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STUDIES IN SPLINE FUNCTIONS AND
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DIVIDED DIFFERENCES AND OTHER NON-LINEAR EXISTENCE PROBLEMS AT EXTREMAL POINTS

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1. PRELIMINARIES AND STATEMENT OF MAIN RESULTS

In certain considerations of best approximation the following result is relevant.

Theorem A. (C. Davis [1], Videnskii [11]).

Let $\{e_i\}_{i=0}^n$ be a sequence of numbers satisfying $(-1)^i(e_i-e_{i-1}) > 0$, $i = 1,2,\ldots,n$. *Then there exists a unique polynomial* $P(t)$ *of degree* n *and unique interpolation points* $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ *having the properties*

$$(1.1)\qquad \begin{aligned} P(t_i) &= e_i \ , \quad i = 0,1,\ldots,n \\ P'(t_i) &= 0 \ , \quad i = 1,\ldots,n-1 \ . \end{aligned}$$

Thus the e_i are the extrema of P in $[0,1]$.

A number of variations of Theorem A occur in [2], [3] and [11] where the prescribed data is divided into two groups involving free and fixed interpolating points. By a method introduced in [4] (see also [6], [7], [9] and [10] for other applications) we shall develop a series of extensions interpolating functionals of higher order divided differences.

Let $f_{(k)}[x_1,\ldots,x_{k+1}]$ denote the k-th divided difference with respect to the points $x_1 < x_2 < \ldots < x_{k+1}$. More explicitly

$$(1.2) \qquad f_{(1)}[x_1,x_2] = \frac{f(x_1)-f(x_2)}{x_1-x_2}$$

and recursively,

$$(1.3) \quad f_{(k)}[x_1,\ldots,x_{k+1}] = \frac{f_{(k-1)}[x_1,\ldots,x_k]-f_{(k-1)}[x_2,\ldots,x_{k+1}]}{x_1-x_{k+1}} .$$

When coincidences occur among the x_i, then derivatives naturally enter into (1.2) and (1.3). For example,

$$f_{(1)}[x,x] = f'(x)$$

$$(1.4) \qquad f_{(2)}[x,x,y] = \frac{f'(x) - \left[\frac{f(x)-f(y)}{x-y}\right]}{x-y}$$

$$f_{(2)}[x,x,x] = \frac{f''(x)}{2} \quad , \text{ etc. } \ldots .$$

We now highlight our findings affirming the existence of a polynomial whose k-th divided difference achieves prescribed values related to certain critical points of the k-th derivative. More precisely,

Theorem 1. *Let* k *be an integer satisfying* $1 \leq k \leq n-1$. *Let* $\{e_i\}_{i=k}^n$ *be given real values oscillating in the manner* $e_ie_{i+1} < 0$, $i = k,\ldots,n-1$. *There exists a polynomial* $P(t)$ *of degree* n, *and abscissa* $\{t_i\}_{i=0}^n$ *of the form* $t_0 = t_1 = \ldots = t_{k-1} = 0 < t_k < \ldots < t_{n-1} < t_n = 1$ *such that*

$$(1.5) \qquad \begin{array}{ll} \text{i)} & P^{(k)}(t_i) = 0 \quad , \quad i = k,\ldots,n-1 \quad , \\ \text{ii)} & P_{(k)}[t_{i-k},\ldots,t_i] = e_i \quad , \quad i = k,\ldots,n . \end{array}$$

We conjecture that $P(t)$ with the properties (1.5) is unique.

Our proof of Theorem 1 invokes the Brouwer fixed point theorem but also exploits special total positivity structure of the basic spline functions.

There is a concept of k^{th} order divided differences for extended complete Chebyshev (ECT) systems. A counterpart of Theorem 1 is available in this setting. The corresponding result is displayed in Section 3. Generalizations are also available to the case of splines with fixed knots, and to classes of Monosplines and Perfect splines with variable knots. We highlight the result for Perfect splines discussed further in §3.

Theorem 3.4. *Let* k *be an integer,* $1 \leq k \leq n-1$. *Given* $\{e_i\}_{i=k}^{n+r}$ *satisfying* $e_i e_{i+1} < 0$, $i = k,\dots,n+r-1$, *then there exists a Perfect spline*

$$p(t) = \sum_{i=0}^{n-1} b_i t^i + c[t^n + 2 \sum_{i=1}^{r} (-1)^i (t-\xi_i)_+^n] ,$$

where $0 < \xi_1 < \dots < \xi_r < 1$, *and abscissa* $\{t_i\}_0^{n+r}$ *of the form* $t_0 = t_1 = \dots = t_{k-1} = 0 < t_k < \dots < t_{n+r-1} < t_{n+r} = 1$ *for which*

i) $p^{(k)}(t_i) = 0$, $i = k,\dots,n+r-1$

ii) $p_{(k)}[t_{i-k},\dots,t_i] = e_i$, $i = k,\dots,n+r$.

Further ramifications of our methodology provide extensions of an existence theorem in [3, Theorem 8] and other applications elaborated in Section 4.

An application extending Theorem A is as follows.

Theorem 2. *Let* k *be any fixed integer,* $1 \leq k \leq n-1$. *Given* $\{e_i\}_{i=k}^{n}$, $e_i e_{i+1} < 0$, $i = k,\dots,n-1$, *then there exists a polynomial* $P(t)$ *of degree* n , *and abscissa* $\{t_i\}_{i=0}^{n}$, $t_0 = t_1 = \dots = t_{k-1} = 0 < t_k < \dots < t_{n-1} < t_n = 1$ *satisfying*

i) $P^{(i)}(0) = 0$, $i = 0,1,\ldots,k-1$,

ii) $P^{(k)}(t_i) = 0$, $i = k,\ldots,n-1$,

iii) $P(t_i) = e_i$, $i = k,\ldots,n$.

Section 2 is devoted to the proofs of existence in Theorems 1 and A (the latter is considerably simpler). A number of direct generalizations of Theorem 1 to other classes of functions is discussed in Section 3. In Section 4, Theorem 1 is cast in a general form and several applications are developed. Some open problems connected with this work are set forth in Section 5.

2. POLYNOMIALS WITH PRESCRIBED k^{TH} ORDER DIVIDED DIFFERENCES: PROOF OF THEOREM 1

For the case involving divided differences of order k , the proof of Theorem 1 utilizes variation diminishing properties attendant to the totally positive nature of the basic spline kernel $M_i(t)$ (see [5, Chap. 10, §4]; an explicit formula for $M_i(t)$ will be stated later). The circumstances $k = 1,2$ can be dealt with directly without reference to the above fact. It is instructive to treat the simpler case $k = 1$.

