) = x^r + x^t$ *does not satisfy* (2.1).

Proof. Assume that $f(x)$ satisfies (2.1). From Theorem 2.1, $f(x)$ satisfies (2.2), and therefore

$$(a^r + a^t)^{1/2}(b^r + b^t)^{1/2} \leqslant a^{r/2}b^{r/2} + a^{t/2}b^{t/2}$$

for all $a, b \geqslant 0$. Set $b = 1$. Then

$$(a^r + a^t)^{1/2} 2^{1/2} \leqslant a^{r/2} + a^{t/2}$$

for all $a \geqslant 0$. Squaring, we obtain

$$(a^{r/2} - a^{t/2})^2 \leqslant 0,$$

which holds only if $a = 0$ or $a = 1$. This contradiction yields our claim. ■

We remark that using similar arguments one can prove that whenever $k > 1$, the function

$$f(x) = \sum_{i=1}^{k} \alpha_i x^{r_i}$$

where $r_i \geqslant 1$, all distinct, and $\alpha_i > 0$, $i = 1, \dots, k$, does not satisfy (2.1).

3. THE INEQUALITY $f(\rho(A_1), \dots, \rho(A_m)) \leqslant \rho(f(A_1, \dots, A_m))$

In this section we study the set of functions $f: \mathbb{R}_+^m \to \mathbb{R}_+$ which satisfy

$$f(\rho(A_1), \dots, \rho(A_m)) \leqslant \rho(f(A_1, \dots, A_m)) \tag{3.1}$$

for all $A_1, \dots, A_m \in P_n^+$ and all $n \in \mathbb{N}$. We do not completely characterize this set. However, we identify various classes of functions satisfying (3.1). Under certain restrictions we are able to completely characterize the set.

Before stating our first main result, we need some simple definitions. Firstly, we say that a function is *bounded above at a point* if there exists a neighborhood of the point such that the function is bounded above in this neighborhood. Secondly, if $f: \mathbb{R}_+^m \to \mathbb{R}_+$, we define $f_k: \mathbb{R}_+ \to \mathbb{R}_+$, $k = 1, \dots, m$, by

$$f_k(x) = f(0, \dots, 0, x, 0, \dots, 0),$$

where x is in the kth coordinate position. We can now state:

THEOREM 3.1. *Assume $f: \mathbb{R}_+^m \to \mathbb{R}_+$ is bounded above at some point $\boldsymbol{\alpha} \in \operatorname{int} \mathbb{R}_+^m$ and $f(\mathbf{0}) = 0$. Then f satisfies (3.1) if and only if*

$$f(x_1, \dots, x_m) = \max_{k=1,\dots,m} f_k(x_k) \tag{3.2}$$

and each $f_k:\mathbb{R}_+ \to \mathbb{R}_+$ *satisfies*

$$f_k(\rho(A)) \leqslant \rho(f_k(A)) \tag{3.3}$$

for all $A \in P_n^+$ *and all* $n \in \mathbb{N}$, *or what is equivalent, each* f_k *satisfies*

$$f_k(a+b) \leqslant f_k(a) + f_k(b), \tag{3.4a}$$

$$f_k(\sqrt{ab}\,) \leqslant \sqrt{f_k(a)f_k(b)} \tag{3.4b}$$

for all $a, b \geqslant 0$.

We will divide the proof of Theorem 3.1 into its two main steps.

