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**CONSTRUCTIVE APPROXIMATION**


# Descartes Systems from Corner Cutting

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**Abstract.** This paper demonstrates that Descartes Systems can be conveniently generated from matrix subdivision algorithms determined by totally positive matrices.

## 1. Introduction

A frequent paradigm in computer graphics is the representation of a curve by means of *control points* and, therefore, the association of the curve with a *control polygon* obtained by joining control points with linear segments. Mathematically, this means a curve representation is specified by scalar-valued blending functions $\psi_1(t), \ldots, \psi_n(t)$ through the formula

$$\Psi(t|\mathbf{c}) := \sum_{i=1}^{n} \mathbf{c}_i \psi_i(t) := (\mathbf{c}, \Psi(t)),$$

where

$$\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_n), \qquad \Psi(t) := (\psi_1(t), \ldots, \psi_n(t)).$$

Here $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are vectors (control points) in some $s$-dimensional linear space, say $\mathbf{R}^s$. The control polygon is then determined by the composite vector $\mathbf{c} \in \mathbf{R}^{sn}$ and we can think of $\mathbf{c}$ geometrically as a polygonal line.

Various algorithms for the manipulation and computation of such curves take the form of successive geometric alterations of the control polygon. In particular, in the case of the Bernstein bases

$$\psi_i^b(t) := \binom{m}{i} t^i (1-t)^{m-i}, \qquad i = 0, 1, \ldots, m,$$

algorithms for evaluating $\Psi^b(t|\mathbf{c})$ either by subdivision or degree elevation or passing from a B-spline representation of a polynomial curve segment to its Bernstein form falls into this category. The common feature shared by these

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algorithms is that the new control polygons are formed by successively replacing two adjacent control points by their convex combinations. We call this *corner cutting* because of its apparent geometric interpretation. A particularly striking example of corner cutting is the method of de Casteljau which gives us a direction for much of what follows in this paper. The de Casteljau algorithm begins with an initial control polygon $\mathbf{c}^0 = (\mathbf{c}_0^0, \dots, \mathbf{c}_m^0)$ and then forms successive averages

$$\mathbf{c}_r^l = \tfrac{1}{2}(\mathbf{c}_r^{l-1} + \mathbf{c}_{r+1}^{l-1}), \qquad r = 0, \dots, m-l, \quad l = 1, \dots, m. \tag{1.1}$$

There are two facts about this recursion which are the subject of generalization here. To explain them we display the de Casteljau points in a triangular array

$$\begin{array}{llllll} \mathbf{c}_0^0 & \cdot & \cdot & \cdot & \cdot & \mathbf{c}_m^0 \\ \mathbf{c}_0^1 & \cdot & \cdot & \cdot & \mathbf{c}_{m-1}^1 & \\ \cdot & & & \cdot & & \\ \cdot & & \cdot & & & \\ \cdot & \cdot & & & & \\ \mathbf{c}_0^m & & & & & \end{array} \tag{1.2}$$

and recall that the lower vertex of the triangle produces the value of the curve at $t = \frac{1}{2}$. Thus

$$\mathbf{c}_0^m = \Psi^b(\tfrac{1}{2}|\mathbf{c}), \qquad \Psi^b(t|\mathbf{c}) := \sum_{j=0}^{m} \mathbf{c}_j^0 \psi_j^b(t). \tag{1.3}$$

Secondly, the sides of the triangle (vertical and diagonal) give a *refined representation* of the curve on the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively. Specifically, we have

$$\Psi^b(t|\mathbf{c}) = \sum_{j=0}^{m} \mathbf{c}_0^j \Psi_j^b(2t), \qquad 0 \le t \le \tfrac{1}{2}, \tag{1.4}$$

and

$$\Psi^b(t|\mathbf{c}) = \sum_{j=0}^{m} \mathbf{c}_j^{m-j} \Psi_j^b(2t-1), \qquad \tfrac{1}{2} \le t \le 1. \tag{1.5}$$

These last relations are the bases of a subdivision scheme for the computation of the *whole* curve $\Psi^b(t|\mathbf{c})$. To explain this we focus on two $m+1 \times m+1$ matrices defined by the equations

$$A_0^b \mathbf{c}^0 := (\mathbf{c}_0^0, \dots, \mathbf{c}_0^m)$$

and

$$A_1^b \mathbf{c}^0 := (\mathbf{c}_0^m, \dots, \mathbf{c}_m^0).$$

These matrices are lower and upper triangular, respectively. They are given explicitly as

$$(A_0^b)_{ij} = 2^{-i}\binom{i}{j}, \qquad i, j = 0, 1, \dots, m, \tag{1.6}$$

where

$$\binom{i}{j} = 0 \qquad \text{if} \quad j > i,$$

and

$$(A_1^b)_{ij} = (A_0^b)_{m-i,m-j}, \qquad i, j = 0, 1, \ldots, m. \tag{1.7}$$

A useful reformulation of the refinement equation (1.4)–(1.5) for the Bernstein representation is the *functional equation* satisfied by the curve

$$\Psi^b(t) := (\Psi_0^b(t), \ldots, \Psi_m^b(t))$$

given by

$$\Psi^b\left(\frac{t+\varepsilon}{2}\right) = (A_\varepsilon^b)^T \Psi^b(t), \qquad 0 \le t \le 1, \quad \varepsilon \in \{0, 1\}.$$

This functional equation was the subject of recent generalization [10]. The idea there was to replace the control polygon $\mathbf{c}^0$ by a new control polygon $\mathbf{c}^1$ determined by the application of two matrices $A_\varepsilon$, $\varepsilon \in \{0, 1\}$, to $\mathbf{c}^0$. Thus $\mathbf{c}^1 = (A_0\mathbf{c}^0, A_1\mathbf{c}^0) = A\mathbf{c}^0$, where

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}$$

and $A_0\mathbf{c}^0$, $A_1\mathbf{c}^0$ are thought to "control" the curve associated with $\mathbf{c}^0$ on the segments $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, respectively. Iterating this procedure leads us to the following subdivision scheme. Suppose $A_0$, $A_1$ are two matrices such that any sequence of products of $A_0$ and $A_1$ applied to any vector converges to a multiple of the vector $\mathbf{e} := (1, 1, \ldots, 1)$ assumed to satisfy $A_\varepsilon \mathbf{e} = \mathbf{e}$, $\varepsilon \in \{0, 1\}$. Then necessarily $A_\varepsilon^T$ has a unique eigenvector $\mathbf{f}_\varepsilon$ normalized so that $(\mathbf{f}_\varepsilon, \mathbf{e}) = 1$. If $A_0^T\mathbf{f}_1 = A_1^T\mathbf{f}_0$ (compatibility relation), then we can unambiguously define a (fundamental) curve $\Psi\colon [0, 1] \to \mathbf{R}^n$ by the formula

$$\lim_{k\to\infty} A_{\varepsilon_k} \cdots A_{\varepsilon_1}\mathbf{c} = \Psi(t|\mathbf{c})\mathbf{e}, \qquad t = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}, \qquad \Psi(t|\mathbf{c}) = (\mathbf{c}, \Psi(t)), \tag{1.8}$$

see [10]. It should be emphasized here that this Matrix Subdivision Scheme (MSS), although motivated by corner cutting as is indeed de Casteljau's algorithm, is itself generally not a corner-cutting procedure.

The characterization of $n \times n$ matrices $A_0$, $A_1$ which admit an MSS, in the sense that there is a continuous curve $\Psi$ satisfying (1.8), remains an open problem. However, when the $A_\varepsilon$, $\varepsilon \in \{0, 1\}$, are *stochastic* necessary and sufficient conditions on $A_0$, $A_1$ are available. (Here and throughout this paper, a matrix $B$ is said to be stochastic if it has nonnegative entries and row sums one.) For our purposes it is convenient to state a simple sufficient condition on two stochastic matrices to admit an MSS, as it serves as a starting point for the observations we make here.

**Theorem 1.1.** *Let $A_0, A_1$ be stochastic matrices each with a positive column. Suppose $\mathbf{f}_0, \mathbf{f}_1$ are the (necessarily unique) eigenvectors of $A_0^T, A_1^T$ corresponding to* *eigenvalue one normalized so that* $(\mathbf{e}, \mathbf{f}_0) = (\mathbf{e}, \mathbf{f}_1) = 1$. *If* $A_1^T\mathbf{f}_0 = A_0^T\mathbf{f}_1$, *then the functional equation*

$$\Psi\left(\frac{t+\varepsilon}{2}\right) = A_\varepsilon^T\Psi(t), \qquad 0 \le t \le 1, \quad \varepsilon \in \{0, 1\}, \tag{1.9}$$

*has a unique continuous solution satisfying* $(\mathbf{e}, \Psi(t)) = 1, 0 \le t \le 1$. *Moreover, it is generated by the subdivision scheme*

$$\lim_{k\to\infty} A_{\varepsilon_k} \cdots A_{\varepsilon_1}\mathbf{c} = (\mathbf{c}, \Psi(t))\mathbf{e}, \qquad t = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}. \tag{1.10}$$

*Also, as a consequence*

$$\lim_{k\to\infty} A_{\varepsilon_1}^T \cdots A_{\varepsilon_k}^T\Psi(x) = \Psi(t) \qquad \textit{for any} \quad x \in [0, 1], \qquad t = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}. \tag{1.11}$$

More can be said about the limiting curve $\Psi$, in particular its smoothness and the surprising relationship of this question to the existence of polynomial components in $\Psi$ [10]. Our intention in the paper is to study features of the fundamental curve $\Psi$ which are motivated by certain properties of the Bernstein-Bézier curve $\Psi^b$.

