Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 532-542, November 2008

ELA
http://math.technion.ac.il/iic/ela

ZERO MINORS OF TOTALLY POSITIVE MATRICES*

ALLAN PINKUS†

Abstract. In this paper the structure of the zero minors of totally positive matrices is studied and applications are presented.

Key words. Totally positive matrices, Zero minors.

AMS subject classifications. 15A48, 15A15.

1. Introduction and Notation. Totally positive (TP) matrices are matrices all of whose minors are nonnegative. In this paper we will study the possible "singularities" of the minors of such matrices. That is, we ask what can be said about the structure of the zero minors of totally positive matrices.

For ease of exposition we assume in this paper that $A = (a_{ij})$ is an $n \times m$ TP matrix with $n \leq m$. We also assume, in our main result, that every n columns of A are linearly independent. (Equivalently, every k rows and every k columns of A are linearly independent for $k = 1, \ldots, n = \min\{n, m\}$.) This assumption will significantly ease our analysis.

We start with some notation, initially with regard to submatrices and minors. For each $1 \leq i_1 < \cdots < i_p \leq n$ and $1 \leq j_1 < \cdots < j_q \leq m$ we let

$$A\begin{bmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_q \end{bmatrix} := (a_{i_k j_\ell})_{k=1}^{p}\,{}_{\ell=1}^{q}$$

denote the $p \times q$ submatrix of A determined by the rows indexed $i_1, \ldots, i_p$ and columns indexed $j_1, \ldots, j_q$. When $p = q$ we let

$$A\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix} := \det\, (a_{i_k j_\ell})_{k,\ell=1}^{p}$$

denote the associated minor, i.e., the determinant of the submatrix. If A is an $n \times n$ matrix, then its *principal submatrices* are the submatrices of the form

$$A\begin{bmatrix} i_1, \ldots, i_p \\ i_1, \ldots, i_p \end{bmatrix}.$$

*Received by the editors April 29, 2008. Accepted for publication November 4, 2008. Handling Editor: Roger A. Horn.

†Department of Mathematics, Technion, Haifa 32000, Israel (pinkus@tx.technion.ac.il).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 532-542, November 2008

That is, principal submatrices are the square submatrices of A whose diagonal elements are diagonal elements of A. The *principal minors* are their determinants

$$A\begin{pmatrix} i_1,\ldots,i_p \\ i_1,\ldots,i_p \end{pmatrix}.$$

Zero entries of TP matrices and zero values of minors are evidence of boundary behavior within the class of TP matrices and, as such, are not arbitrary in nature. A zero entry of an $n \times m$ TP matrix A and a zero minor of this TP matrix portends linear dependence or "throws a shadow". That is, under suitable linear independence assumptions all minors of the same order to the right and above it, or to the left and below it, are also zero. Let us define these notions more exactly. We start with individual entries. The *right shadow* of the entry a_{ij} is the submatrix $(a_{rs})_{r=1}^{i}{}_{s=j}^{m}$, and the *left shadow* of the entry a_{ij} is the submatrix $(a_{rs})_{r=i}^{n}{}_{s=1}^{j}$. For minors we have the following definition. Assume that we are given a submatrix of A composed of r consecutive rows and columns, namely,

$$A\begin{bmatrix} i+1,\ldots,i+r \\ j+1,\ldots,j+r \end{bmatrix}.$$

Then its *right shadow* is the $(i+r)\times(m-j)$ submatrix

$$A\begin{bmatrix} 1,\ldots,i+r \\ j+1,\ldots,m \end{bmatrix},$$

and its *left shadow* is the $(n-i)\times(j+r)$ submatrix

$$A\begin{bmatrix} i+1,\ldots,n \\ 1,\ldots,j+r \end{bmatrix}.$$

With these definitions we can now state our main result.