<u>Proof of the case $k = 1$.</u> Let $\{e_i\}_{i=1}^n$ be given satisfying $e_i(-1)^{i+1} > 0$, $i = 1,\ldots,n$. Consider the simplex

$\Delta = \{\underline{\xi}:\underline{\xi} = (\xi_1,\ldots,\xi_n),\ \xi_i \geq 0\ ,\ \sum_{i=1}^{n} \xi_i = 1\}$, and

associate with each $\underline{\xi}$ the sequence $t_0 = 0$, $t_i = \sum_{j=1}^{i} \xi_j$, $i=1,\ldots,n$. For each $\underline{\xi} \in \Delta$, we determine the unique polynomial of degree n , $P(t;\underline{\xi})$ exhibiting extrema exactly at the $\{t_i\}_1^{n-1}$ and suitably normalized. Formally, $P(t;\underline{\xi})$ is characterized by the properties

(1) $P(0;\underline{\xi}) = 0$

(2) $P'(t_i;\underline{\xi}) = 0$, $i = 1,\ldots,n-1$

(3) $P'(t;\underline{\xi}) > 0$, $t < t_1$

(4) The sum of the squares of the coefficients of $P(t;\underline{\xi})$ is 1 .

(Where coincidences occur, viz. $t_i = t_{i+1} = \ldots = t_{i+\ell}$, we interpret (2) reflecting higher multiplicities at t_i in the standard manner.)

Consider the first divided difference of $P(t;\underline{\xi})$ with respect to t_{i-1},t_i . $f_i(\underline{\xi}) = P_{(1)}[t_{i-1},t_i;\underline{\xi}]$, $i=1,\ldots,n$, and set $g_i(\underline{\xi}) = \dfrac{f_i(\underline{\xi})}{e_i}$, $i = 1,\ldots,n$. Obviously $g_i(\underline{\xi})$ is a continuous function of $\underline{\xi} \in \Delta$. Define $g(\underline{\xi}) = \max\limits_{i=1,\ldots,n} g_i(\underline{\xi})$, and $F_i(\underline{\xi}) = g(\underline{\xi}) - g_i(\underline{\xi})$, $i=1,\ldots,n$.

Since $f_i(\underline{\xi}) = P_{(1)}[t_{i-1},t_i;\underline{\xi}] = \frac{d}{dt} P(\tilde{t};\underline{\xi})$, for some $\tilde{t} \in [t_{i-1},t_i]$, and $P'(t;\underline{\xi})$ is of one strict sign in each segment (t_{i-1},t_i) , it follows that $f_i(\underline{\xi}) = 0$ iff $\xi_i = 0$. The normalization condition (4) guarantees $g(\underline{\xi}) > 0$ for all $\underline{\xi} \in \Delta$.

If there exists some $\underline{\xi}^* \in \Delta$ for which $\sum\limits_{i=1}^{n} F_i(\underline{\xi}^*) = 0$, then $g_i(\underline{\xi}^*) = g(\underline{\xi}^*) = c > 0$, independent of i and $\dfrac{P(t;\underline{\xi}^*)}{c}$ is the required polynomial.

Now, assume no such $\underline{\xi}$ can be found so that for all

$\underline{\xi} \in \Delta$, $\sum_{i=1}^{n} F_i(\underline{\xi}) > 0$. We then construct the well-defined continuous mapping Φ of Δ into itself with $\Phi(\underline{\xi}) = \underline{\xi}'$, having components

$$\xi_i' = \frac{F_i(\underline{\xi})}{\sum_{j=1}^{n} F_j(\underline{\xi})} , \qquad i = 1,\ldots,n .$$

Appeal to the Brouwer fixed point theorem produces a point $\tilde{\underline{\xi}} \in \Delta$ satisfying

$$\tilde{\xi}_i = \frac{F_i(\tilde{\underline{\xi}})}{\sum_{j=1}^{n} F_j(\tilde{\underline{\xi}})} , \quad i = 1,\ldots,n . \tag{2.1}$$

From the definition of F_i , there certainly exists some $i_0 \in \{1,\ldots,n\}$ for which $F_{i_0}(\tilde{\underline{\xi}}) = 0$. By (2.1), then $\tilde{\xi}_{i_0} = 0$ implying $f_{i_0}(\tilde{\underline{\xi}}) = 0$ as noted earlier, and so $g_{i_0}(\tilde{\underline{\xi}}) = 0$ and finally $g(\tilde{\underline{\xi}}) = 0$, contrary to the fact that $g(\underline{\xi}) > 0$ for all $\underline{\xi} \in \Delta$. To avoid this contradiction, we must have $\sum_{i=1}^{n} F_i(\underline{\xi}^*) = 0$ for some $\underline{\xi}^* \in \Delta$ and the argument is complete in this case.

Q.E.D.

Slight adjustments of the same procedure validates the result of the theorem for the case $k = 2$. However, new difficulties arise for the cases $k \geq 3$. We focus henceforth on the general problem where more recondite techniques with totally positive kernels are used. In order to facilitate the further analysis we record, for easy reference, the following definitions and facts.

Definition 2.1 . Let $\underline{x} = (x_1,\ldots,x_m)$ be a real vector of m components. Then $S^-(\underline{x})$ counts the number of actual sign changes in the sequence $x_1,\ldots,x_m$ with zero terms discarded.

Let $u(t)$ be continuous. We introduce the notation

$$(2.2)\quad S^-\{u(t):[0,1]\} = \sup_{\substack{0\leq t_1<t_2<\ldots<t_m\leq 1 \\ \text{all } m}} S^-(u(t_1),\ldots,u(t_m)) .$$

Theorem 2.1. [5, Chap. 5]. Let $K_i(t)$ be a totally positive kernel with respect to the variables $0 \leq t \leq 1$ and $i = k, k+1,\ldots,n$. Let $u(t)$ be continuous on $[0,1]$ and determine

$$x_i = \int_0^1 K_i(t)u(t)dt \quad , \quad i = k,\ldots,n .$$

Then

$$(2.3)\qquad S^-(x_k,x_{k+1},\ldots,x_n) \leq S^-\{u(t):[0,1]\} .$$

Moreover, if equality holds in (2.3), then the arrangement of the sign in $\underline{x}$ and $u(t)$ coincide, i.e., the first non-zero sign in the sequence $\{x_i\}_k^n$ agrees with the initial non-trivial sign of $u(t)$ as t traverses $[0,1]$.