It was observed quite awhile ago by I. J. Schoenberg that the Bernstein polynomial bases have the property that they are variation diminishing on (0, 1), in the strong sense that

$$Z(\Psi^b(\cdot|\mathbf{c})) \le S^-(\mathbf{c}), \qquad \mathbf{c} \in \mathbf{R}^{m+1}. \tag{1.12}$$

Here $Z(f)$ counts the number of zeros of $f$ on (0, 1) counting multiplicities and $S^-(\mathbf{c})$ is the number of sign changes in the components of the vector $\mathbf{c} = (c_0, \ldots, c_m)$, where zero entries are discarded. The proof of this fact is elementary and can be based on Descartes' rule of signs. To see this we write

$$\Psi^b(t|\mathbf{c}) = (1-t)^m \sum_{j=0}^{m} c_j \binom{m}{j}\left(\frac{t}{1-t}\right)^j$$

so that (1.12) follows from Descartes' rule of signs

$$Z\left(\sum_{j=0}^{m} a_j t^j\right)\bigg|_{(0,\infty)} \le S^-(a_0, \ldots, a_m).$$

This property of the Bernstein polynomials has a more or less equivalent form in certain determinantal inequalities, namely

$$\Psi^b\begin{pmatrix} i_1, \ldots, i_s \\ x_1, \ldots, x_s \end{pmatrix} := \det_{l,j=1,\ldots,s} (\Psi_{i_l}^b(x_j)) \ge 0 \tag{1.13}$$

for $0 \le i_1 < \cdots < i_s \le m, 0 \le x_1 < \cdots < x_s \le 1$. As we shall soon see, equality in (1.13) holds if and only if $x_1 = 0$ and $i_1 > 0$ or $x_s = 1$ and $i_s < m$.

The actual relationship between determinantal inequalities and the strong variation diminishing property is that for any continuous curve $\Psi : [0, 1] \to \mathbf{R}^m$

$$\Psi\begin{pmatrix} i_1, \ldots, i_s \\ x_1, \ldots, x_s \end{pmatrix} > 0,$$

for all $1 \le i_1 < \cdots < i_s \le m$, $0 < x_1 < \cdots < x_s < 1$, and all $s$ if and only if

$$Z(\Psi(\cdot|\mathbf{c})) \le S^-(\mathbf{c})$$

(here $Z(f)$ counts only simple zeros of $f$ on $(0, 1)$) and whenever $Z(\Psi(\cdot|\mathbf{c})) = S^-(\mathbf{c})$ then the sign of $\Psi(t|\mathbf{c})$ for $t$ near zero is the same as the sign of the first nonzero component of $\mathbf{c}$ [7, p. 223]. The inequalities (1.13) for the Bernstein curve says more in that the exact criteria for strict equality on $[0, 1]$ is available.

The question arises as to whether or not there are other triangular arrays (1.2) with associated fundamental curve $\Psi$ which satisfy all these three properties, (1.3), (1.4), (1.5), and (1.13). We will show, in contrast to the observation in [2], that there is a wide class of curves having these properties.

Our analysis of this question focuses on the $2n \times n$ matrix

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}.$$

We will show that the essential property is that $A$ is *totally positive* (TP), that is, all its minors are nonnegative, and both $A_0$ and $A_1$ are nonsingular. The fact that these properties hold for the Bernstein polynomials follows from the factorization of

$$A^b = \begin{bmatrix} A_0^b \\ A_1^b \end{bmatrix}$$

implied by the de Casteljau's procedure. Specifically, $A^b$ can be factored as a product of one-banded matrices with nonnegative elements. Since each one-banded factor is easily seen to be totally positive, by the Cauchy-Binet formula [7], so too is the matrix $A^b$.

We now turn to some properties of the curve $\Psi$ of Theorem 1.1 when $A$ is TP.

## 2. Descartes Systems from Subdivision

This section contains a proof of the following theorem. Its geometric interpretation as a corner-cutting algorithm is discussed in Section 3.

**Theorem 2.1.** *Let $A_0$, $A_1$ be nonsingular $n \times n$ stochastic matrices such that*

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}$$

*is TP. Suppose further that the first row of $A_0$ is $(1, 0, \ldots, 0)$, the last row of $A_1$ is $(0, \ldots, 0, 1)$, and the last row of $A_0$ and the first row of $A_1$ are the same. Then there exists a unique continuous solution $\Psi$: $[0, 1] \to \mathbf{R}^n$ to the functional equation*

$$\Psi\left(\frac{t + \varepsilon}{2}\right) = A_\varepsilon^T \Psi(t), \qquad 0 \le t \le 1, \quad \varepsilon \in \{0, 1\},$$

*satisfying* $(\mathbf{e}, \Psi(t)) = 1$. *Furthermore*, $\Psi$ *is constructed as*

$$\lim_{k\to\infty} A_{\varepsilon_k} \cdots A_{\varepsilon_1}\mathbf{c} = (\mathbf{c}, \Psi(t))\mathbf{e}, \qquad t = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k},$$

*and moreover*

$$\Psi\begin{pmatrix} i_1, \ldots, i_s \\ x_1, \ldots, x_s \end{pmatrix} \geq 0$$

*for* $1 \leq i_1 < \cdots < i_s \leq n$, $0 \leq x_1 < \cdots < x_s \leq 1$, *where equality holds if and only if either* $x_1 = 0$, $i_1 > 1$, *or* $x_s = 1$, $i_s < n$.

We present the proof of this result in a series of observations which contain further useful information about MSS when

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}$$

is TP. We begin with some necessary facts about TP matrices and related matters.

**Lemma 2.1.** *Let $A$ be a nonnegative $n \times n$ matrix such that $A_{ij} > 0$ for $i \leq j$ and suppose $\mathbf{x}$ is an eigenvector with nonnegative components corresponding to the largest eigenvalue $\lambda_0$ of $A$. If $x_k = 0$, then $x_l = 0$ for all $l \geq k$.*

**Proof.** Since

$$0 = \lambda_0 x_k = \sum_{j=1}^{n} A_{kj} x_j$$

we get $A_{kj}x_j = 0$ and so $x_j = 0$ for $j \geq k$. ■

*Remark 2.1.* Similarly, if $A_{ij} > 0$ for $i \geq j$ and $x_k = 0$, then $x_l = 0$ for all $l \leq k$.

**Lemma 2.2.** *Let $A_0, A_1$ be nonsingular $n \times n$ stochastic matrices such that*

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}$$

*is totally positive. Then $A_0^T, A_1^T$ have unique eigenvectors $\mathbf{x}^0, \mathbf{x}^1$, corresponding to eigenvalue one normalized to satisfy $(\mathbf{e}, \mathbf{x}^0) = (\mathbf{e}, \mathbf{x}^1) = 1$, respectively, and $(A_0)_{ij}(A_1)_{ji} > 0$ for $j \leq i$.*

**Proof.** Since $A_0, A_1$ are nonsingular and totally positive their principal minors are nonsingular, i.e.,

$$A_0\begin{pmatrix} i_1, \ldots, i_s \\ i_1, \ldots, i_s \end{pmatrix}, A_1\begin{pmatrix} j_1, \ldots, j_s \\ j_1, \ldots, j_s \end{pmatrix} > 0,$$

see p. 89 of [7]. Thus, in particular, the diagonal elements of $A_0$ and $A_1$ are positive. Consequently, since for $1 \le j \le i \le n$

$$0 \le A\begin{pmatrix} i & j+n \\ j & i \end{pmatrix} = \begin{vmatrix} (A_0)_{ij} & (A_0)_{ii} \\ (A_1)_{jj} & (A_1)_{ji} \end{vmatrix}$$

we conclude that $(A_0)_{ij}(A_1)_{ji} > 0$ as asserted. Specializing this observation we have that the first column of $A_0$ and last column of $A_1$ are strictly positive. Since both $A_0$ and $A_1$ are stochastic their highest eigenvalue is one and $A_0^T, A_1^T$ have unique corresponding eigenvectors as claimed. ∎

In preparation for the main result about the functional equation we note the following fact.