THEOREM 1.1. *Let A be an $n \times m$ TP matrix, $n \le m$. Assume that every n columns of A are linearly independent. Let $(\alpha_k,\beta_k,r_k)_{k=1}^{\ell}$ be the set of all triples such that for each $k \in \{1,\ldots,\ell\}$*

$$A\begin{pmatrix} \alpha_k+1,\ldots,\alpha_k+r_k \\ \beta_k+1,\ldots,\beta_k+r_k \end{pmatrix} = 0, \qquad k=1,\ldots,\ell,$$

and no principal minors of the $r_k \times r_k$ submatrix

$$A\begin{bmatrix} \alpha_k+1,\ldots,\alpha_k+r_k \\ \beta_k+1,\ldots,\beta_k+r_k \end{bmatrix} \tag{1}$$

vanish. For such minors we have either

$$\alpha_k+m-n<\beta_k,$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 532-542, November 2008

ELA
http://math.technion.ac.il/iic/ela

or

$$\alpha_k > \beta_k.$$

Moreover

$$A\begin{pmatrix} i_1,\dots,i_p \\ j_1,\dots,j_p \end{pmatrix} = 0$$

if and only if for some $k \in \{1,\dots,\ell\}$ *there is an* $r_k \times r_k$ *principal minor of*

$$A\begin{bmatrix} i_1,\dots,i_p \\ j_1,\dots,j_p \end{bmatrix}$$

that lies in the right shadow of the submatrix (1) if $\alpha_k + m - n < \beta_k$*, or in its left shadow if* $\alpha_k > \beta_k$*.*

That is, all vanishing minors are derived in the above specific manner from the given set of vanishing minors based on consecutive rows and columns.

2. Preliminaries. Theorem 1.1 is based upon various known results. For completeness, we present these results here. The first result can be found in de Boor, Pinkus [1, p. 85].

PROPOSITION 2.1. *If* A *is an* $n \times m$ *TP matrix and*

$$\operatorname{rank} A\begin{bmatrix} i+1,\dots,i+r \\ j+1,\dots,j+r \end{bmatrix} = r-1,$$

then at least one of the following holds. Either the rows $i+1,\dots,i+r$ *or the columns* $j+1,\dots,j+r$ *of* A *are linearly dependent, or the right or left shadow of*

$$A\begin{bmatrix} i+1,\dots,i+r \\ j+1,\dots,j+r \end{bmatrix}$$

has rank $r-1$*.*

The second result is the following. A different proof may be found in Karlin [10, p. 89].

PROPOSITION 2.2. *Let* A *be an* $n \times n$ *TP nonsingular matrix. Then all principal minors of* A *are strictly positive.*

Proof. We first prove directly that $a_{rr} > 0$ for all $r \in \{1,\dots,n\}$. Assume that $a_{rr} = 0$. From Proposition 2.1 we have four options. But all four options contradict the nonsingularity of A. Obviously we cannot have that the rth row or column of A is zero. Thus either the left or right shadow of a_{rr} is zero. Assume that it is the right

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 532-542, November 2008

ELA
http://math.technion.ac.il/iic/ela

shadow which vanishes. Then $a_{ij} = 0$ for all $i \leq r$ and all $j \geq r$, implying that the first r rows of A are linearly dependent. This is a contradiction and therefore $a_{rr} > 0$.

We derive the general result by applying an induction argument on the size of the minor and using Sylvester's Determinant Identity. We assume that for any TP nonsingular $n \times n$ matrix (any n) all principal minors of order at most $p-1$ are strictly positive ($p < n$). We prove that this same result holds for all principal minors of order p . We have proven the case $p = 1$. For any $1 \leq i_1 < \cdots < i_p \leq n$ set

$$b_{k\ell} = A \begin{pmatrix} i_1, \ldots, i_{p-1}, k \\ i_1, \ldots, i_{p-1}, \ell \end{pmatrix},$$