Let

$$(2.4)\quad M_i(t;\underline{x})\,(={}_kM_i(t;\underline{x})) = M_i(t;x_{i-k},\ldots,x_i)$$

(the k appearing as a left subscript is suppressed where no ambiguity in the interpretation is conceivable) be the basic (B) spline polynomial of order k associated with the abscissae $x_{i-k} \leq \ldots \leq x_i$. An explicit expression of $M_i(t;\underline{x})$ is

$$(2.5)\quad M_i(t;\underline{x})\,(=M_i(t)) = \frac{\begin{vmatrix} u_0(x_{i-k}) & \cdots & u_0(x_i) \\ \vdots & & \vdots \\ u_{k-1}(x_{i-k}) & \cdots & u_{k-1}(x_i) \\ \varphi_k(x_{i-k};t) & \cdots & \varphi_k(x_i;t) \end{vmatrix}}{\begin{vmatrix} u_0(x_{i-k}) & \cdots & u_0(x_i) \\ \vdots & & \vdots \\ u_k(x_{i-k}) & \cdots & u_k(x_i) \end{vmatrix}}$$

where $u_i(x) = x^i$, $i = 0,1,\ldots,k$ and $\varphi_k(x;t) = k(x-t)_+^{k-1}$.

<u>Theorem 2.2 . [5, Chap. 10]. The kernel $M_i(t)$ is positive on $x_{i-k} < t < x_i$ and vanishes outside this range. Moreover, $M_i(t)$ is totally positive with respect to $-\infty < t < \infty$, and $i = k,k+1,\ldots$.</u>

For any $f(t)$ of continuity class C^k there is available the representation formula for the k-th divided difference

$$(2.6)\quad \begin{aligned} f_{(k)}[x_{i-k},\ldots,x_i] &= \frac{1}{k!}\int_{x_{i-k}}^{x_i} M_i(t)f^{(k)}(t)\,dt \\ &= \frac{1}{k!}\int_0^1 M_i(t;\underline{x})f^{(k)}(t)\,dt \end{aligned}$$

provided $0 \leqslant x_{i-k} < x_i \leqslant 1$.

With this preparation, we now present:

<u>Proof of Theorem 1.</u> Let the interpolation data $\{e_i\}_{i=k}^n$ be given, assuming for definiteness that $e_i(-1)^{i+k} > 0$, $i = k,\ldots,n$. Let Δ be the simplex $\Delta = \{\underline{\xi}:\underline{\xi} = (\xi_k,\ldots,\xi_n)$, $\xi_i \geqslant 0$, $\sum_{i=k}^{n} \xi_i = 1\}$ and associate with each $\underline{\xi} \in \Delta$ the sequence $t_0 = t_1 = \ldots = t_{k-1} = 0$, $t_i = \sum_{j=k}^{i} \xi_j$, $i=k,\ldots,n$.

For each $\underline{\xi} \in \Delta$, we form the unique polynomial $P(t;\underline{\xi})$ of degree n determined by the conditions

1. $\frac{d^i}{dt^i} P(0;\underline{\xi}) = P^{(i)}(0;\underline{\xi}) = 0$, $i = 0,1,\ldots,k-1$,
2. $P^{(k)}(t_i;\underline{\xi}) = 0$, $i = k,\ldots,n-1$,
3. $P^{(k)}(t;\underline{\xi}) > 0$, $t < t_k$,
4. The sum of the squares of the coefficients of $P(t;\underline{\xi})$ is 1.

Thus $P(t;\underline{\xi})$ vanishes at 0 with multiplicity k and the zeros of the k-th derivative are located at $t_k, t_{k+1}, \ldots, t_{n-1}$.

Construct next the sequence of k-th divided differences of $P(t;\underline{\xi})$ with respect to $\{t_0, t_1, \ldots, t_n\}$, (viz., $P_{(k)}[t_{i-k}, \ldots, t_i; \underline{\xi}] = f_i(\underline{\xi})$, $i = k, k+1, \ldots, n$ (cf. (1.3) and (1.4)). Observe also that

(2.7) $P^{(k)}(t;\underline{\xi})$ maintains a strict positive sign in (t_0, t_k), $0 = t_0 = t_{k-1} < t_k$ unless $t_k = t_{k-1}$ and then, of course, the value is zero.

Define

$$g_i(\underline{\xi}) = \max\{0, \frac{f_i(\underline{\xi})}{e_i}\} \quad , \quad i = k,\ldots,n$$

$$g(\underline{\xi}) = \min_{i=k,\ldots,n} g_i(\underline{\xi})$$

$$F_i(\underline{\xi}) = g_i(\underline{\xi}) - g(\underline{\xi}) \quad , \quad i = k,\ldots,n \; .$$

(Note that $g(\underline{\xi})$ is a minimum and not a maximum as in the discussion of the case $k = 1$.) Manifestly, each $F_i(\underline{\xi})$ is a continuous function of $\underline{\xi} \in \Delta$.

Suppose there exists a $\underline{\xi}^* \in \Delta$ satisfying $\sum_{j=k}^{n} F_j(\underline{\xi}^*) = 0$

so that each term $F_j(\underline{\xi}^*) = 0$. Then $g_j(\underline{\xi}^*) = g(\underline{\xi}^*) = c$, $j = k,\ldots,n$. By definition $g(\underline{\xi}^*) \geq 0$. Consider the feasibility of $g(\underline{\xi}^*) = 0$. Since always $f_k(\underline{\xi}) \geq 0$ and $e_k > 0$, it necessarily follows that $f_k(\underline{\xi}^*) = 0$. But since $f_i(\underline{\xi}) = P^{(k)}(\tilde{t};\underline{\xi})$ for some $\tilde{t} \in (t_{i-k},t_i)$, we find that the equation $f_k(\underline{\xi}^*) = 0$ is equivalent to $t_k^* = 0$ (consult (2.7)). Thus, $t_k^* = t_{k-1}^* = 0$ and in this case, the prescription of $f_{k+1}(\underline{\xi}^*)$ reveals that on $[t_1^*,t_{k+1}^*]$, $P^{(k)}(t;\underline{\xi}^*) \leq 0$ and in particular $f_{k+1}(\underline{\xi}^*) \leq 0$. Again since $g_{k+1}(\underline{\xi}^*) = 0$ and $e_{k+1} < 0$ it follows that $f_{k+1}(\underline{\xi}^*) = 0$ and concomitantly $t_{k+1}^* = t_k^* = 0$. Continuing in this fashion leads to the result $\sum_{i=k}^{n} \xi_i^* = 0$, an absurdity. Hence, the event $g(\underline{\xi}^*) = 0$ is precluded and therefore $g(\underline{\xi}^*) = c > 0$, and $\dfrac{P(t;\underline{\xi}^*)}{c}$ is the requisite polynomial of (1.5), provided we verify $0 < t_k^* < \ldots < t_{n-1}^* < 1$. This property will be confirmed at the final steps of the proof.

Our next objective is to establish the existence of a $\underline{\xi}^*$ in Δ satisfying $\sum_{i=k}^{n} F_i(\underline{\xi}^*) = 0$. Suppose to the contrary that for all $\underline{\xi} \in \Delta$, the inequality $\sum_{i=k}^{n} F_i(\underline{\xi}) > 0$ is present.