**Lemma 2.3.** *Let $A_0, A_1$ be nonsingular $n \times n$ stochastic matrices such that*

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}$$

*is totally positive. Suppose further that*

$$A_1^T\mathbf{x}^0 = A_0^T\mathbf{x}^1,$$

*where $\mathbf{x}^0, \mathbf{x}^1$ are the unique eigenvectors of $A_0^T, A_1^T$ as referred to in Lemma 2.2. Then*

$$\mathbf{x}^0 = (1, 0, \ldots, 0), \qquad \mathbf{x}^1 = (0, \ldots, 0, 1)$$

*and*

$$(A_0)_{nj} = (A_1)_{1j}, \quad (A_0)_{1j} = \delta_{1j}, \quad (A_1)_{nj} = \delta_{nj}, \qquad j = 1, \ldots, n.$$

**Proof.** Let $k$ be the largest integer $\le n$ such that $(\mathbf{x}^0)_k > 0$. Then Lemma 2.2 allows us to apply Lemma 2.1 to $A_0^T$ and conclude that $(\mathbf{x}^0)_j > 0$, $j \le k$, and, of course, by definition we have $(\mathbf{x}^0)_j = 0$, $j > k$. Similarly, we let $r$ be the least integer $\ge 1$ such that $(\mathbf{x}^1)_r > 0$. Hence just as before $(\mathbf{x}^1)_j = 0$, $j < r$, and $(\mathbf{x}^1)_j > 0$, $j \ge r$.

We consider the vector $\mathbf{x} = (\mathbf{x}^1, -(\mathbf{x}^0)_1, \ldots, -(\mathbf{x}^0)_k) \in \mathbf{R}^{n+k}$. Then $\mathbf{x}\tilde{A} = \mathbf{0}$ where $\tilde{A}$ is the $(n+k) \times n$ submatrix consisting of the first $n+k$ rows of $A$. We recall the fact [7, p. 230] that, for any TP $m \times n$ matrix $B$ of rank $n$, the equation $\mathbf{y}B = \mathbf{0}$ for some $\mathbf{y} \in \mathbf{R}^m$ implies that $S^+(\mathbf{y}) \ge n$. ($S^+(\mathbf{c})$ is the maximum number of sign changes in the components of the vector $\mathbf{c} = (c_0, \ldots, c_m)$, where zero entries are given arbitrary sign.) Therefore, since $S^+(\mathbf{x}) = r$ we have $r \ge n$, i.e., $r = n$. Similarly, we obtain $k = 1$. The form of $\mathbf{x}^0$ and $\mathbf{x}^1$ implies the remaining claims of the lemma. ∎

**Proposition 2.1.** *Suppose $A_0, A_1$ are nonsingular stochastic matrices and*

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}$$

*is totally positive. Then the functional equation*

$$(2.1)\qquad \begin{aligned} \Psi(t) &= A_0^T\Psi(2t), && 0 \le t \le \tfrac{1}{2},\\ \Psi(t) &= A_1^T\Psi(2t-1), && \tfrac{1}{2} \le t \le 1, \end{aligned}$$

*has a nontrivial continuous solution if and only if*

$$(A_0)_{1j} = \delta_{1j}, \quad (A_1)_{nj} = \delta_{nj}, \quad (A_0)_{nj} = (A_1)_{1j}, \qquad j = 1, \ldots, n.$$

*In this case,* $(\mathbf{e}, \Psi(t)) = 1$, $0 \le t \le 1$, $\Psi(0) = (1, 0, \ldots, 0)$, *and* $\Psi(1) = (0, \ldots, 0, 1)$.

**Proof.** Suppose $A_0$, $A_1$ satisfy these conditions. Then $\mathbf{x}^0 = (1, 0, \ldots, 0)$ and $\mathbf{x}^1 = (0, \ldots, 0, 1)$ are the unique eigenvectors of $A_0^T$, $A_1^T$ for eigenvalue one (both have positive columns) and

$$A_1^T\mathbf{x}^0 = A_0^T\mathbf{x}^1.$$

Hence Theorem 1.1 implies that the limit

$$\lim_{k\to\infty} A_{\varepsilon_k} \cdots A_{\varepsilon_1}\mathbf{c} = (\mathbf{c}, \Psi(t))\mathbf{e}$$

exists where $\Psi$ is a continuous curve on $[0, 1]$ satisfying $(\mathbf{e}, \Psi(t)) = 1$, $0 \le t \le 1$, and the functional equation (2.1).

Conversely, if $\Psi$ satisfies the functional equation, then it follows that

$$(2.2)\qquad \Psi(t) = \lim_{k\to\infty} A_{\varepsilon_1}^T \cdots A_{\varepsilon_k}^T\Psi(x), \qquad t = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k},$$

for any $x, t \in [0, 1]$. Thus if we set $\mathbf{x}^0 := \Psi(0)$ and $\mathbf{x}^1 := \Psi(1)$ we get $\mathbf{x}^0, \mathbf{x}^1 \ne \mathbf{0}$ and $A_1^T\mathbf{x}^0 = A_0^T\mathbf{x}^1$ as well as $A_\varepsilon^T\mathbf{x}^\varepsilon = \mathbf{x}^\varepsilon$, $\varepsilon \in \{0, 1\}$. Equation (2.2) implies that $(\mathbf{e}, \Psi(x))$ is a nonzero constant which we normalize to be one. Hence by Lemma 2.3 all the desired properties of $A_0$ and $A_1$ follow. ∎

Next we turn to the principal consequence of our running hypothesis that

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}$$

is totally positive. The following result and Proposition 2.1 embody Theorem 2.1.

**Proposition 2.2.** *Assume that the statements of Proposition 2.1 hold. Let* $1 \le i_1 < \cdots < i_r \le n$ *and* $0 \le x_1 < \cdots < x_r \le 1$, $1 \le r \le n$. *Then*

$$\Psi\begin{pmatrix} i_1, \ldots, i_r \\ x_1, \ldots, x_r \end{pmatrix} \ge 0,$$

*where equality holds if and only if either* $x_1 = 0$, $i_1 > 1$ *or* $x_r = 1$, $i_r < n$.

**Proof.** We prove this result by induction on $r$. We begin with the case $r = 1$. Thus we will establish the inequalities:

$$\psi_1(t) > 0 \qquad \text{if and only if} \quad t \in [0, 1),$$

$$\psi_i(t) > 0 \qquad \text{if and only if} \quad t \in (0, 1), \qquad 2 \le i \le n-1,$$

and

$$\psi_n(t) > 0 \qquad \text{if and only if} \quad t \in (0, 1].$$

We have already pointed out that $\Psi(0) = (1, 0, \ldots, 0)$ and $\Psi(1) = (0, \ldots, 0, 1)$ and so by (2.2) (choosing $x = 0$) we get $\Psi(t) \geq 0$ for all $t \in [0, 1]$. Also, from the form of $\Psi(0)$, $\Psi(1)$, we can restrict ourselves to $t \in (0, 1)$. To show $\psi_1(t) > 0$ for $t \in (0, 1)$ we expand $t$ in its binary representation

$$t = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}.$$

Choose the least integer $l \geq 1$ such that $\varepsilon_r = 1$, $r < l$. Then $\varepsilon_l = 0$ and $y_l := 2^{l-1}(t - 1/2 - \cdots - 1/2^{l-1}) \in [0, \frac{1}{2})$. For $l = 1$, $y_1 = t \in (0, \frac{1}{2}]$ and therefore

$$\psi_1(t) = \sum_{k=1}^{n} (A_0)_{k1} \psi_k(2t).$$

If $\psi_1(t) = 0$, then, by Lemma 2.2, $\Psi(2t) = \mathbf{0}$ and therefore by (2.2) (with $x = 2t$) $\Psi = \mathbf{0}$, a contradiction. When $l \geq 2$ we use the equation

$$\psi_1(t) = \sum_{k=1}^{n} (A_1^{l-1})_{k1} \psi_k(y_l)$$

and therefore

$$\psi_1(t) \geq ((A_1)_{11})^{l-1} \psi_1(y_l) > 0.$$

Thus $\psi_1(t) > 0$ for $t \in (0, 1)$. Similarly to show that $\psi_n(t) > 0$, $t \in (0, 1)$, we let $l$ be the least positive integer $l \geq 1$ such that $\varepsilon_r = 0$, $r < l$. Then $\varepsilon_l = 1$ and $z_l := 2^{l-1} t \in (\frac{1}{2}, 1]$. If $l = 1$, then $t \in [\frac{1}{2}, 1)$ and we use the equation

$$\psi_n(t) = \sum_{k=1}^{n} (A_1)_{kn} \psi_k(2t - 1)$$

which implies $\psi_n(t) > 0$ because $(A_1)_{kn} > 0$, $1 \leq k \leq n$, and $\Psi(x) \neq \mathbf{0}$ for all $x \in [0, 1]$. When $l \geq 2$ we use

$$\psi_n(t) = \sum_{k=1}^{n} (A_0^{l-1})_{kn} \psi_k(z_l) \geq ((A_0)_{nn})^{l-1} \psi_n(z_l) > 0.$$

Let us now consider the other components of $\Psi$. For $t \in (0, \frac{1}{2})$ and $2 \leq i \leq n - 1$ we use the inequality

$$\psi_i(t) = \sum_{k=1}^{n} (A_0)_{ki} \psi_k(2t) \geq (A_0)_{ni} \psi_n(2t) > 0$$

while for $t \in [\frac{1}{2}, 1)$ we employ

$$\psi_i(t) = \sum_{k=1}^{n} (A_1)_{ki} \psi_k(2t - 1) \geq (A_1)_{1i} \psi_1(2t - 1) > 0.$$

This takes care of the case $r = 1$.