for $k, \ell \in \{1, \ldots, n\} \backslash \{i_1, \ldots, i_{p-1}\}$, and let $B = (b_{k\ell})$. As an immediate consequence of Sylvester's Determinant Identity and our induction hypothesis it follows that B is totally positive and nonsingular. Thus the diagonal entries of B are strictly positive. As

$$0 < b_{i_p i_p} = \left[A \begin{pmatrix} i_1, \ldots, i_{p-1} \\ i_1, \ldots, i_{p-1} \end{pmatrix} \right]^{p-2} A \begin{pmatrix} i_1, \ldots, i_p \\ i_1, \ldots, i_p \end{pmatrix}$$

and, by our induction hypothesis, we have

$$A \begin{pmatrix} i_1, \ldots, i_{p-1} \\ i_1, \ldots, i_{p-1} \end{pmatrix} > 0$$

it therefore follows that

$$A \begin{pmatrix} i_1, \ldots, i_p \\ i_1, \ldots, i_p \end{pmatrix} > 0. \qquad \square$$

An immediate consequence of Proposition 2.2 is the following.

Proposition 2.3. *Let A be an $n \times m$ TP matrix, $n \leq m$, and assume that every n columns of A are linearly independent. If $\alpha \leq \beta \leq \alpha + m - n$ then*

$$A \begin{pmatrix} \alpha + 1, \ldots, \alpha + r \\ \beta + 1, \ldots, \beta + r \end{pmatrix} > 0,$$

for $r = 1, \ldots, \min\{n - \alpha, m - \beta\}$, $\alpha = 0, \ldots, n - r$.

Proof. The minor in question is a principal minor of the $n \times n$ nonsingular TP matrix

$$A \begin{bmatrix} 1, \ldots, n \\ \beta - \alpha + 1, \ldots, \beta - \alpha + n \end{bmatrix}.$$

Apply Proposition 2.2. $\square$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 532-542, November 2008

The next result can be found in Gantmakher, Krein [3, p. 453], see also Karlin [10, p. 88].

PROPOSITION 2.4. *Let A be an $n \times m$ TP matrix, and let $\mathbf{a}^k$ denote the kth row of A, $k = 1, \ldots, n$. Given $1 = i_1 < \cdots < i_{r+1} = n$, assume the $r+1$ vectors $\mathbf{a}^{i_1}, \ldots, \mathbf{a}^{i_{r+1}}$ are linearly dependent, while the r vectors $\mathbf{a}^{i_1}, \ldots, \mathbf{a}^{i_r}$ and $\mathbf{a}^{i_2}, \ldots, \mathbf{a}^{i_{r+1}}$ are each linearly independent. Then A is necessarily of rank r.*

We will use the following consequence of Proposition 2.4, also to be found in Gantmakher, Krein [3, p. 454], see also Karlin [10, p. 89].

PROPOSITION 2.5. *Let A be an $n \times m$ TP matrix. Assume $1 = i_1 < \cdots < i_{r+1} = n$ and $1 = j_1 < \cdots < j_{r+1} = m$. If*

$$A\begin{pmatrix} i_1, \ldots, i_{r+1} \\ j_1, \ldots, j_{r+1} \end{pmatrix} = 0,$$

while

$$A\begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix}, \; A\begin{pmatrix} i_2, \ldots, i_{r+1} \\ j_2, \ldots, j_{r+1} \end{pmatrix} > 0,$$

then A is of rank r.