Consider the continuous map of Δ into itself defined by $\Phi(\underline{\xi}) = \underline{\xi}'$ having components

$$\xi_i' = \frac{F_{i+1}(\underline{\xi})}{\sum_{j=k}^{n} F_j(\underline{\xi})} \quad , \quad i = k,\ldots,n$$

where by convention $F_{n+1}(\underline{\xi}) \equiv F_k(\underline{\xi})$. (Notice the shift of the index in the definition of ξ_i' .) The Brouwer fixed point theorem guarantees the occurrence of a $\hat{\underline{\xi}} \in \Delta$ satisfying

$$(2.8) \qquad \hat{\xi}_i = \frac{F_{i+1}(\hat{\underline{\xi}})}{\sum_{j=k}^{n} F_j(\hat{\underline{\xi}})} , \quad i = k,\ldots,n .$$

The definition of $F_j(\hat{\underline{\xi}})$ entails the existence of at least one index i_0 such that $F_{i_0}(\hat{\underline{\xi}}) = 0$. Let i_0 be the <u>smallest</u> $i \in \{k,\ldots,n\}$ with this property. We distinguish several possibilities.

a) $i_0 = k$:

Thus $F_k(\hat{\underline{\xi}}) = 0$, which by virtue of (2.8) implies $\hat{\xi}_n = 0$. We consider two subcases.

First assume $g(\hat{\underline{\xi}}) = 0$. We adapt the earlier argument precluding this event. In fact, $g(\hat{\underline{\xi}}) = 0$ and $F_k(\hat{\underline{\xi}}) = 0$ imply that $\hat{\xi}_k = 0$. This in turn by (2.8) requires $F_{k+1}(\hat{\underline{\xi}}) = 0$. Since $g(\hat{\underline{\xi}}) = 0$, we obtain $\hat{\xi}_{k+1} = 0$. Continuing in this manner, we arrive at $\sum_{i=k}^{n} \hat{\xi}_i = 0$, an absurdity. Thus it remains to consider the possibility $0 < g(\hat{\underline{\xi}}) = \min_j \left[\max\{0, \frac{f_j(\hat{\underline{\xi}})}{e_j}\} \right]$. Since $\{e_j\}_k^n$ strictly alternate in sign we have $f_j(\hat{\underline{\xi}})(-1)^{j+k} > 0$, $j = k,\ldots,n$ and therefore

$$(2.9) \qquad S^-(f_k(\hat{\underline{\xi}}),\ldots,f_n(\hat{\underline{\xi}})) = n-k .$$

On the other hand, since $\hat{\xi}_n = 0$, $t_{n-1} = 1$ and $u(t) = P^{(k)}(t;\hat{\underline{\xi}})$ can change sign only at the points $t_k, t_{k+1},\ldots,t_{n-2}$ in $[0,1]$, we have

(2.10) $\quad S^-\{u(t):[0,1]\} \leq n-1-k$.

But, in accordance with (2.6), (note that $f_i(\hat{\underline{\xi}}) \neq 0$ implies $t_{i-k} < t_i$) ,

$$f_i(\hat{\underline{\xi}}) = \frac{1}{k!}\int_0^1 M_i(t;\hat{\underline{\xi}})\, u(t)dt \quad .$$

Applying Theorems 2.1 and 2.2 we secure the inequality

(2.11) $\quad S^-(f_k(\hat{\underline{\xi}}),\ldots,f_n(\hat{\underline{\xi}})) \leq S^-\{u(t):[0,1]\}$.

The relations (2.9) and (2.10) are obviously incompatible with (2.11). Thus case a) is impossible.

b) $\quad i_0 = k+1$; that is $F_{k+1}(\hat{\underline{\xi}}) = 0$.

The first equation of (2.8) shows that $\hat{\xi}_k = 0$. However $\hat{\xi}_k = 0$ iff $f_k(\hat{\underline{\xi}}) = 0$, as is clear from the construction of $f_k(\hat{\underline{\xi}})$. Then, we have $F_k(\hat{\underline{\xi}}) = 0$, contradicting the definition of i_0 .

c) $\quad i_0 = k+2$:

Thus $F_{k+2}(\hat{\underline{\xi}}) = 0$, and $\hat{\xi}_{k+1} = 0$ from (2.8). The fact of $\hat{\xi}_{k+1} = 0$, i.e., $t_k = t_{k+1}$, and $f_{k+1}(\hat{\underline{\xi}}) = P_{(k)}[t_1,\ldots,t_{k+1};\hat{\underline{\xi}}] = P^{(k)}(\tilde{t};\hat{\underline{\xi}})$ for some $\tilde{t} \in [0,t_{k+1}]$ entails $f_{k+1}(\hat{\underline{\xi}}) \geq 0$ (see 3). However, $e_{k+1} < 0$ and hence $F_{k+1}(\hat{\underline{\xi}}) = 0$, again contradicting the definition of i_0 .

d) $\quad i_0 \geq k+3$:

By the definition of i_0 , we have $g_j(\hat{\underline{\xi}}) > g(\hat{\underline{\xi}}) \geq 0$ and therefore $f_j(\hat{\underline{\xi}})(-1)^{j+k} > 0$ for $j = k,\ldots,i_0-1$, yielding

(2.12) $\quad S^-(f_k(\hat{\underline{\xi}}),\ldots,f_{i_0-1}(\hat{\underline{\xi}})) = i_0-k-1$.

Since $F_{i_0}(\hat{\underline{\xi}}) = \hat{\xi}_{i_0-1} = 0$, and $F_j(\hat{\underline{\xi}}) = c\hat{\xi}_{j-1} > 0$, $j = k+1,\ldots,i_0-1$, we deduce that $P^{(k)}(t;\hat{\underline{\xi}})$ exhibits i_0-k-2 changes of sign on $[0,t_{i_0-1}]$.

A contradiction now ensues as in the discussion of case a) from the fact that $t_{i_0-2} = t_{i_0-1}$ and the representation

$$(2.13)\quad f_j(\hat{\underline{\xi}}) = \frac{1}{k!}\int_0^{t_{i_0-1}} M_j(t;\hat{\underline{\xi}})P^{(k)}(t;\hat{\underline{\xi}})dt\ , \quad j=k,\ldots,i_0-1\ .$$

Thus in all cases we arrive at a contradiction and therefore there necessarily exists a $\underline{\xi}^* \in \Delta$ satisfying $\sum_{i=k}^{n} F_i(\underline{\xi}^*) = 0$ and $g(\underline{\xi}^*) > 0$. To finally prove that $0 < t_k^* < \ldots < t_{n-1}^* < 1$, we once again apply the analysis of case a). The proof of the theorem is complete.

Q.E.D.