We now assume inductively that

$$\Psi\binom{i_1,\ldots,i_l}{x_1,\ldots,x_l} \geq 0,$$

for $1 \leq i_1 < \cdots < i_l \leq n$, $0 \leq x_1 < \cdots < x_l \leq 1$, and all $l \leq r-1$, where equality holds if and only if either $x_1 = 0$, $i_1 > 0$ or $x_l = 1$, $i_l < n$. We consider a typical minor of order $r$

$$\Psi\binom{i_1,\ldots,i_r}{t_1,\ldots,t_r},$$

where $1 \leq i_1 < \cdots < i_r \leq n$ and $0 \leq t_1 < \cdots < t_r \leq 1$. If $t_1 = 0$, then because $\psi_i(0) = \delta_{i1}$, $i = 1, 2, \ldots, n$, we get

$$\Psi\binom{i_1, i_2,\ldots,i_r}{0, t_2,\ldots,t_r} = \delta_{i_1 1}\Psi\binom{i_2,\ldots,i_r}{t_2,\ldots,t_r}$$

and similarily if $t_r = 1$,

$$\Psi\binom{i_1,\ldots,i_{r-1}, i_r}{t_1,\ldots,t_{r-1}, 1} = \delta_{i_r n}\Psi\binom{i_1,\ldots,i_{r-1}}{t_1,\ldots,t_{r-1}}.$$

Therefore the induction hypothesis allows us to assume $0 < t_1 < \cdots < t_r < 1$.

The first possibility we consider is $\frac{1}{2} \in [t_1, t_r]$. We dismiss the cases where $\frac{1}{2}$ is an endpoint of $[t_1, t_r]$ as follows. For $\frac{1}{2} = t_1 < \cdots < t_r < 1$ we use the Cauchy-Binet formula and the functional equation to obtain

$$\begin{aligned}\Psi\binom{i_1,\ldots,i_r}{t_1,\ldots,t_r} &= \sum_{1 \leq j_1 < \cdots < j_r \leq n} A_1\binom{j_1,\ldots,j_r}{i_1,\ldots,i_r}\Psi\binom{j_1,\ldots,j_r}{2t_1-1,\ldots,2t_r-1} \\ &= \sum_{2 \leq j_2 < \cdots < j_r \leq n} A_1\binom{1, j_2,\ldots,j_r}{i_1,\ldots,i_r}\Psi\binom{j_2,\ldots,j_r}{2t_2-1,\ldots,2t_r-1}.\end{aligned}$$

If $1 < i_1$, then since the last $r$ rows of the $r+1 \times r$ matrix

$$T := A_1\Psi\begin{bmatrix}1, i_1,\ldots,i_r \\ i_1,\ldots,i_r\end{bmatrix}$$

are linearly independent and the first nonzero, there exists some $2 \leq j_2^0 < \cdots < j_r^0 \leq n$, $j_l^0 \in \{i_1,\ldots,i_r\}$, $2 \leq l \leq r$, such that the first row and rows $j_2^0,\ldots,j_r^0$ of $T$ are linearly independent. When $i_1 = 1$ then we may set $j_k^0 = i_k$, $k = 2,\ldots,r$. Therefore we have by the TP property of $A_1$ and the induction hypothesis

$$\Psi\binom{i_1,\ldots,i_r}{t_1,\ldots,t_r} \geq A_1\binom{1, j_2^0,\ldots,j_r^0}{i_1,\ldots,i_r}\Psi\binom{j_2^0,\ldots,j_r^0}{2t_2-1,\ldots,2t_r-1} \geq 0.$$

Similarly, if $\frac{1}{2}$ is the right endpoint of $[t_1, t_r]$ we use the equation

$$\Psi\binom{i_1,\ldots,i_r}{t_1,\ldots,t_r} = \sum_{1 \leq j_1 < \cdots < j_{r-1} \leq n-1} A_0\binom{j_1,\ldots,j_{r-1}, n}{i_1,\ldots,i_r}\Psi\binom{j_1,\ldots,j_{r-1}}{2t_1,\ldots,2t_{r-1}}.$$

Just as before we choose $j_1^0, \ldots, j_{r-1}^0 \in \{i_1, \ldots, i_r\}$, $1 \le j_1^0 < \cdots < j_{r-1}^0 \le n-1$, such that by induction it follows that

$$\Psi\begin{pmatrix} i_1, \ldots, i_r \\ t_1, \ldots, t_r \end{pmatrix} \ge A_0\begin{pmatrix} j_1^0, \ldots, j_{r-1}^0, n \\ i_1, \ldots, i_r \end{pmatrix}\Psi\begin{pmatrix} j_1^0, \ldots, j_{r-1}^0 \\ 2t_1, \ldots, 2t_{r-1} \end{pmatrix} > 0.$$

The case when $\frac{1}{2} \in (t_1, t_r)$ is more involved. Here we choose an integer $l$, $1 \le l < r$, such that

$$t_1 < \cdots < t_l \le \tfrac{1}{2} < t_{l+1} < \cdots t_r.$$

(When $l = 1$, we need only consider the possibility that $t_1 < \frac{1}{2} < t_2$ because we already considered the case $t_1 = \frac{1}{2}$.) We now use the functional equation and factor the $n \times r$ matrix $\Psi\begin{bmatrix} 1, \ldots, n \\ t_1, \ldots, t_r \end{bmatrix}$ as

$$\Psi\begin{bmatrix} 1, \ldots, n \\ t_1, \ldots, t_r \end{bmatrix} = A^T C, \qquad A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix},$$

where $C$ is the $2n \times r$ (block) matrix

$$C := \begin{bmatrix} \Psi\begin{bmatrix} 1, \ldots, n \\ 2t_1, \ldots, 2t_l \end{bmatrix} & 0 \\ 0 & \Psi\begin{bmatrix} 1, \ldots, n \\ 2t_{l+1} - 1, \ldots, 2t_r - 1 \end{bmatrix} \end{bmatrix}.$$

By the Cauchy-Binet formula we have

$$\Psi\begin{pmatrix} i_1, \ldots, i_r \\ t_1, \ldots, t_r \end{pmatrix} = \sum_{1 \le j_1 < \cdots < j_r \le 2n} A\begin{pmatrix} j_1, \ldots, j_r \\ i_1, \ldots, i_r \end{pmatrix} C\begin{pmatrix} j_1, \ldots, j_r \\ 1, \ldots, r \end{pmatrix}. \tag{2.3}$$

If $k := |\{j_1, \ldots, j_r\} \cap \{1, \ldots, n\}| > l$, then by taking linear combinations of its first $k$ rows the matrix

$$C\begin{bmatrix} j_1, \ldots, j_r \\ 1, \ldots, r \end{bmatrix}$$

has a zero row and therefore a zero determinant. Similarly,

$$C\begin{pmatrix} j_1, \ldots, j_r \\ 1, \ldots, r \end{pmatrix} = 0$$

if $|\{j_1, \ldots, j_r\} \cap \{n+1, \ldots, 2n\}| > r - l$. Hence (2.3) becomes

$$\Psi\begin{pmatrix} i_1, \ldots, i_r \\ t_1, \ldots, t_r \end{pmatrix} = \sum_{\substack{1 \le j_1 < \cdots < j_l \le n \\ 1 \le k_1 < \cdots < k_{r-l} \le n}} A\begin{pmatrix} j_1, \ldots, j_l, k_1 + n, \ldots, k_{r-l} + n \\ i_1, \ldots, i_r \end{pmatrix}$$
$$\times \Psi\begin{pmatrix} j_1, \ldots, j_l \\ 2t_1, \ldots, 2t_l \end{pmatrix}\Psi\begin{pmatrix} k_1, \ldots, k_{r-l} \\ 2t_{l+1} - 1, \ldots, 2t_r - 1 \end{pmatrix}.$$