PROPOSITION 3.2. *Assume* $f:\mathbb{R}_+^m \to \mathbb{R}_+$, *and* $f(\mathbf{0}) = 0$. *Then* f *satisfies* (3.1) *if and only if* (3.2) *and* (3.3) *hold.*

Proof. $\Rightarrow$: Assume f satisfies (3.1) and $f(\mathbf{0}) = 0$. It easily follows that each f_k satisfies (3.3). To prove that (3.2) holds, set

$$A_k = \begin{pmatrix} 0 & b_{1k} \\ b_{2k} & 0 \end{pmatrix}, \qquad k = 1,\dots,m.$$

Then $\rho(A_k) = (b_{1k}b_{2k})^{1/2}$, $k = 1,\dots,m$. From (3.1), using $f(\mathbf{0}) = 0$, we obtain

$$f\big((b_{11}b_{21})^{1/2},\dots,(b_{1m}b_{2m})^{1/2}\big) \leqslant f^{1/2}(b_{11},\dots,b_{1m})f^{1/2}(b_{21},\dots,b_{2m}).$$

Thus in particular

$$f(0,\dots,0,a_k,0,\dots,0) \leqslant f^{1/2}(0,\dots,0,a_k,0,\dots,0)f^{1/2}(a_1,\dots,a_m).$$

If $f_k(a_k) = f(0,\dots,0,a_k,0,\dots,0) > 0$, then

$$f_k(a_k) \leqslant f(a_1,\dots,a_m).$$

Thus

$$\max_{k=1,\dots,m} f_k(a_k) \leqslant f(\mathbf{a}).$$

Now let A_k denote the $m \times m$ matrix all of whose entries are zero except for the (k,k) entry equal to a_k. Thus $\rho(A_k) = a_k$, $k = 1,\dots,m$, and

$$\rho(f(A_1,\dots,A_m)) = \max_{k=1,\dots,m} f_k(a_k).$$

From (3.1)

$$f(\mathbf{a}) \leqslant \max_{k=1,\dots,m} f_k(a_k).$$

Therefore (3.2) holds.

$\Leftarrow$: Assume that the $\{f_k\}_{k=1}^m$ satisfy (3.3), and f is defined by (3.2). From (3.2) and (3.3),

$$f(\rho(A_1),\dots,\rho(A_m)) = \max_{k=1,\dots,m} f_k(\rho(A_k)) \leqslant \max_{k=1,\dots,m} \rho(f_k(A_k)).$$

Again from (3.2),

$$f_k(A_k) \leqslant f(A_1,\dots,A_m)$$

for each $k = 1,\dots,m$, where the inequality is understood elementwise. Since $A \leqslant B$ implies $\rho(A) \leqslant \rho(B)$, we obtain

$$\max_{k=1,\dots,m} \rho(f_k(A_k)) \leqslant \rho(f(A_1,\dots,A_m)).$$

This together with a previous inequality implies (3.1). ■

PROPOSITION 3.3. *Assume* $f:\mathbb{R}_+ \to \mathbb{R}_+$ *is bounded above at some point* $\alpha \in (0,\infty)$, *and* $f(0) = 0$. *Then* f *satisfies* (3.3) *if and only if for every* $a, b \geqslant 0$

$$f(a+b) \leqslant f(a) + f(b), \tag{3.4a}$$

$$f(\sqrt{ab}) \leqslant \sqrt{f(a)f(b)}. \tag{3.4b}$$

Proof. $\Rightarrow$: If f satisfies (3.3) and $f(0) = 0$, then from (3.3) applied to the matrices

$$A = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$$

we obtain (3.4a) and (3.4b), respectively.

$\Leftarrow$: We now assume that $f:\mathbb{R}_+ \to \mathbb{R}_+$ is bounded above at some $\alpha \in (0,\infty)$, $f(0)=0$, and f satisfies (3.4). We first claim that f is continuous on $(0,\infty)$ and

$$f(a^{\lambda}b^{1-\lambda}) \leqslant f^{\lambda}(a)f^{1-\lambda}(b) \tag{3.5}$$

for all $a,b \geqslant 0$ and $\lambda \in (0,1)$. To this end let us assume that $f \not\equiv 0$. We note that $f(x)>0$ for all $x>0$. For if $f(b)=0$ for some $b>0$, then from (3.4b) we obtain $f(\sqrt{ab})=0$ for all $a \geqslant 0$, implying that $f \equiv 0$. Set $g(x)=\ln f(e^{x})$. Since $f(x)>0$ for all $x>0$, g is well defined on all $\mathbb{R}$. Furthermore, from (3.4b)

$$g\left(\frac{c+d}{2}\right) \leqslant \frac{g(c)+g(d)}{2}$$

for all $c,d \in \mathbb{R}$. Thus g is midconvex. By assumption g is bounded above at $\ln\alpha$. This implies that g is both convex and continuous on all $\mathbb{R}$. Thus f is continuous on $(0,\infty)$ and satisfies (3.5).