3. Proof of Theorem 1.1.

We are now in a position to prove Theorem 1.1

Proof of Theorem 1.1. Assume that

$$A\begin{pmatrix} \alpha_k + 1, \ldots, \alpha_k + r_k \\ \beta_k + 1, \ldots, \beta_k + r_k \end{pmatrix} = 0,$$

and no principal minors of

$$A\begin{bmatrix} \alpha_k + 1, \ldots, \alpha_k + r_k \\ \beta_k + 1, \ldots, \beta_k + r_k \end{bmatrix}$$

vanish. From Proposition 2.3 we have for each $k \in \{1, \ldots, \ell\}$ either

$$\alpha_k + m - n < \beta_k,$$

or

$$\alpha_k > \beta_k.$$

By assumption, every set of r_k rows and columns is linearly independent. Thus, from Proposition 2.1, each such vanishing minor either throws a right or a left shadow. If $\alpha_k + m - n < \beta_k$, then it must throw a right shadow since the left shadow of

$$A\begin{bmatrix} \alpha_k + 1, \ldots, \alpha_k + r_k \\ \beta_k + 1, \ldots, \beta_k + r_k \end{bmatrix}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 532-542, November 2008

ELA
http://math.technion.ac.il/iic/ela

is

$$A\begin{bmatrix}\alpha_k+1,\ldots,n\\1,\ldots,\beta_k+r_k\end{bmatrix},$$

which contains the nonsingular $r_k\times r_k$ minor

$$A\begin{bmatrix}\alpha_k+1,\ldots,\alpha_k+r_k\\\alpha_k+1,\ldots,\alpha_k+r_k\end{bmatrix}.$$

But this contradicts Proposition 2.2 (or Proposition 2.3). Similarly, if $\alpha_k>\beta_k$ then

$$A\begin{bmatrix}\alpha_k+1,\ldots,\alpha_k+r_k\\\beta_k+1,\ldots,\beta_k+r_k\end{bmatrix}$$

must throw a left shadow.

Now if

$$A\begin{bmatrix}i_1,\ldots,i_p\\j_1,\ldots,j_p\end{bmatrix}$$

contains an $r_k\times r_k$ principal minor that lies either in the right shadow of one of the

$$A\begin{bmatrix}\alpha_k+1,\ldots,\alpha_k+r_k\\\beta_k+1,\ldots,\beta_k+r_k\end{bmatrix}$$

if $\alpha_k+m-n<\beta_k$, or in its left shadow if $\alpha_k>\beta_k$, then from Proposition 2.1 that principal minor must vanish. It now follows from Proposition 2.2 that

$$A\begin{pmatrix}i_1,\ldots,i_p\\j_1,\ldots,j_p\end{pmatrix}=0.$$

This proves the easier direction of the theorem.

Let us now assume that

$$A\begin{pmatrix}i_1,\ldots,i_p\\j_1,\ldots,j_p\end{pmatrix}=0 \tag{2}$$

for some choice of $1\le i_1<\cdots<i_p\le n$ and $1\le j_1<\cdots<j_p\le m$. If $p=1$ the theorem is a direct consequence of Proposition 2.1. As such we assume that $p>1$. We may also assume, in what follows, that no principal minors of this minor vanish. (Otherwise replace the minor (2) by a principal minor with the same property.) As

$$A\begin{pmatrix}i_1,\ldots,i_p\\j_1,\ldots,j_p\end{pmatrix}=0$$

while

$$A\begin{pmatrix}i_1,\ldots,i_{p-1}\\j_1,\ldots,j_{p-1}\end{pmatrix},\ A\begin{pmatrix}i_2,\ldots,i_p\\j_2,\ldots,j_p\end{pmatrix}>0$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 532-542, November 2008

ELA
http://math.technion.ac.il/iic/ela

it follows from Proposition 2.5 that the $(i_p - i_1 + 1) \times (j_p - j_1 + 1)$ TP matrix

$$A \begin{bmatrix} i_1, i_1+1, \ldots, i_p \\ j_1, j_1+1, \ldots, j_p \end{bmatrix}$$

(composed of consecutive rows and columns) has rank $p-1$.

We claim that

$$j_p - i_1 \le p-2 \qquad \text{or} \qquad i_p - j_1 \le p - 2 - m + n.$$

Assume not. Then

$$j_p - i_1 \ge p-1 \qquad \text{and} \qquad i_p - j_1 \ge p - 1 - m + n. \tag{3}$$

Set $\alpha = \max\{i_1, j_1 - m + n\} - 1$ and $\beta = \max\{i_1, j_1\} - 1$. From this definition it follows that $\alpha \le \beta \le \alpha + m - n$.