The $2r \times r$ matrix

$$A\begin{bmatrix} i_1,\ldots,i_r,i_1+n,\ldots,i_r+n \\ i_1,\ldots,i_r \end{bmatrix}$$

has the property that its first $r$ rows as well as its last $r$ rows are linearly independent. Hence for any choice of $l$ rows among its first $r$ rows there is a choice of $r-l$ row vectors from the last rows for which the resulting set of vectors is linearly independent. Thus for any choice of integers $1 \le j_1^0 < \cdots < j_l^0 \le n$ in $\{i_1,\ldots,i_r\}$ there are integers $1 \le k_1^0 < \cdots < k_{r-l}^0 \le n$ in $\{i_1,\ldots,i_r\}$ such that

$$A\begin{pmatrix} j_1^0,\ldots,j_l^0,k_1^0+n,\ldots,k_{r-l}^0+n \\ i_1,\ldots,i_r \end{pmatrix} > 0.$$

The other qualification we must make in our choice is that if $t_l = \frac{1}{2}$ we choose $j_l^0 = n$. This is easily done since the last row of $A_0$ is nonzero. Therefore we obtain by induction

$$\Psi\begin{pmatrix} i_1,\ldots,i_r \\ t_1,\ldots,t_r \end{pmatrix} \ge A\begin{pmatrix} j_1^0,\ldots,j_l^0,k_1^0+n,\ldots,k_{r-l}^0+n \\ i_1^0,\ldots,i_r^0 \end{pmatrix}$$
$$\times \Psi\begin{pmatrix} j_1^0,\ldots,j_l^0 \\ 2t_1,\ldots,2t_l \end{pmatrix}\Psi\begin{pmatrix} k_1^0,\ldots,k_{r-l}^0 \\ 2t_{l+1}-1,\ldots,2t_r-1 \end{pmatrix} > 0.$$

There remain the two cases $t_r < \frac{1}{2}$ or $t_1 > \frac{1}{2}$. In the first instance, we consider the binary expansion of the vector $\mathbf{t} = (t_1,\ldots,t_r)$

$$\mathbf{t} = \sum_{k=1}^{\infty} \boldsymbol{\varepsilon}^k 2^{-k},$$

where $\boldsymbol{\varepsilon}^k = (\varepsilon_1^k,\ldots,\varepsilon_r^k)$, $\varepsilon_i^k \in \{0, 1\}$. In the case at hand $\boldsymbol{\varepsilon}^1 = \mathbf{0}$. We let $m_1$ be the largest integer $\ge 2$ such that $\boldsymbol{\varepsilon}^k = \mathbf{0}$ for $k < m_1$. Thus $\boldsymbol{\varepsilon}^{m_1} \ne \mathbf{0}$ and so its last component must be one. Either the first component of $\boldsymbol{\varepsilon}_1^m$ is zero or we have $\boldsymbol{\varepsilon}^{m_1} = (1,\ldots,1)$. In the latter case we let $m_2$ be the largest integer greater than $m_1$ such that $\boldsymbol{\varepsilon}^k = (1,\ldots,1)$, $m_1 \le k < m_2$. Continuing in this way we can find a dyadic fraction $\tau \in [0, 1]$ such that $y_i = 2^\mu(t_i - \tau) \in [0, 1]$, the first $\mu$ binary digits of each $t_i$, $i = 1,\ldots,r$, agree with $\tau$, and $\frac{1}{2} \in [y_1, y_r]$. We take $\mu$ *minimal* so that this holds. Therefore

$$\Psi\begin{pmatrix} i_1,\ldots,i_r \\ t_1,\ldots,t_r \end{pmatrix} = \sum_{1 \le j_1 < \cdots < j_r \le n} (A_{\varepsilon_1^\mu} \cdots A_{\varepsilon_1^1})\begin{pmatrix} j_1,\ldots,j_r \\ i_1,\ldots,i_r \end{pmatrix}\Psi\begin{pmatrix} j_1,\ldots,j_r \\ y_1,\ldots,y_r \end{pmatrix},$$

where $\varepsilon_1^1 = 0$. When $0 < y_1 < y_r < 1$, then we use what we have already proved to conclude that

$$\Psi\begin{pmatrix} j_1,\ldots,j_r \\ y_1,\ldots,y_r \end{pmatrix} > 0$$

for all $1 \le j_1 < \cdots < j_r \le n$. Since $A_{\varepsilon_1^\mu} \cdots A_{\varepsilon_1^1}$ is TP and

$$(A_{\varepsilon_1^\mu} \cdots A_{\varepsilon_1^1})\begin{pmatrix} i_1,\ldots,i_r \\ i_1,\ldots,i_r \end{pmatrix} > 0$$

we obtain our desired result because $\mu$ was minimially chosen with $\frac{1}{2} \in [y_1, y_r]$. It may be verified that we must have $0 < y_1 < y_r < 1$. For if $y_1 = 0$, then since $0 < t_1 < \cdots < t_r < 1$ we must have had $y_1 = \frac{1}{2}$ at some previous stage. Similarly, if $y_r = 1$, then at some previous stage we must have had $y_r = \frac{1}{2}$. This completes this case. The remaining case $t_1 > \frac{1}{2}$ is the same.

This proves the proposition and Theorem 2.1 as well. ∎

We end this section with some remarks on possible extensions of this result. The first possibility we consider is the iteration of more than two matrices. Thus the functional equation takes the form

$$\Psi(t) = A_i^T \Psi(pt - i), \qquad \frac{i}{p} \le t \le \frac{i+1}{p}, \qquad i = 0, 1, \ldots, p-1,$$

and the iteration is based on $p$-adic expansions

$$t = \sum_{k=1}^{\infty} \varepsilon_k p^{-k}, \qquad \varepsilon_k \in \{0, 1, \ldots, p-1\}, \qquad \lim_{k\to\infty} A_{\varepsilon_k} \cdots A_{\varepsilon_1} \mathbf{c} = (\mathbf{c}, \Psi(t))\mathbf{e}.$$

This case is also considered in [10]. The analysis necessary to extend Theorem 2.1 is not essentially different from what we have already provided. It leads to the following result.

**Theorem 2.2.** *Let $A_\varepsilon$, $\varepsilon \in \{0, 1, \ldots, p-1\}$ be nonsingular matrices such that the $pn \times n$ matrix*

$$A = \begin{bmatrix} A_0 \\ \vdots \\ A_{p-1} \end{bmatrix}$$

*is totally positive. Suppose further that the first row of $A_0$ is $(1, 0, \ldots, 0)$, the last row of $A_{p-1}$ is $(0, \ldots, 0, 1)$, and the last row of $A_i$ and the first row of $A_{i+1}$, $i = 0, 1, \ldots, p-2$, are the same. Then there exists a unique continuous solution $\Psi\colon [0, 1] \to \mathbf{R}^n$ to the functional equation*

$$\Psi\left(\frac{t+\varepsilon}{p}\right) = A_\varepsilon^T \Psi(t), \qquad 0 \le t \le 1, \quad \varepsilon \in \{0, 1, \ldots, p-1\},$$

*satisfying $(\mathbf{e}, \Psi(t)) = 1$, $0 \le t \le 1$. Furthermore, $\Psi(t)$ can be constructed as*

$$\lim_{k\to\infty} A_{\varepsilon_k} \cdots A_{\varepsilon_1} \mathbf{c} = (\mathbf{c}, \Psi(t))\mathbf{e}, \qquad t = \sum_{k=1}^{\infty} \varepsilon_k p^{-k}, \quad \varepsilon_k \in \{0, 1, \ldots, p-1\},$$

*and moreover*

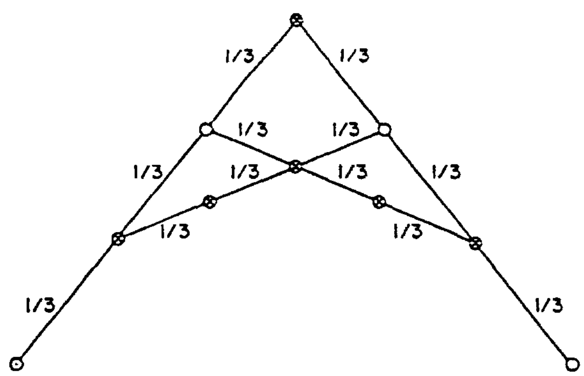
$$\Psi\begin{pmatrix} i_1, \ldots, i_s \\ x_1, \ldots, x_s \end{pmatrix} \ge 0$$

*if $1 \le i_1 < \cdots < i_s \le n$, $0 \le x_1 < \cdots < x_s \le 1$ where equality holds if and only if either $x_1 = 0$, $i_1 > 1$ or $x_s = 1$, $i_s < n$.*