Set $h(x)=x/f(x)$ for $x>0$, and $h(0)=0$. Since $f(x)>0$ for all $x>0$, h is well defined. From (3.5) we easily obtain

$$h^{\lambda}(a)h^{1-\lambda}(b) \leqslant h(a^{\lambda}b^{1-\lambda}) \tag{3.6}$$

for all $a,b \geqslant 0$ and $\lambda \in (0,1)$. Also, h is an increasing function. The proof of this fact is as follows. From (3.4a), $f(nx) \leqslant nf(x)$ for all $n \in \mathbb{N}$. Let $t>1$ and $n^{\lambda}=t$ for some $n \in \mathbb{N}$ and $\lambda \in (0,1)$. Then using (3.5)

$$\begin{aligned} f(tx) &= f(n^{\lambda}x) = f\left((nx)^{\lambda}x^{1-\lambda}\right) \leqslant f^{\lambda}(nx)f^{1-\lambda}(x) \\ &\leqslant n^{\lambda}f^{\lambda}(x)f^{1-\lambda}(x) = tf(x). \end{aligned}$$

Thus

$$h(x)=\frac{x}{f(x)} \leqslant \frac{tx}{f(tx)} = h(tx)$$

for all $t>1$ and $x \geqslant 0$, i.e., h is increasing.

To prove (3.3) we initially note that (3.3) trivially holds if $\rho(A)=0$ [since $f(0)=0$]. Thus we assume that $\rho(A)>0$, implying that $f(\rho(A))>0$. We have from the definition of $\mu(h(A))$ the existence of $\{i_1,\dots,i_s\} \subseteq \{1,\dots,n\}$, $1 \leqslant s \leqslant n$ $(i_{s+1} = i_1)$ such that

$$\mu(h(A)) = \left[\prod_{j=1}^{s} h(a_{i_j i_{j+1}})\right]^{1/s}.$$

Since h is increasing and satisfies (3.6), we also obtain

$$\mu(h(A)) = \left[\prod_{j=1}^{s} h(a_{i_j i_{j+1}})\right]^{1/s} \leqslant h\left[\left(\prod_{j=1}^{s} a_{i_j i_{j+1}}\right)^{1/s}\right]$$
$$\leqslant h(\mu(A)) \leqslant h(\rho(A)).$$

Thus from (1.4) and the easily checked equality $h(A)f(A) = A$, we have

$$\frac{\rho(A)}{\rho(f(A))} = \frac{\rho(h(A)f(A))}{\rho(f(A))} \leqslant \mu(h(A)) \leqslant h(\rho(A)) = \frac{\rho(A)}{f(\rho(A))},$$

i.e., $f(\rho(A)) \leqslant \rho(f(A))$. ∎

REMARK 3.1. The converse (more difficult) part of Proposition 3.3 can also be proven via the analysis of Section 2. As therein, we set $\rho = \rho(A)$. If $\rho = 0$, then (3.3) holds. For $\rho > 0$ we obtain, as in the proof of Proposition 2.4, the existence of an $\alpha \in \mathbb{R}$ for which

$$f(x) \geqslant f(\rho)\rho^{-\alpha}x^{\alpha}$$

for all $x > 0$. From (3.4a), we infer that $\alpha \leqslant 1$. It may then be shown that for all such α

$$\rho(A^{\alpha}) \geqslant \rho(A)^{\alpha},$$

where we set $a_{ij}^{\alpha} = 0$ if $a_{ij} = 0$. Finally

$$\rho(f(A)) \geqslant f(\rho)\rho^{-\alpha}\rho(A^{\alpha}) \geqslant f(\rho) = f(\rho(A)).$$