We also claim that $i_1 \le \alpha+1 < \cdots < \alpha+p \le i_p$ and $j_1 \le \beta+1 < \cdots < \beta+p \le j_p$. To see this, note that if $\alpha = i_1 - 1$, then $\alpha + 1 = i_1$ and $\alpha + p = i_1 + p - 1 \le i_p$ (since $i_1, \ldots, i_p$ are increasing integers). If $\alpha+1 = j_1 - m + n$, then from its definition $\alpha + 1 \ge i_1$, and from (3) $\alpha + p = j_1 - m + n + p - 1 \le i_p$. Similarly, if $\beta = j_1 - 1$, then $\beta + 1 = j_1$ and $\beta + p = j_1 + p - 1 \le j_p$. If $\beta + 1 = i_1$, then from its definition $\beta + 1 \ge j_1$, and (3) ensures that $\beta + p = i_1 + p - 1 \le j_p$. As

$$A \begin{bmatrix} i_1, i_1+1, \ldots, i_p \\ j_1, j_1+1, \ldots, j_p \end{bmatrix}$$

has rank $p-1$, it follows that

$$A \begin{pmatrix} \alpha+1, \ldots, \alpha+p \\ \beta+1, \ldots, \beta+p \end{pmatrix} = 0.$$

But $\alpha \le \beta \le \alpha + m - n$, which contradicts Proposition 2.3. Thus $j_p - i_1 \le p-2$ or $i_p - j_1 \le p - 2 - m + n$.

Let us assume that $j_p - i_1 \le p - 2$. The matrix

$$A \begin{bmatrix} i_1, i_1+1, \ldots, i_1-p+1 \\ j_p-p+1, j_p-p+2, \ldots, j_p \end{bmatrix}$$

composed of p consecutive rows and columns has rank at most $p-1$. Thus by our assumptions of linear independence and Proposition 2.1 this submatrix throws a right or left shadow. From the analysis in the first part of the proof of this theorem we see that it throws a left shadow since $i_1 > j_p - p + 1$. That is,

$$A \begin{bmatrix} i_1, \ldots, n \\ 1, \ldots, j_p \end{bmatrix}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 532-542, November 2008

ELA
http://math.technion.ac.il/iic/ela

is of rank $p-1$. As

$$A\begin{bmatrix} i_1,\ldots,i_p \\ j_1,\ldots,j_p \end{bmatrix}$$

lies in this submatrix, we see that

$$A\begin{bmatrix} i_1,\ldots,i_p \\ j_1,\ldots,j_p \end{bmatrix}$$

lies in the left shadow of

$$A\begin{bmatrix} \alpha+1,\ldots,\alpha+p \\ \beta+1,\ldots,\beta+p \end{bmatrix}, \tag{4}$$

where $\alpha = i_1 - 1$ and $\beta = j_p - p$. In fact, from our assumption that no principal minors of

$$A\begin{bmatrix} i_1,\ldots,i_p \\ j_1,\ldots,j_p \end{bmatrix}$$

vanish, it follows that (4) has rank $p-1$.

The case where $i_p - j_1 \le p-2-m+n$ is handled similarly. That is, it follows that

$$A\begin{bmatrix} i_1,\ldots,i_p \\ j_1,\ldots,j_p \end{bmatrix}$$

is in the right shadow of the matrix

$$A\begin{bmatrix} i_p-p+1, i_p-p+2,\ldots,i_p \\ j_1, j_1+1,\ldots,j_1+p-1 \end{bmatrix}$$

of rank $p-1$. ☐

4. Some Applications. The following five corollaries are immediate consequences of Theorem 1.1. The first four we state without proof. All these results (except for Corollary 4.4, which we could not find in the literature) were proved by very much more complicated methods.