As a simple example of the above we consider subdivision for the quadratic Bernstein-Bézier curve by *trisection*. The matrices in this case are

$$A_0=\begin{pmatrix}1&0&0\\ \frac{2}{3}&\frac{1}{3}&0\\ \frac{4}{9}&\frac{4}{9}&\frac{1}{9}\end{pmatrix},\qquad A_1=\begin{pmatrix}\frac{4}{9}&\frac{4}{9}&\frac{1}{9}\\ \frac{2}{9}&\frac{5}{9}&\frac{2}{9}\\ \frac{1}{9}&\frac{4}{9}&\frac{4}{9}\end{pmatrix},\qquad A_2=\begin{pmatrix}\frac{1}{9}&\frac{4}{9}&\frac{4}{9}\\ 0&\frac{1}{3}&\frac{2}{3}\\ 0&0&1\end{pmatrix}$$

and geometrically the process proceeds as a corner-cutting scheme:




Our next remarks are a useful weakening of our hypotheses in Theorem 2.1. Let $\tilde{A}_0, \ldots, \tilde{A}_{p-1}$ be a family of matrices satisfying the hypotheses of Theorem 2.2. Assume that $Y$ is a stochastic nonsingular totally positive matrix and set

$$A_i = Y^{-1}\tilde{A}_i Y, \qquad i = 0, 1, \ldots, p-1. \tag{2.4}$$

Then associated with the matrices $A_i$, $i = 0, 1, \ldots, p-1$, is a fundamental curve $\Psi\colon [0, 1] \to \mathbf{R}^n$ which satisfies the functional equation

$$\Psi\left(\frac{t+\varepsilon}{p}\right) = A_\varepsilon^T \Psi(t), \qquad 0 \le t \le 1, \quad \varepsilon \in \{0, 1, \ldots, p-1\},$$

and

$$\lim_{k\to\infty} A_{\varepsilon_k} \cdots A_{\varepsilon_1}\mathbf{c} = (\mathbf{c}, \Psi(t))\mathbf{e}, \qquad t = \sum_{k=1}^{\infty} \varepsilon_k p^{-k}, \quad \varepsilon_k \in \{0, 1, \ldots, p-1\}.$$

This curve is given by $\Psi = Y^T\tilde{\Psi}$, where $\tilde{\Psi}$ is the fundamental curve associated with the matrices $\tilde{A}_i$, $i = 0, 1, \ldots, p-1$.

Although the matrix

$$A = \begin{bmatrix} A_0 \\ \vdots \\ A_{p-1} \end{bmatrix}$$

is not generally totally positive, the curve $\Psi$ inherits positivity from $\tilde{\Psi}$ and more.

**Proposition 2.3.** *Assume the* $A_i$, $i = 0, 1, \ldots, p-1$, *are given by* (2.4) *where the*

*$\tilde{A}_0, \ldots, \tilde{A}_{p-1}$ satisfy the hypotheses of Theorem 2.2, and $Y$ is a stochastic nonsingular totally positive matrix. Let $\Psi$ denote its fundamental curve. Then*

$$\Psi\begin{pmatrix} i_1, \ldots, i_s \\ x_1, \ldots, x_s \end{pmatrix} \geq 0$$

*for all $1 \leq i_1 < \cdots < i_s \leq n$ and $0 \leq x_1 < \cdots < x_s \leq 1$ with equality if and only if $x_1 = 0$, $i_1 > k$ or $x_s = 1$, $i_s < l$, where*

$$k = \max\{j\colon y_{1j} > 0\},$$
$$l = \min\{j\colon y_{nj} > 0\}.$$

**Proof.** As previously noted, $\Psi = Y^T\tilde{\Psi}$ where $\tilde{\Psi}$ satisfies the conclusion of Theorem 2.2. Thus

$$\Psi\begin{pmatrix} i_1, \ldots, i_s \\ x_1, \ldots, x_s \end{pmatrix} = \sum_{1 \leq j_1 < \cdots < j_s \leq n} Y\begin{pmatrix} j_1, \ldots, j_s \\ i_1, \ldots, i_s \end{pmatrix} \tilde{\Psi}\begin{pmatrix} j_1, \ldots, j_s \\ x_1, \ldots, x_s \end{pmatrix}. \tag{2.5}$$

Since $Y$ is TP and $\tilde{\Psi}\begin{pmatrix} j_1, \ldots, j_s \\ x_1, \ldots, x_s \end{pmatrix} \geq 0$ for all ordered $\{j_r\}_{r=1}^s$ and $\{x_r\}_{r=1}^s$, it immediately follows that

$$\Psi\begin{pmatrix} i_1, \ldots, i_s \\ x_1, \ldots, x_s \end{pmatrix} \geq 0$$

for all $1 \leq i_1 < \cdots < i_s \leq n$ and $0 \leq x_1 < \cdots < x_s \leq 1$. Moreover because $Y$ is nonsingular,

$$Y\begin{pmatrix} i_1, \ldots, i_s \\ i_1, \ldots, i_s \end{pmatrix} > 0,$$

and thus

$$\Psi\begin{pmatrix} i_1, \ldots, i_s \\ x_1, \ldots, x_s \end{pmatrix} > 0,$$

if

$$\tilde{\Psi}\begin{pmatrix} i_1, \ldots, i_s \\ x_1, \ldots, x_s \end{pmatrix} > 0.$$

It therefore remains to consider the cases where $x_1 = 0$ and $i_1 > 1$ or $x_s = 1$ and $i_s < n$.

Assume $x_1 = 0$ and $i_1 > k$. From the properties of $\tilde{\Psi}$, (2.5) reduces to

$$\Psi\begin{pmatrix} i_1, i_2, \ldots, i_s \\ 0, x_2, \ldots, x_s \end{pmatrix} = \sum_{2 \leq j_2 < \cdots < j_s \leq n} Y\begin{pmatrix} 1, j_2, \ldots, j_s \\ i_1, i_2, \ldots, i_s \end{pmatrix} \tilde{\Psi}\begin{pmatrix} 1, j_2, \ldots, j_s \\ 0, x_2, \ldots, x_s \end{pmatrix}.$$

Because $i_1 > k$, we have $y_{1i_r} = 0$, $r = 1, \ldots, k$. Thus

$$\Psi\begin{pmatrix} i_1, i_2, \ldots, i_s \\ 0, x_2, \ldots, x_s \end{pmatrix} = 0.$$

Similarly

$$\Psi\begin{pmatrix} i_1, \ldots, i_s \\ x_1, \ldots, x_s \end{pmatrix} = 0$$

if $x_s = 1$ and $i_s < l$.

It remains to consider the case where $x_1 = 0$ and $1 < i_1 \leq k$ and/or $x_s = 1$ and $l \leq i_s < n$. By definition $y_{1k} > 0$ and $y_{1r} = 0$ for all $r > k$. Since $Y$ is TP and nonsingular, $y_{ii} > 0$, $i = 1, \ldots, n$. If $y_{1r} = 0$ for some $1 < r < k$, then

$$Y\begin{pmatrix} 1 & r \\ r & k \end{pmatrix} < 0,$$

a contradiction. Thus $y_{1r} > 0$, $r = 1, \ldots, k$. Similarly $y_{nr} > 0$, $r = l, \ldots, n$.

Assume for the moment that $x_1 = 0$ and $1 < i_1 \leq k$ while $x_s < 1$. Thus

$$\Psi\begin{pmatrix} i_1, i_2, \ldots, i_s \\ 0, x_2, \ldots, x_s \end{pmatrix} = \sum_{2 \leq j_2 < \cdots < j_s \leq n} Y\begin{pmatrix} 1, j_2, \ldots, j_s \\ i_1, i_2, \ldots, i_s \end{pmatrix} \tilde{\Psi}\begin{pmatrix} 1, j_2, \ldots, j_s \\ 0, x_2, \ldots, x_s \end{pmatrix}.$$

The last $s$ rows of the $(s+1) \times s$ matrix

$$Y\begin{bmatrix} 1, i_1, \ldots, i_s \\ i_1, \ldots, i_s \end{bmatrix}$$

are linearly independent and the first row is not identically zero. Thus there exist $j_2' < \cdots < j_s'$ in $\{i_1, \ldots, i_s\}$ such that

$$Y\begin{pmatrix} 1, j_2', \ldots, j_s' \\ i_1, i_2, \ldots, i_s \end{pmatrix} > 0.$$

Since

$$\tilde{\Psi}\begin{pmatrix} 1, j_2', \ldots, j_s' \\ 0, x_2, \ldots, x_s \end{pmatrix} > 0,$$

we obtain the desired result. The similar analysis proves the strict positivity in the case where $x_s = 1$, $l \leq i_s < n$, and $x_1 > 0$.