To prove Theorem 3.1, we apply Propositions 3.2 and 3.3, and note that since f is bounded above at some point $\boldsymbol{\alpha} \in \operatorname{int} \mathbb{R}_+^m$ and $f(\mathbf{0}) = 0$, we obtain from (3.2) that f_k is bounded above at some point $\alpha_k \in (0,\infty)$ and $f_k(0)=0$ for each $k=1,\dots,m$.

We will return to the study of functions of the above form. Firstly however let us note some general properties of functions satisfying (3.1).

PROPOSITION 3.4. *Let* $f, g: \mathbb{R}_+^m \to \mathbb{R}_+$, $h: \mathbb{R}_+ \to \mathbb{R}_+$ *and* $h_1,\dots,h_m: \mathbb{R}_+^p \to \mathbb{R}_+$ *all satisfy* (3.1). *Then so do*:

(i) $h(f(\mathbf{x}))$ *if* h *is increasing.*
(ii) $f(h_1(\mathbf{x}),\dots,h_m(\mathbf{x}))$ *if* f *is componentwise increasing.*
(iii) $\max\{f(\mathbf{x}), g(\mathbf{x})\}$.
(iv) $f(c\mathbf{x})$ *and* $cf(\mathbf{x})$ *for any* $c>0$.

Proof. The proofs of these facts are simple consequences of (3.1). ∎

From Proposition 3.4(iii), all functions of the form

$$f(x_1,\dots,x_m) = \max\{f_1(x_1),\dots,f_m(x_m)\}$$

satisfy (3.1) if the $f_k: \mathbb{R}_+ \to \mathbb{R}_+$ satisfy (3.1) [(3.3)]. Theorem 3.1 does not give us all functions f_k satisfying (3.3). To see this more explicitly, we first prove:

LEMMA 3.5. *Assume* $f: \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (3.3). *Let*

$$g(x) = \begin{cases} f(x), & x>0, \\ c, & x=0, \end{cases}$$

where $c \geqslant f(0)$. *Then* g *satisfies* (3.3).

Proof. If $\rho(A)=0$, then *every* function $g: \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$g(\rho(A)) \leqslant \rho(g(A)).$$

For since $\rho(A)=0$, we have that $a_{ii}=0$ for all i. Thus $g(A) \geqslant B$, where $b_{11}=g(a_{11})=g(0)$, and $b_{ij}=0$ for all $(i,j)\neq(1,1)$. Therefore

$$g(\rho(A)) = g(0) = \rho(B) \leqslant \rho(g(A)).$$

Assume $\rho(A) > 0$. Now $g(A) \geqslant f(A)$ and thus

$$g(\rho(A)) = f(\rho(A)) \leqslant \rho(f(A)) \leqslant \rho(g(A)).$$ ■

In the proof of Proposition 3.3, the condition $f(0) = 0$ was used to obtain (3.4b) from (3.3). We needed (3.4), but not $f(0) = 0$ in the proof of the converse direction of Proposition 3.3. Lemma 3.5 illustrates this point. We now incorporate this fact into one direction of Proposition 3.3, and state certain properties of the functions contained therein.

PROPOSITION 3.6. *If $f: \mathbb{R}_+ \to \mathbb{R}_+$ is bounded above at some point in $(0,\infty)$ and satisfies* (3.4) *for all $a, b > 0$ ($f(0)$ arbitrary), then f satisfies* (3.3). *Furthermore, f is necessarily continuous on $(0,\infty)$ and satisfies the following*:

(i) $\lim_{\varepsilon \to 0^+} f(\varepsilon)$ *and* $\lim_{x\to\infty} f(x)$ *exist (and may be infinite).*

(ii) *If* $\lim_{\varepsilon \to 0^+} f(\varepsilon) = 0$, *then f is strictly increasing on $(0,\infty)$ and* $\lim_{x\to\infty} f(x) = \infty$.