The existence of the polynomial affirmed in Theorem A is immediate from the method of proof of the case $k = 1$, where we now define

$$f_i(\underline{\xi}) = \int_{t_{i-1}}^{t_i} P'(t;\underline{\xi})dt\ , \text{ and } g_i(\underline{\xi}) = \frac{f_i(\underline{\xi})}{e_i-e_{i-1}}\ , \ i=1,\ldots,n\ .$$

3. EXTENSIONS OF THEOREM 1 TO VARIOUS CLASSES OF SPLINE FUNCTIONS AND CHEBYSHEV SYSTEMS

a. Extended Complete Chebyshev (ECT) Systems

We say that $\{u_i(t)\}_{i=0}^n$ is an ECT-system on $[a,b]$ if

$u_i(t) \in C^n[a,b]$, and any $u(t) = \sum_{i=0}^{p} a_i u_i(t)$ admits at most p zeros on $[a,b]$, counting multiplicities, for $\{a_i\}_0^p$ real, where $\sum_{i=0}^{p} a_i^2 > 0$, for each $p = 0,1,\ldots,n$. Associated with any ECT-system is a natural sequence of first-order differential operators

$$D_j f(t) = \frac{d}{dt} \frac{f(t)}{w_j(t)} , \quad j = 0,1,\ldots,n ,$$

where the $\{w_i(t)\}_{i=0}^n$ are suitable positive functions (see [7]). Define $D^0 = I$, and $D^j = D_{j-1} \cdots D_1 D_0$, $j=1,\ldots,n+1$.

The fundamental solution $\varphi_n(t;x)$ of the differential operator D^{n+1} admits the representation

$$\varphi_n(t;x) = \begin{cases} w_0(t)\int_x^t w_1(\xi_1) \int_x^{\xi_1} w_2(\xi_2) \cdots \int_x^{\xi_{n-1}} w_n(\xi_n) d\xi_n \cdots d\xi_1 , & t \geq x \\ 0 & t < x . \end{cases}$$

The analogue of the k-th divided difference in the case of an ECT-system $\{u_i(t)\}_{i=0}^n$ is

$$f_{(k)}[y_i,\ldots,y_{i+k}] = \frac{\tilde{U}\begin{pmatrix} 0,1,\ldots,k-1,f \\ y_i,y_{i+1},\ldots,y_{i+k} \end{pmatrix}}{\tilde{U}\begin{pmatrix} 0,1,\ldots,k-1,k \\ y_i,y_{i+1},\ldots,y_{i+k} \end{pmatrix}} =$$

$$= \frac{\begin{vmatrix} u_0(y_i) & \cdots & u_0(y_{i+k}) \\ \vdots & & \vdots \\ u_{k-1}(y_i) & \cdots & u_{k-1}(y_{i+k}) \\ f(y_i) & \cdots & f(y_{i+k}) \end{vmatrix}}{\begin{vmatrix} u_0(y_i) & \cdots & u_0(y_{i+k}) \\ \vdots & & \vdots \\ u_k(y_i) & \cdots & u_k(y_{i+k}) \end{vmatrix}} .$$

We also have the representation formula

$$f_{(k)}[y_i,\ldots,y_{i+k}] = \frac{1}{k!} \int_{y_i}^{y_{i+k}} M_i(t;\underline{y}) D^k f(t)\,dt ,$$

where

$$M_i(t;\underline{y}) = \frac{\varepsilon \cdot \tilde{U}\begin{pmatrix} 0,1,\ldots,k-1,t \\ y_i,y_{i+1},\ldots,y_{i+k} \end{pmatrix}}{\tilde{U}\begin{pmatrix} 0,1,\ldots,k-1,k \\ y_i,y_{i+1},\ldots,y_{i+k} \end{pmatrix}} ,$$

$\tilde{U}\binom{j}{y_i} = u_j(y_i)$, $\tilde{U}\binom{t}{y_i} = \varphi_{k-1}(y_i;t)$, and ε is some positive constant independent of the $\{y_j\}$ and t (see [5, Chap.10, §4]).

The following is the analogue of Theorem 1.

<u>Theorem 3.1 .</u> <u>Consider any fixed integer</u> k , $1 \leq k \leq n-1$. <u>Let</u> $\{e_i\}_{i=k}^{n}$ <u>be given such that</u> $e_i e_{i+1} < 0$, $i = k,\ldots,n-1$. <u>Let</u> $\{u_i(t)\}_{i=0}^{n}$ <u>be an ECT-system on</u> $[a,b]$. <u>Then there exists a "polynomial"</u> $u(t) = \sum_{i=0}^{n} a_i u_i(t)$, <u>and points</u> $\{t_i^*\}_{i=0}^{n}$ <u>where</u>

$$t_0^* = t_1^* = \dots = t_{k-1}^* = a < t_k^* < \dots < t_{n-1}^* < t_n^* = b$$

such that

i) $D^k u(t_i^*) = 0$, $i = k,\dots,n-1$

ii) $u_{(k)}[t_{i-k}^*,\dots,t_i^*] = e_i$, $i = k,\dots,n$.

b. Splines with Fixed Knots

A spline of degree n with r fixed knots $\{\xi_i\}_{i=1}^r$, where $0 < \xi_1 < \dots < \xi_r < 1$, is a function of the form $S(t) = \sum_{i=0}^n a_i t^i + \sum_{i=1}^r c_i (t-\xi_i)_+^n$. The following version of Theorem 1 persists for splines.

Theorem 3.2 . Given $\{e_i\}_{i=k}^{n+r}$ such that $e_i e_{i+1} < 0$, $i = k,\dots,n+r-1$, and knots $\{\xi_i\}_{i=1}^r$, $0<\xi_1<\dots<\xi_r<1$, then there exists a spline $S(t) = \sum_{i=0}^n a_i t^i + \sum_{i=1}^r c_i (t-\xi_i)_+^n$, and points $\{t_i^*\}_{i=0}^{n+r}$, where $t_0^* = \dots = t_{k-1}^* = 0 < t_k^* < \dots < t_{n+r-1}^* < t_{n+r}^* = 1$ for which

i) $S^{(k)}(t_i^*) = 0$, $i = k,\dots,n+r-1$,

ii) $S_{(k)}[t_{i-k}^*,\dots,t_i^*] = e_i$, $i = k,\dots,n+r$,

for each specific $k = 1,\dots,n-1$.