Finally let us assume that $x_1 = 0$, $1 \leq i_1 \leq k$, and $x_s = 1$, $l \leq i_s \leq n$ $(s \geq 2)$. We first digress to prove a general result. Assume $B$ is an $m \times s$ $(s \geq 2)$ TP ($TP_2$) matrix of rank at least 2. If the first and last rows of $B$ are not identically zero, then they are necessarily linearly independent. For since $B$ is of rank at least 2 there exists an $i \in \{2, \ldots, m\}$ and $1 \leq j_1 < j_2 \leq s$ such that

$$B\begin{pmatrix} 1 & i \\ j_1 & j_2 \end{pmatrix} > 0.$$

If the first and last rows are linearly dependent, the last row must be a positive (because of the TP property) multiple of the first row, and $i < m$. Thus

$$B\begin{pmatrix} m & i \\ j_1 & j_2 \end{pmatrix} > 0.$$

But then

$$B\begin{pmatrix} i & m \\ j_1 & j_2 \end{pmatrix} < 0.$$

contradicting the TP property.

The $(s+2) \times s$ matrix

$$Y\begin{bmatrix} 1, i_1, \ldots, i_s, n \\ i_1, \ldots, i_s \end{bmatrix}$$

is TP and of rank $s$ since

$$Y\begin{pmatrix} i_1, \ldots, i_s \\ i_1, \ldots, i_s \end{pmatrix} > 0.$$

Because $i_1 \leq k$ and $i_s \geq l$, both the first and last rows are not identically zero. Thus the first and last rows are linearly independent. There therefore exist $\{j'_2, \ldots, j'_{s-1}\} \subseteq \{i_1, \ldots, i_s\}$ such that

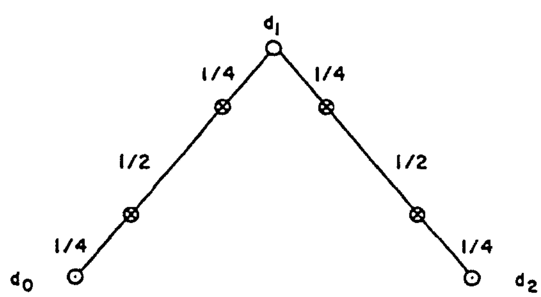
$$Y\begin{pmatrix} 1, j'_2, \ldots, j'_{s-1}, n \\ i_1, i_2, \ldots, i_{s-1}\, i_s \end{pmatrix} > 0.$$

Since

$$\tilde{\Psi}\begin{pmatrix} 1, j'_2, \ldots, j'_{s-1}, n \\ 0, x_2, \ldots, x_{s-1}, 1 \end{pmatrix} > 0,$$

we obtain the strict positivity of the associated minor of $\Psi$. ■

As an example of this observation we consider the Chaiken algorithm




Here the matrices are

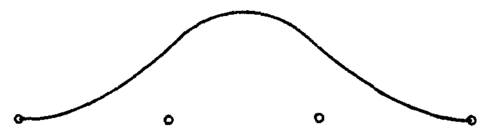
$$A_0 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}, \qquad A_1 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

The fundamental curve $\Psi$: $[0, 1] \to \mathbf{R}^3$ is easily seen to be

$$\Psi(t) = \begin{pmatrix} \frac{1}{2}(1-t)^2 \\ t(1-t) + \frac{1}{2} \\ \frac{1}{2}t^2 \end{pmatrix}$$

and the components of $\Psi$ form the pieces of the *quadratic B-spline* $\varphi$ given by

$$\varphi(x) := \begin{cases} \psi_3(x), & 0 \le x \le 1, \\ \psi_2(x-1), & 1 \le x \le 2, \\ \psi_1(x-2), & 2 \le x \le 3, \end{cases}$$



The $Y$ matrix in this case is

$$Y = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and

$$\tilde{A}_0 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}, \qquad \tilde{A}_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

are the Bernstein–Bézier subdivision matrices.

Clearly $Y$ is totally positive and it is straightforward to confirm (2.4). The matrix $Y$ converts a quadratic polynomial expressed in B-spline form to its Bernstein-Bézier form, specifically we have

$$\Psi(t) = Y^T \begin{pmatrix} (1-t)^2 \\ 2t(1-t) \\ t^2 \end{pmatrix} = Y^T \tilde{\Psi}(t).$$

Note that

$$\Psi\begin{pmatrix} i_1, \ldots, i_s \\ x_1, \ldots, x_s \end{pmatrix} > 0$$

unless $s = 1$ and $x_1 = 0$, $i_1 = 3$ or $x_1 = 1$, $i_1 = 1$. This example admits generalization to arbitrary degree B-splines, but we do not dwell upon it here.

We make one further observation in this section. For this purpose, we recall the definition of stationary subdivision [9]. We are given a *mask* $\{a_j : j \in \mathbf{Z}\}$ which is assumed to have only a finite number of nonzero terms. Given control points $\{\mathbf{c}_j^0 : j \in \mathbf{Z}\}$ we form new control points $\{\mathbf{c}_j^1 : j \in \mathbf{Z}\}$ by the rule

$$\mathbf{c}_i^1 = \sum_{-\infty}^{\infty} a_{i-2k} \mathbf{c}_k^0.$$

If we suppose for convenience that the nonzero elements of the mask are confined to $\{a_0, \ldots, a_n\}$, $n \ge 1$, then we may express a step of stationary subdivision in MSS form as

$$\begin{pmatrix} c_1^1 \\ \vdots \\ c_{-n+2}^0 \end{pmatrix} = \begin{pmatrix} a_1 & a_3 & \cdots & a_{2n-1} \\ a_0 & a_2 & \cdots & a_{2n-2} \\ \vdots & \vdots & & \vdots \\ a_{-n+2} & a_{-n+4} & \cdots & a_n \end{pmatrix} \begin{pmatrix} c_0^0 \\ \vdots \\ c_{-n+1}^0 \end{pmatrix},$$

$$\begin{pmatrix} c_0^1 \\ \vdots \\ c_{-n+1}^1 \end{pmatrix} = \begin{pmatrix} a_0 & a_2 & \cdots & a_{2n-2} \\ a_{-1} & a_1 & \cdots & a_{2n-3} \\ \vdots & \vdots & & \vdots \\ a_{-n+1} & a_{-n+3} & \cdots & a_{n-1} \end{pmatrix} \begin{pmatrix} c_0^0 \\ \vdots \\ c_{-n+1}^0 \end{pmatrix},$$

Therefore, $A_0 = (A_{ij}^0)_{i,j=1}^n$, $A_1 = (A_{ij}^1)_{i,j=1}^n$ with

$$A_{ij}^0 := a_{2j-i}, \quad A_{ij}^1 := a_{2j-i-1}, \qquad i, j = 1, \ldots, n.$$

We assume that $A_0, A_1$ are nonsingular and so the functional equation for $\Psi$,

$$\Psi\left(\frac{t+\varepsilon}{2}\right) = A_\varepsilon^T \Psi(t), \qquad \varepsilon \in \{0, 1\}, \quad 0 \le t \le 1, \tag{2.6}$$

has the equivalent form

$$\varphi\left(\frac{x}{2}\right) = \sum_{j=0}^{n} a_j \varphi(x-j), \qquad -\infty < x < \infty, \tag{2.7}$$

where $\varphi(x) = 0$, $x \notin [0, n]$, and otherwise it is given by the formulas

$$\varphi(x) = \psi_{n-l}(x-l), \qquad l \le x \le l+1, \quad l = 0, 1, \ldots, n-1,$$

$$\Psi(x) = (\psi_1(x), \ldots, \psi_n(x)),$$

see [9]. When $\sum_{i=-\infty}^{\infty} a_{2i} = \sum_{i=-\infty}^{\infty} a_{2i-1} = 1$, then $A_0, A_1$ have row sums one and if the corresponding MSS converges, then $\varphi$ is continuous [9]. This also follows from the hypotheses that the functional equation (2.6) has a continuous solution, $a_0 \ne 1, a_n \ne 1$, and the nonsingularity of $A_0, A_1$. To see this first note that $\psi_n(0) = 0, \psi_1(1) = 0$ by (2.6) and so $\varphi$ is continuous at $x = 0$ and $x = n$. For the integers interior to the support of the mask we introduce the $n \times n+1$ matrix

$$B = \begin{bmatrix} a_1 & a_0 & \cdots & a_{-n+2} & a_{-n+1} \\ a_3 & a_2 & \cdots & a_{-n+4} & a_{-n+3} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{2n-1} & a_{2n-2} & \cdots & a_n & a_{n-1} \end{bmatrix}$$

The first $n$ columns and $n$ rows of $B$ are $A_0^T$, while its last $n$ columns and $n$ rows are $A_1^T$. Thus from the functional equation (2.6)

$$B \begin{pmatrix} \psi_1(1) \\ \psi_2(1) - \psi_1(0) \\ \vdots \\ \psi_n(1) - \psi_{n-1}(0) \\ -\psi_n(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\psi_1(1) = \psi_n(0) = 0$ the nonsingularity of $A_0$ and $A_1$ imply

$$\psi_{i+1}(1) = \psi_i(0), \qquad i = 1, \ldots, n-1,$$

which are the conditions for continuity of $\varphi$ at $x = 1, \ldots, n-1$.