This first corollary was proven in Gasca, Peña [5] and then reproved in Gladwell [8]. We recall that a matrix is *strictly totally positive* if all its minors are strictly positive.

Corollary 4.1. *Assume A is an $n\times n$ totally positive matrix. Then A is strictly totally positive if*

$$A\begin{pmatrix} 1,\ldots,p \\ n-p+1,\ldots,n \end{pmatrix} > 0$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 532-542, November 2008

and

$$A\begin{pmatrix} n-p+1,\ldots,n \\ 1,\ldots,p \end{pmatrix} > 0$$

for $p = 1,\ldots,n$.

A similar result is the following, which can be found in Gasca, Micchelli, Peña [4]. A matrix $A = (a_{ij})$ is said to be (r,s)*-banded* if $a_{ij} = 0$ whenever $i - j > s$ or $j - i > r$. That is, the only possible non-zero entries of A lie on the diagonals $(a_{i,i+k})$ for $k = -s,\ldots,r$. We say that A is *strictly* (r,s)*-banded* if, in addition, $a_{j+s,j} \neq 0$ and $a_{i,i+r} \neq 0$ for all possible j and i.

Corollary 4.2. *Let A be an $n \times n$ strictly (r,s)-banded totally positive matrix. Then*

$$A\begin{pmatrix} i_1,\ldots,i_q \\ j_1,\ldots,j_q \end{pmatrix} > 0$$

whenever

$$s \geq i_k - j_k \geq -r, \qquad k = 1,\ldots,q$$

if and only if

$$A\begin{pmatrix} 1,\ldots,p \\ n-p+1,\ldots,n \end{pmatrix} > 0, \qquad p = n-r+1,\ldots,n,$$

and

$$A\begin{pmatrix} n-p+1,\ldots,n \\ 1,\ldots,p \end{pmatrix} > 0, \qquad p = n-s+1,\ldots,n.$$

A nonsingular triangular $n \times n$ totally positive matrices can be regarded as a strictly $(n-1,0)$- or $(0,n-1)$-banded matrix (except that the strictness is applied only to the 0-band). We state the next result for an upper triangular matrix. This result was first proven in Shapiro, Shapiro [11], but also follows from Gasca, Micchelli, Peña [4].

Corollary 4.3. *Assume A is an $n \times n$ upper triangular totally positive matrix. Then*

$$A\begin{pmatrix} i_1,\ldots,i_q \\ j_1,\ldots,j_q \end{pmatrix} > 0$$

for all choices of $i_k \leq j_k$, $k = 1,\ldots,q$, and all q if and only if

$$A\begin{pmatrix} 1,\ldots,p \\ n-p+1,\ldots,n \end{pmatrix} > 0$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 532-542, November 2008

for $p = 1, \ldots, n$.

In the infinite case, (r, s)-strictly banded totally positive matrices are as strictly totally positive as they can possibly be.

Corollary 4.4. *Assume that* A *is an infinite or biinfinite matrix, i.e.,* $A = (a_{ij})_{i,j=1}^{\infty}$ *or* $A = (a_{ij})_{i,j=-\infty}^{\infty}$*, that is totally positive and strictly* (r, s)*-banded. Then*

$$A\begin{pmatrix} i_1, \ldots, i_q \\ j_1, \ldots, j_q \end{pmatrix} > 0$$

if and only if

$$s \geq i_k - j_k \geq -r, \qquad k = 1, \ldots, q.$$

As we have seen in Proposition 2.2, if a totally positive matrix satisfies

$$A\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix} > 0$$

then $a_{i_k, j_k} > 0$, $k = 1, \ldots, p$. The converse need not hold. If the converse holds, then A is said to be an *almost strictly totally positive* matrix. This class of matrices was first defined in Gasca, Micchelli, Peña [4]. They proved the following result.