(iii) *If* $\lim_{\varepsilon\to 0^+} f(\varepsilon) = c$ *where $0 < c < \infty$, then there exists a K, $0 \leqslant K \leqslant \infty$, such that $f(x) = c$ for all $x \in (0, K]$, while f is strictly increasing on (K,∞). Furthermore, if $K < \infty$, then* $\lim_{x\to\infty} f(x) = \infty$.

(iv) *If* $\lim_{\varepsilon\to 0^+} f(\varepsilon) = \infty$, *then one of the following holds*:

(1) *f is strictly decreasing on $(0,\infty)$.*
(2) *There exists a K, $0 < K < \infty$, and $c > 0$ such that f is strictly decreasing on $(0, K)$ and $f(x) = c$ for all $x \in [K,\infty)$.*
(3) *There exist K and L, $0 < K \leqslant L < \infty$, and $c > 0$ such that f is strictly decreasing on $(0, K)$, f is strictly increasing on (L,∞), and $f(x) = c$ for all $x \in [K, L]$. Furthermore,* $\lim_{x\to\infty} f(x) = \infty$.

Proof. The fact that f satisfies (3.3) and is continuous on $(0,\infty)$ follows from the proof of Proposition 3.3. [Assume $f(0) = 0$ and then use Lemma 3.5 to obviate this assumption.] Properties (i)–(iv) follow from the fact that $g(x) = \ln f(e^x)$ is convex on $\mathbb{R}$, as was shown in the proof of Proposition 3.3. ■

Each of the possibilities as enumerated in (iv) of Proposition 3.6 can occur. Examples of such are easily constructed from the next few results. Note also that if (ii) of Proposition 3.6 holds and $f(0) = 0$, then f^{-1} exists, $f^{-1}: \mathbb{R}_+ \to \mathbb{R}_+$, and f^{-1} satisfies (2.2) and thus (2.1). Conversely, if $f: \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (2.1) and $f(x) > 0$ for all $x > 0$, then f is continuous and strictly increasing on $\mathbb{R}_+$ and $\lim_{x\to\infty} f(x)=\infty$ [since $nf(x)\leqslant f(nx)$]. Thus $f^{-1}:\mathbb{R}_+\to\mathbb{R}_+$ satisfies (3.4) and thus (3.3).

We also have:

PROPOSITION 3.7. *If $f,g:\mathbb{R}_+\to\mathbb{R}_+$ are both continuous on $(0,\infty)$ and satisfy (3.4) for all $a,b>0$, then $f+g$ satisfies (3.3).*

Proof. Obviously $f+g$ satisfies (3.4a). A simple calculation proves (3.4b). ■

We remark that we have replaced "bounded above at some point in $(0,\infty)$" by "continuous on $(0,\infty)$" due to Proposition 3.6.

This result together with (iv) of Proposition 3.4 implies that the set of functions satisfying the conditions of Proposition 3.7 is convex. As examples of functions satisfying the above, we have the class of functions $f(x)=x^r$ for $x>0$ and $f(0)\geqslant 0$, where $-\infty<r\leqslant 1$. For this class equality holds in (3.4b). The inequality (3.4a) holds for $r\leqslant 0$ simply because the functions are decreasing on $(0,\infty)$. The inequality (3.4a) holds for $0<r\leqslant 1$ from the fact that $(a^r+b^r)^{1/r}$ is a decreasing function of $r\in(0,\infty)$. Alternatively, it follows because the inverse function satisfies Theorem 2.1.