Proof. The $n+r+1$ functions $\{1,t,\dots,t^n,(t-\xi_1)_+^n,\dots,(t-\xi_r)_+^n\}$ are linearly independent and constitute a weak complete Chebyshev system on $[0,1]$ (see e.g., [5, Chap.10]). Implementing a standard perturbation technique (consult [5,p.103]), there exists a spline $S(t)$ and points $\{t_i^*\}_{i=0}^{n+r}$ such that

i) and ii) of Theorem 3.2 hold. Because of the smoothing procedure, we may a-priori only conclude that

$$t_0^* = \dots = t_{k-1}^* = 0 \leqslant t_k^* \leqslant \dots \leqslant t_{n+r-1}^* \leqslant 1 .$$

To corroborate the strictness $0 < t_k^* < \dots < t_{n+r-1}^* < t_{n+r}^* = 1$, we again invoke the more detailed methodology of Theorem 1. Indeed, since $e_i e_{i+1} < 0$, $i = k,\dots,n+r-1$, $S^-(e_k,\dots,e_{n+r}) = n+r-k$. Furthermore, $s^{(k)}(t)$ has at most $n+r-k$ sign changes on $[0,1]$. (Note that since $s_{(k)}[t_{i-k}^*,\dots,t_i^*] = e_i \neq 0$, $i = k,\dots,n+r$, no sequence exceeding k consecutive t^*'s coalesce.) However, if $t_{i-1}^* = t_i^*$ for some $i = k,\dots,n+r$, then by the construction of $s(t)$, we find that $s^{(k)}(t)$ has fewer than $n+r-k$ sign changes on $[0,1]$. In view of the representation

$$s_{(k)}[t_i^*,\dots,t_{i+k}^*] = \frac{1}{k!}\int_0^1 M_i(t)s^{(k)}(t)dt$$

coupled to the total positivity nature of $M_i(t)$ (with reference to Theorem 2.1), the desired result clearly follows.

c. <u>Generalized Splines with Fixed Knots</u>

Let $\{u_i(t)\}_{i=0}^n$ be an ECT-system on $[a,b]$ and $\varphi_n(t,\xi)$ be as defined in part a) of this section. Let $\{\xi_i\}_{i=1}^r$ be r fixed points satisfying $a < \xi_1 < \dots < \xi_r < b$. Then a "generalized spline" is a function of the form

$$s(t) = \sum_{i=0}^{n} a_i u_i(t) + \sum_{i=1}^{r} b_i \varphi_n(t,\xi_i) .$$

The corresponding result to Theorem 3.2 holds for the case of generalized splines.

d. Monosplines

A monospline with r free knots on $[0,1]$ is of the form

$$M(t) = \frac{t^n}{n} - \sum_{i=0}^{n-1} a_i t^i - \sum_{i=1}^{\ell} \sum_{j=0}^{\mu_i} c_{ij}(t-\xi_i)_+^{n-1-j} ,$$

where $0 \leqslant \mu_i \leqslant n-1$, and $\sum_{i=1}^{\ell} (\mu_i+1) \leqslant r$.

Recall the following result, see [8] and references therein. Given any $n+2r$ points on $[0,1]$ (with no block exceeding n coincident values), there exists a unique monospline which vanishes exactly at the prescribed points. For this monospline, we have $r = \ell$, $\mu_i = 0$, $i = 1,\ldots,\ell$, and $c_i > 0$, $i = 1,\ldots,r$. As in the case of splines, there arise technical problems in the concept of zeros of large multiplicity requiring modifications in carrying out the proof of Theorem 1. We again implement a standard smoothing procedure. We do not enter into details.

The final theorem reads as follows.

Theorem 3.3 . Let k be an integer, $1 \leqslant k \leqslant n-2$. Given $\{e_i\}_{i=k}^{n+2r}$ such that $e_i e_{i+1} < 0$, $i = k,\ldots,n+2r-1$, then there exist $\lambda \neq 0$ and $\{t_i^*\}_{i=0}^{n+2r}$ satisfying

$$t_0^* = t_1^* = \cdots = t_{k-1}^* = 0 < t_k^* < \cdots < t_{n+2r-1}^* < t_{n+2r}^* = 1 ,$$

and a monospline $M(t) = \frac{t^n}{n} - \sum_{i=0}^{n-1} a_i^* t^i - \sum_{i=1}^{r} c_i^*(t-\xi_i^*)_+^{n-1}$

where $c_i^* > 0$, $i = 1,\ldots,r$, and $0 < \xi_1^* < \cdots < \xi_r^* < 1$ satisfying

1) $M^{(k)}(t_i^*) = 0$, $i = k,\ldots,n+2r-1$

2) $M_{(k)}[t_{i-k}^*,\ldots,t_i^*] = \lambda e_i$, $i = k,\ldots,n+2r$.

e. Perfect Splines

A perfect spline in $[0,1]$ is a function of the form:

$$p(t) = \sum_{i=0}^{n-1} a_i t^i + c[t^n + 2 \sum_{i=1}^{r} (-1)^i (t-\xi_i)_+^n] ,$$

where $0 < \xi_1 < \ldots < \xi_r < 1$.

Given $n+r$ points (encompassing no more than n equal values), then there exists a unique non-trivial perfect spline (up to a multiplicative constant) vanishing at these points (e.g., see [6]).

The analogue of a perfect spline for an extended totally positive kernel $K(t,\xi)$ on $[0,1] \times [0,1]$ has the structure

$$(3.1) \quad h(t) = \sum_{i=0}^{n-1} b_i \frac{\partial^i K(t,0)}{\partial \xi^i} + c \sum_{i=1}^{r+1} (-1)^{i+1} \int_{\xi_{i-1}}^{\xi_i} K(t,\xi)d\xi .$$

This is the smoothed version of an ordinary perfect spline. There exist unique $\{\xi_i\}_{i=0}^{r+1}$, $0=\xi_0<\xi_1<\ldots<\xi_r<\xi_{r+1}=1$, $\{b_i\}_{i=0}^{n-1}$ and c such that $h(t)$ of the above form (up to a multiplicative constant) vanishes at any $n+r$ prescribed points. The proof of Theorem 3.4 (stated in §1) proceeds as in Theorem 1 proving the result first in the case (3.1).

4. A GENERAL FORMULATION AND APPLICATIONS

We pinpoint some of the key ingredients in the analysis of Theorem 1 for purposes of enunciating a more general result bearing some further interesting applications.

Let $\underline{x} = (x_1,\ldots,x_n)$ be an n-tuple of real numbers with $0 = x_0 \leq x_1 \leq \ldots \leq x_n \leq x_{n+1} = 1$. Associated with $\underline{x}$ we have $\{L_i(t,\underline{x})\}_{i=1}^{n+1}$, and a family of functions $\mathcal{F}$ possessing the following properties.

1) For each $\underline{x}$ there exists a unique function $F(t,\underline{x}) \in \mathcal{F}$ continuous for $t \in [0,1]$, except possibly at the $\{x_i\}_0^{n+1}$, changing sign at the x_j, $j = 1,\ldots,n$, and only there, and not vanishing identically in (x_i, x_{i+1}) if $x_i < x_{i+1}$. (For the case of coincidences among the x_i, the appropriate definition applies.)