Corollary 4.5. *Let* A *be an* $n \times n$ *nonsingular totally positive matrix. Assume that*

$$A\begin{pmatrix} i+1, \ldots, i+p \\ j+1, \ldots, j+p \end{pmatrix} > 0$$

if $a_{i+k, j+k} > 0$, $k = 1, \ldots, p$, *for all possible* i, j *and* p*. Then* A *is almost strictly totally positive.*

Proof. Assume that

$$A\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix} = 0.$$

From Theorem 1.1 there exist (α, β, r) such that

$$A\begin{pmatrix} \alpha+1, \ldots, \alpha+r \\ \beta+1, \ldots, \beta+r \end{pmatrix} = 0,$$

no principal minor of

$$A\begin{bmatrix} \alpha+1, \ldots, \alpha+r \\ \beta+1, \ldots, \beta+r \end{bmatrix}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 532-542, November 2008

vanishes, and some $r \times r$ principal submatrix of

$$A\begin{bmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{bmatrix}$$

lies in the right shadow of

$$A\begin{bmatrix} \alpha+1, \ldots, \alpha+r \\ \beta+1, \ldots, \beta+r \end{bmatrix}$$

if $\alpha < \beta$, or in its left shadow if $\alpha > \beta$.

The assumption of the corollary implies that $r = 1$. Thus we have an $\alpha \neq \beta$ such that for some $k \in \{1, \ldots, p\}$ the a_{i_k,j_k} lies in the right shadow of $a_{\alpha+1,\beta+1} = 0$ if $\alpha < \beta$, or in its left shadow if $\alpha > \beta$. This implies that

$$a_{i_k,j_k} = 0. \quad \square$$

As a further consequence of Theorem 1.1, paralleling Corollaries 4.1–4.3, one can show that for a nonsingular totally positive matrix it is not necessary to verify all the conditions in the statement of Corollary 4.5 in order to determine if the matrix A is almost strictly totally positive. If we know the zero entries of A then Theorem 1.1 permits us to determine a minimal number of such conditions that must be verified. This result was obtained using other methods in Gasca, Peña [6], Gladwell [9], and Gasca, Peña [7].

REFERENCES

[1] C. de Boor and A. Pinkus. The approximation of a totally positive band matrix by a strictly banded totally positive one. *Linear Algebra Appl.*, 42:81–98, 1982.
[2] F. R. Gantmacher. *The Theory of Matrices*. Gostekhizdat, Moscow-Leningrad, 1953; English transl. as *Matrix Theory*, Chelsea, New York, 2 vols., 1959.
[3] F. R. Gantmakher and M. G. Krein. Sur les matrices complètement non négatives et oscillatoires. *Composito Math.*, 4:445–476, 1937.
[4] M. Gasca, C. A. Micchelli, and J. M. Peña. Almost strictly totally positive matrices. *Numerical Algorithms*, 2:225–236, 1992.
[5] M. Gasca and J. M. Peña. Total positivity and Neville elimination. *Linear Algebra Appl.*, 165:25–44, 1992.
[6] M. Gasca and J. M. Peña. On the characterization of almost strictly totally positive matrices. *Adv. Comp. Math.*, 3:239–250, 1995.
[7] M. Gasca and J. M. Peña. Characterizations and decompositions of almost strictly totally positive matrices. *SIAM J. Matrix Anal.*, 28:1–8, 2006.
[8] G. M. L. Gladwell. Total positivity and the QR algorithm. *Linear Algebra Appl.*, 271:257–272, 1998.
[9] G. M. L. Gladwell. Inner totally positive matrices. *Linear Algebra Appl.*, 393:179–195, 2004.
[10] S. Karlin. *Total Positivity*, Volume 1. Stanford University Press, Stanford, CA, 1968.
[11] B. Z. Shapiro and M. Z. Shapiro. On the boundary of totally positive upper triangular matrices. *Linear Algebra Appl.*, 231:105–109, 1995.