There are functions which satisfy (3.1) but do not satisfy (3.2), or more explicitly, functions $f:\mathbb{R}_+\to\mathbb{R}_+$ which satisfy (3.3) but do not satisfy (3.4b) [which we obtained using the fact that $f(0)=0$].

THEOREM 3.8. *If $f:\mathbb{R}_+^m\to\mathbb{R}_+$ satisfies either*

(i) *f is componentwise decreasing, or*
(ii) *there exists a $c>0$ such that $c\leqslant f(\mathbf{x})\leqslant 2c$ for all $\mathbf{x}\in\mathbb{R}_+^m$,*

then f satisfies (3.1).

Proof. (i): From the Perron-Frobenius theorem, $\rho(A_k)\geqslant a_{11}^k$, $k=1,\dots,m$. Thus $f(\rho(A_1),\dots,\rho(A_m))\leqslant f(a_{11}^1,\dots,a_{11}^m)$. Let B be the $n\times n$ matrix where $b_{11}=f(a_{11}^1,\dots,a_{11}^m)$ and $b_{ij}=0$ for all $(i,j)\neq(1,1)$. Then $f(A_1,\dots,A_m)\geqslant B$ and

$$f(\rho(A_1),\dots,\rho(A_m))\leqslant f\big(a_{11}^1,\dots,a_{11}^m\big)=\rho(B)\leqslant\rho\big(f(A_1,\dots,A_m)\big).$$

(ii): Equality in (3.1) holds if $n=1$. Let $A_1,\dots,A_m\in P_n^+$, $n\geqslant 2$. By assumption, $f(\rho(A_1),\dots,\rho(A_m))\leqslant 2c$. Let J denote the $n\times n$ matrix all of whose entries are 1. Then by assumption $f(a_{ij}^1,\dots,a_{ij}^m)\geqslant c$ and therefore

$f(A_1,\ldots,A_m) \geqslant cJ$. Thus $\rho(f(A_1,\ldots,A_m)) \geqslant \rho(cJ) = cn$. Since $n \geqslant 2$, we have our result. ∎

REMARK 3.2. Note that each of the above two sets of functions is a convex cone. Thus each of the three identified sets of functions satisfying (3.1) is a convex cone. This is *not* true of the set of all functions satisfying (3.1). As an example it suffices to consider $m = 1$. Let $f(x) = x$, and g satisfy $1 \leqslant g(x) \leqslant 2$ where $g(1) = g(2) = g(3) = 1$ and $g((5+\sqrt{5})/2) = 2$. Set

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

It is easily checked that $f + g$ does not satisfy (3.1).

REMARK 3.3. The above three sets of functions, i.e., those satisfying either (i) or (ii) of Theorem 3.8, or those obtained via (3.2) and (3.4), are not all the functions satisfying (3.1). Using Proposition 3.4(iii) we may construct additional functions satisfying (3.1). For example $f(x) = \max\{x,(1+x)^{-1}\}$ satisfies (3.1) and is in none of these three sets.

In the previous results, we delineated functions satisfying (3.1). In the converse direction our results are more sparse except in the case of Theorem 3.1, where we demand that $f(\mathbf{0}) = 0$. In general, if f satisfies (3.1), then it necessarily satisfies (3.4a), but not (3.4b). We end this paper with an additional consequence of (3.1).

PROPOSITION 3.9. *Let* $f: \mathbb{R}_+^m \to \mathbb{R}_+$ *satisfy* (3.1), *and assume that* f *is bounded above on every rectangle in* $\mathbb{R}_+^m$. *If there exists a sequence* $\{\mathbf{x}_r\}_{r=1}^{\infty}$ *for which* $\lim_{r\to\infty} \mathbf{x}_r = \mathbf{y}$, *where* $y \in \operatorname{int} R_+^m$ *and* $\lim_{r\to\infty} f(\mathbf{x}_r) = 0$, *then* f *is componentwise decreasing.*