2) $$\lim_{(x_i - x_{i-1}) \to 0} \int_{x_{i-1}}^{x_i} L_j(t,\underline{x}) F(t,\underline{x})\,dt = 0 \quad \text{for each}$$

$j = 1,\ldots,n+1$, and $\left| \int_{x_{i-1}}^{x_i} L_j(t,\underline{x}) F(t,\underline{x})\,dt \right| \leq C$ for some constant C independent of $\underline{x}$ and $1 \leq j \leq n+1$.

3) $f_i(\underline{x}) = \int_0^{x_i} L_i(t,\underline{x}) F(t,\underline{x})\,dt$ is a continuous function of $\underline{x}$.

4) $\int_{x_{i-1}}^{x_i} L_i(t,\underline{x})\,dt > 0$ if $x_{i-1} < x_i$, $i = 1,\ldots,n+1$.

5) $L_i(t,\underline{x}) = 0$ for $t > x_i$.

6) $L_i(t,\underline{x})$ is totally positive (TP) on $I \times T$, where $I = \{1,2,\ldots,n+1\}$ and $T = [0,1]$.

In the above framework, the analysis of Theorem 1 extends without difficulty to establish

Theorem 4.1. *Under the above assumptions, and given* $\{e_i\}_{i=1}^{n+1}$, *strictly oscillating in the manner that* $e_i e_{i+1} < 0$, $i = 1,\ldots,n$, *there exists* $\lambda^* \neq 0$, *and* $\underline{x}^*$

satisfying $x_0^* = 0 < x_1^* < \ldots < x_n^* < x_{n+1}^* = 1$ endowed with the interpolation characteristics

$$\lambda^* e_i = \int_0^{x_i^*} L_i(t,\underline{x}^*)F(t,\underline{x}^*)dt \ , \quad i = 1,\ldots,n+1 \ .$$

This theorem can be regarded as a generalization of a result in [3, Theorem 8]. A number of additional applications are now easily forthcoming.

a) Theorem 4.2. With each k, $1 \leq k \leq n-1$, and prescribed $\{e_i\}_{i=k}^n$, $e_i e_{i+1} < 0$, $i = k,\ldots,n-1$, and ℓ, $0 \leq \ell < k$, there exists a polynomial $P(t)$ of degree n, and $\{t_i^*\}_{i=0}^n$, where $0 = t_0^* = \ldots = t_{k-1}^* < t_k^* < \ldots < t_{n-1}^* < t_n^* = 1$ having the properties

1) $P^{(i)}(0) = 0$, $i = \ell,\ldots,k-1$,

2) $P^{(k)}(t_i^*) = 0$, $i = k,\ldots,n-1$,

3) $P_{(\ell)}[t_{i-\ell}^*,\ldots,t_i^*] = e_i$, $i = k,\ldots,n$.

Remark. Theorem 2 (stated in the introduction) is the case $\ell = 0$.

Proof. We adhere to the notation introduced in the proof of Theorem 1. Recall from (2.6) that

$$(4.1) \quad P_{(\ell)}[t_{i-\ell},\ldots,t_i;\underline{\xi}] = \frac{1}{\ell!}\int_{t_{i-\ell}}^{t_i} {}_{\ell}M_i(t;\underline{\xi})P^{(\ell)}(t;\underline{\xi})dt$$

where ${}_{\ell}M_i(t;\underline{\xi})$ is the basic (B) spline (2.4) involving $\ell+1$ consecutive abscissa. Taylor's formula gives

$$(4.2) \quad P^{(\ell)}(t;\underline{\xi}) = \frac{1}{(k-\ell-1)!}\int_0^1 (t-x)_+^{k-\ell-1} P^{(k)}(x;\underline{\xi})dx$$

since $P^{(i)}(0;\underline{\xi}) = 0$, $i = \ell,\ldots,k-1$.

Inserting (4.2) in (4.1) and rearranging produces

$$P_{(\ell)}[t_{i-\ell},\ldots,t_i;\underline{\xi}]$$

$$= \frac{1}{\ell!(k-\ell-1)!}\int_0^{t_i} {}_\ell M_i(t;\underline{\xi})\left(\int_0^{t_i}(t-x)_+^{k-\ell-1} P^{(k)}(x;\underline{\xi})dx\right)dt$$

$$= \frac{1}{\ell!(k-\ell-1)!}\int_0^{t_i}\left(\int_0^{t_i} {}_\ell M_i(t;\underline{\xi})(t-x)_+^{k-\ell-1}dt\right) P^{(k)}(x;\underline{\xi})dx .$$

Define $L_i(x,\underline{\xi}) = \int_0^{t_i} {}_\ell M_i(t;\underline{\xi})(t-x)_+^{k-\ell-1}dt$. Properties 1), 2) and 3) are easily checked. Manifestly $\int_{t_{i-1}}^{t_i} L_i(x,\underline{\xi})dx > 0$ if $t_{i-1} < t_i$, and $L_i(x,\underline{\xi}) \equiv 0$ if $x > t_i$. The total positivity nature of $L_i(x,\underline{\xi})$ emanates from the basic composition formula [5, p.98], and the theorem is proven.

b) We next consider the situation of spline interpolation at their knots, with fixed coefficients but variable knots. We discuss this in the framework of Theorem 4.1.

<u>Proposition 4.1.</u> <u>Given</u> $\{e_i\}_{i=1}^{r+1}$, $e_ie_{i+1} < 0$, $i=1,\ldots,r$, <u>and</u> $\{d_i\}_{i=0}^{r}$ <u>such that</u> $(\sum_{j=0}^{i} d_j)(-1)^i > 0$, $i = 0,1,\ldots,r$, <u>then there exist</u> $\lambda^* \neq 0$ <u>and nodes</u> $\{\xi_i^*\}_{i=1}^{r}$ <u>satisfying</u> $0 < \xi_1^* < \ldots < \xi_r^* < \xi_{r+1}^* = 1$ <u>such that the function</u>

$$S(x) = d_0x^n + \sum_{i=1}^{r} d_i(x-\xi_i^*)_+^n$$

<u>satisfies</u> $S(\xi_i^*) = \lambda^* e_i$, $i = 1,\ldots,r+1$.

Proof. Let $L_i(t,\underline{\xi}) = (\xi_i - t)_+^{n-1}$, $i = 1,\dots,r+1$, and $F(t,\underline{\xi}) = \sum_{i=0}^{j-1} d_i$ on (ξ_{j-1},ξ_j) , $j = 1,\dots,r+1$, where $\xi_0 = 0$, $\xi_{r+1} = 1$. Since $\int_{\xi_{j-1}}^{\xi_j} (\xi_i - t)_+^{n-1}\, dt =$

$$= -\frac{(\xi_i - \xi_j)_+^n}{n} + \frac{(\xi_i - \xi_{j-1})_+^n}{n}$$

, the result follows.

Another proof of Proposition 4.1 is available by inducting on the number of knots.

5. OPEN PROBLEMS

I. The major unresolved problem for Theorem 1 and its extensions is that of uniqueness. There is some evidence pointing to uniqueness. Uniqueness is known for Theorem A, but this is indeed the simplest case.

II. In Theorem 1 and its extensions of Section 3, we specified $t_0 = t_1 = \dots = t_{k-1} = 0$, and $t_n = 1$. We do not know whether the result of Theorem 1 persists if ℓ $(1 \leq \ell \leq k)$ and $k+1-\ell$ points are prescribed at 0 and 1 , respectively. For the case $k = 3$ (the first non-trivial case), we shall show that Theorem 1 holds where we specify 2 points at 0 and 2 points at 1 .

Theorem 5.1 . *Given* $\{e_i\}_{i=2}^{n-1}$ *such that* $e_i(-1)^i > 0$, $i = 2,\dots,n-1$, *then there exists a polynomial* $P(t)$ *of degree* n , *and* $\{t_i^*\}_{i=0}^{n}$ *of the form*

$$0 = t_0^* = t_1^* < t_2^* < \dots < t_{n-2}^* < t_{n-1}^* = t_n^* = 1$$

such that

1) $P^{(3)}(t_i^*) = 0$, $i = 2,\ldots,n-2$

2) $P_{(3)}[t_{i-2}^*, t_{i-1}^*, t_i^*, t_{i+1}^*] = e_i$, $i = 2,\ldots,n-1$.

Proof. Let $\Delta = \{\underline{\xi} : \underline{\xi} = (\xi_2,\ldots,\xi_{n-1})$, $\xi_i \geq 0$, $\sum_{i=2}^{n-1} \xi_i = 1\}$.

Associated with $\underline{\xi} \in \Delta$ we define $t_0 = t_1 = 0$, $t_{n-1} = t_n = 1$, and $t_i = \sum_{j=2}^{i} \xi_j$, $i = 2,\ldots,n-2$. With each $\underline{\xi} \in \Delta$, we determine the unique polynomial $P(t;\underline{\xi})$ of degree n satisfying

1) $P(0;\underline{\xi}) = P'(0;\underline{\xi}) = P''(0;\underline{\xi}) = 0$;

2) $P^{(3)}(t_i;\underline{\xi}) = 0$, $i = 2,\ldots,n-2$;

3) $P^{(3)}(t;\underline{\xi}) > 0$, $t < t_2$;

4) sum of squares of the coefficients of $P(t;\underline{\xi})$ is one.

Define $f_i(\underline{\xi}) = P_{(3)}[t_{i-2},t_{i-1},t_i,t_{i+1};\underline{\xi}]$, $i=2,\ldots,n-1$,

$$g_i(\underline{\xi}) = \frac{f_i(\underline{\xi})}{e_i} , \quad i = 2,\ldots,n-1 , \quad g(\underline{\xi}) = \max_{i=2,\ldots,n-1} g_i(\underline{\xi}) ,$$

and $F_i(\underline{\xi}) = g(\underline{\xi}) - g_i(\underline{\xi})$, $i = 2,\ldots,n-1$.

We claim that $g(\underline{\xi}) \geq 0$ for all $\underline{\xi} \in \Delta$. Assume to the contrary that $g(\underline{\xi}) < 0$ for some $\underline{\xi} \in \Delta$. Since the e_i strictly alternate, we find that $S^-(f_2(\underline{\xi}),\ldots,f_{n-1}(\underline{\xi})) = n-3$, while $P^{(3)}(t;\underline{\xi})$ has at most $n-3$ sign changes on $[0,1]$. On account of Theorem 2.1 and these facts, we see that $P^{(3)}(t;\underline{\xi})$ exhibits exactly $n-3$ sign changes. However, the orientation of the signs in $\{f_2(\underline{\xi}),\ldots,f_{n-1}(\underline{\xi})\}$ and $P^{(3)}(t;\underline{\xi})$ are reversed in violation of the last statement of Theorem 2.1. Thus, $g(\underline{\xi}) \geq 0$ must hold for all $\underline{\xi} \in \Delta$ as claimed.

Next, suppose there exists a $\underline{\xi}^* \in \Delta$ for which $\sum_{i=2}^{n-1} F_i(\underline{\xi}^*) = 0$. In this circumstance we assert $g(\underline{\xi}^*) > 0$. Suppose $g(\underline{\xi}^*) = 0$. Because each $F_i(\underline{\xi}^*) = 0$, also $f_i(\underline{\xi}^*) = 0$ is present, $i = 2,\ldots,n-1$.

Now $f_i(\underline{\xi}^*) = P_{(3)}[t_{i-2},t_{i-1},t_i,t_{i+1};\underline{\xi}^*]$, $i = 2,\ldots,n-1$

$$= P^{(3)}(\tilde{t};\underline{\xi}^*) \ , \quad \text{for some} \quad \tilde{t} \in [t_{i-2},t_{i+1}] \ .$$

Moreover in $[t_{i-2},t_{i+1}]$, $P^{(3)}(\tilde{t};\underline{\xi}^*)$ has the sign $(-1)^i$ only on (t_{i-1},t_i) . Therefore it follows that either $t_0 = t_1 = 0 < t_2 < \ldots < t_{n-2} < t_{n-1} = t_n = 1$, or $\sum_{i=2}^{n-1} \xi_i = 0$. Since the second alternative is untenable, $t_0 = t_1 = 0 < t_2 < \ldots < t_{n-2} < t_{n-1} = t_n = 1$ holds. However, it is fairly direct to check that $f_i(\underline{\xi}^*) = 0$, $i = 2,\ldots,n-1$, is impossible, by using the precise total positivity properties of the kernel $M_i(t)$.

Thus it remains to prove the existence of a $\underline{\xi}^* \in \Delta$ for which $\sum_{i=2}^{n-1} F_i(\underline{\xi}^*) = 0$. (The requirement $0 = t_0 = t_1 = t_2 < \ldots < t_{n-2} < t_{n-1} = t_n = 1$ is verified as in the proof of Theorem 1.)

Suppose $\sum_{i=2}^{n-1} F_i(\underline{\xi}) > 0$ for all $\underline{\xi} \in \Delta$, and consider the mapping

$$\xi_i' = \frac{F_i(\underline{\xi})}{\sum_{j=2}^{n-1} F_j(\underline{\xi})} \ , \quad i = 2,\ldots,n-1 \ .$$

A contradiction results as in Theorem 1.

Q.E.D.

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