Proof. Let $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R}_+^m$, and

$$A_k = \begin{pmatrix} a_k & b_k \\ c_k & a_k \end{pmatrix}, \qquad k = 1,\ldots,m.$$

Then $\rho(A_k) = a_k + \sqrt{b_k c_k}$, $k = 1,\ldots,m$, and from (3.1),

$$f\big(\mathbf{a} + (\mathbf{b}\circ\mathbf{c})^{1/2}\big) \leqslant f(\mathbf{a}) + [f(\mathbf{b})f(\mathbf{c})]^{1/2}.$$

Let $\mathbf{d} \in \mathbb{R}_+^m$. Set $\mathbf{b} = \mathbf{x}_r$ and $\mathbf{c} = \mathbf{d}^2 \circ \mathbf{x}_r^{-1}$. Then

$$f(\mathbf{a}+\mathbf{d}) \leqslant f(\mathbf{a}) + \left[f(\mathbf{x}_r) f(\mathbf{d}^2 \circ \mathbf{x}_r^{-1}) \right]^{1/2}.$$

Letting $r \uparrow \infty$ and using the hypothesis, we obtain

$$f(\mathbf{a}+\mathbf{d}) \leqslant f(\mathbf{a})$$

for all $\mathbf{a}, \mathbf{d} \in \mathbb{R}_+^m$. Thus f is componentwise decreasing. ∎

A similar result holds if $\lim_{r \to \infty} f(\mathbf{x}_r) = 0$ where $\lim_{r \to \infty} (\mathbf{x}_r)_k = y_k$ and $0 < y_k \leqslant \infty$, $k = 1, \dots, m$.

REMARK 3.4. If $f: \mathbb{R}_+^m \to \mathbb{R}_+$ satisfies

$$\rho(f(A_1, \dots, A_m)) = f(\rho(A_1), \dots, \rho(A_m))$$

for all $A_k \in P_n^+$, $k = 1, \dots, m$, and $n \in \mathbb{N}$, then

$$f(x_1, \dots, x_m) = c x_k$$

for some $c \geqslant 0$ and $k \in \{1, \dots, m\}$. This may be proven as follows. From (2.2a),

$$f(\mathbf{a}) + f(\mathbf{b}) \leqslant f(\mathbf{a}+\mathbf{b})$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^m$, implying that $f(\mathbf{0}) = 0$ and f is bounded above at some $\boldsymbol{\alpha} \in \operatorname{int} \mathbb{R}_+^m$. From Theorem 3.1,

$$f(x_1, \dots, x_m) = \max_{k=1,\dots,m} f_k(x_k)$$

and

$$f_k(a_k + b_k) \leqslant f_k(a_k) + f_k(b_k)$$

for each $k \in \{1, \dots, m\}$ and all $a_k, b_k \in \mathbb{R}_+$. From the above two inequalities,

$$\max_{k=1,\dots,m} f_k(a_k) + \max_{k=1,\dots,m} f_k(b_k) \leqslant \max_{k=1,\dots,m} f_k(a_k+b_k)$$
$$\leqslant \max_{k=1,\dots,m} \left[f_k(a_k) + f_k(b_k) \right].$$

Thus

$$\max_{k=1,\dots,m} f_k(a_k) + \max_{k=1,\dots,m} f_k(b_k) = \max_{k=1,\dots,m} \left[f_k(a_k) + f_k(b_k) \right].$$

It now follows that there exists at most one $k \in \{1,\dots,m\}$ for which $f_k \not\equiv 0$. Assume $f_k \not\equiv 0$ and $f_i \equiv 0$ for all $i \neq k$. From the above,

$$f_k(a) + f_k(b) = f_k(a+b)$$

for all $a, b \in \mathbb{R}_+$. From Proposition 3.6, $f_k \in C(0,\infty)$. As is easily checked, f is continuous at zero. This functional equality now implies that $f_k(x) = cx$ for all $x \geqslant 0$, where $c = f(1) > 0$.

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