In [9] it was proved that when $a_0, \ldots, a_n > 0, n \geq 2$, and $\sum_{-\infty}^{\infty} a_{2j} = \sum_{-\infty}^{\infty} a_{2j-1} = 1$, there exists a unique solution to the functional equation (2.7) which is continuous and satisfies $\sum_{j=-\infty}^{\infty} \varphi(x-j) = 1$ (see [4] for *multivariate* versions of this result). It was left open in [9] as to whether or not $\varphi(x) \geq 0$, with strict inequality if and only if $x \in (0, n)$. We prove that this is indeed the case.

**Proposition 2.4.** *Let* $\sum_{j=-\infty}^{\infty} a_{2j} = \sum_{j=-\infty}^{\infty} a_{2j-1} = 1$, $a_j > 0$, $j = 0, 1, \ldots, n$, *zero otherwise*, $n \geq 2$. *Then there exists a unique continuous function* $\varphi(x)$, $-\infty < x < \infty$, *satisfying the functional equation*

$$\varphi\left(\frac{x}{2}\right) = \sum_{i=-\infty}^{\infty} a_i \varphi(x-i), \qquad -\infty < x < \infty,$$

*which is strictly positive on* $(0, n)$, *and zero otherwise.*

**Proof.** We need to show that for $n \geq 2$

$$\psi_i(x) > 0, \qquad x \in [0, 1], \quad i = 2, \ldots, n-1,$$
$$\psi_1(x) > 0, \qquad x \in [0, 1),$$
$$\psi_n(x) > 0, \qquad x \in (0, 1].$$

We begin with the case where $n$ is odd, $n = 2m+1$, $m \geq 1$. In this case, the $(m+1)$st column of $A_0$ is $(a_{2m+1}, \ldots, a_1)^T$ and the $(m+1)$st of $A_1$ is $(a_{2m}, \ldots, a_0)^T$. Hence from Theorem 1.1 we conclude

$$\Psi(t) = \lim_{k \to \infty} A_{\varepsilon_1}^T \cdots A_{\varepsilon_k}^T \Psi(0), \qquad t = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k},$$

and so $A_0^T \Psi(0) = \Psi(0)$. $\Psi(0)$ is the nonnegative eigenvector of $A_0^T$. Hence $\psi_i(t) \geq 0$, $i = 1, \ldots, n$, $t \in [0, 1]$. Now, it follows that $\psi_{m+1}(t) > 0$, $t \in [0, \frac{1}{2}]$, because

$$\psi_{m+1}(t) = \sum_{k=1}^{2m+1} A_{k,m+1}^0 \psi_k(2t) > 0$$

and since there is no $t_0 \in [0, 1]$ with $\Psi(t_0) = 0$. Similarly, we can show $\psi_{m+1}$ is positive on $[\frac{1}{2}, 1]$. When $n = 2m$ the same argument shows that $\psi_m(t) > 0$ for $t \in [0, 1]$. Thus we have established that on some open interval of length greater than one $\varphi$ is positive. Now it is an easy matter to "propogate" positivity by the use of the functional equation (2.7). Specifically this equation implies that whenever $\varphi(x) > 0$ on some interval $I_1 := (a, b) \subseteq (0, n)$ of length greater than one then $\varphi(x) > 0$ on the interval $\bigcup_{j=0}^{n} (j + (a, b))/2 = (a/2, (n+b)/2) = \frac{1}{2} I_1 + \frac{1}{2}(0, n)$. The iteration $I_{k+1} = \frac{1}{2} I_k + \frac{1}{2}(0, n)$, $k = 1, 2, \ldots$, clearly converges to $(0, n)$ thereby establishing the positivity of $\varphi$ on $(0, n)$. ∎

In [9] it was shown that when $\{a_j: -\infty < j < \infty\}$ is a Polyá frequency sequence

$$S^-\left(\sum_{-\infty}^{\infty} c_j\varphi(\cdot - j)\right) \le S^-(\{c_j: -\infty < j < \infty\}).$$

Here $S^-(\{c_j: -\infty < j < \infty\})$ is the number of strict sign changes in the vector $\{\ldots, c_{-1}, c_0, c_1, \ldots\}$, and similarily $S^-(f)$ counts the number of sign changes of a function $f$ on $(-\infty, \infty)$. We conjecture that in fact under the same hypothesis $\varphi$ satisfies the following determinantal inequalities. Let $K(x, y) := \varphi(x - y)$. Then

$$K\begin{pmatrix} x_1, \ldots, x_r \\ i_1, \ldots, i_r \end{pmatrix} \ge 0$$

and strict inequality holds if and only if $\prod_{l=1}^{r} K(x_l, i_l) > 0$. The techniques used in the proof of Theorem 2.1 do not seem to carry over to this problem.

*Added in Proof*: In the meantime the conjecture has been proved by T. N. T. Goodman and C. A. Micchelli, *On Refinement Equations Determined by Polyá Frequency Sequences*, preprint, 1990.

## 3. Corner Cutting and Total Positivity

In this section we give a geometric interpretation to our central hypothesis that the $2n \times n$ matrix

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}$$

is TP. But first we demonstrate that the variation-diminishing property of the associated fundamental curve follows easily from this condition. For this purpose, we observe that the successive control polygons generated by MSS can be described in the following way. We define inductively rectangular matrices $A^k$, $k = 0, 1, 2, \ldots$, of size $2^{k+1}n \times 2^k n$. For $k = 0$ we set $A^0 = A$ and generally

$$A^{k+1} = \begin{bmatrix} A^k & 0 \\ 0 & A^k \end{bmatrix}.$$

It follows that $A^k$ is TP whenever $A$ is TP. Next we generate successive control polygons by the formula

$$\mathbf{d}^{k+1} = A^k \mathbf{d}^k, \qquad \mathbf{d}^0 = \mathbf{c} \in \mathbf{R}^n. \tag{3.1}$$

Thus $\mathbf{d}^k = (\mathbf{d}_0^k, \ldots, \mathbf{d}_{2^k-1}^k) \in \mathbf{R}^{2^k n}$ and by construction $\mathbf{d}_l^k = A_{\varepsilon_{k-1}} \cdots A_{\varepsilon_0}\mathbf{c}$, where $l = \varepsilon_{k-1} + 2\varepsilon_{k-2} + \cdots + 2^{k-1}\varepsilon_0$, $\varepsilon_j \in \{0, 1\}$, $j = 0, 1, \ldots, k$.

Let $\Psi$ be the fundamental curve for MSS based on $A_0$ and $A_1$. *If* $r$ is any integer such that

$$S^-\left(\sum_{j=1}^{n} c_j\psi_j\right) = r,$$

we can find points $0 < t_1 < \cdots < t_{r+1} < 1$ such that the function $f(t) := \sum_{j=1}^{n} c_j\psi_j(t)$ alternates in sign thereon, i.e., $f(t_i)f(t_{i+1}) < 0$, $i = 1, \ldots, r$. We now choose integers $l_k^i$ such that $0 \le l_k^1 < \cdots < l_k^{r+1} < 2^k$, $k = 1, 2, \ldots$, and

$$\lim_{k \to \infty} \frac{l_k^i}{2^k} = t_i, \qquad i = 1, \ldots, r + 1.$$

Using the variation diminishing property of totally positive matrices, see [7], we get by (1.10) for $k$ sufficiently large,

$$r = S^-(d_{l_k^1}^k, \ldots, d_{l_k^{r+1}}^k) \le S^-(\mathbf{d}^k) \le S^-(\mathbf{c}).$$

In other words when $A$ is TP we conclude that

$$S^-\left(\sum_{j=1}^n c_j \psi_j\right) \le S^-(c_1, \ldots, c_n).$$

We now turn to the main subject of this section. We demonstrate that each step of the iteration (3.1) can be viewed as a *corner-cutting* procedure. This leads us to the factorization of rectangular TP matrices as a product of a certain type of one-banded matrices. To explain what we have in mind we recall some terminology from [6].

Given the control polygon $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_n) \in \mathbf{R}^{sn}$ *cutting $k$ corners from the right*, $1 \le k \le n - 1$, means forming the new control polygon $\mathbf{d} = (\mathbf{d}_1, \ldots, \mathbf{d}_n) \in \mathbf{R}^{sn}$ by

$$\begin{aligned} &\mathbf{d}_j = \mathbf{c}_j, \qquad j = 1, \ldots, n - k, \\ &\mathbf{d}_j = \lambda_j \mathbf{c}_{j-1} + (1 - \lambda_j)\mathbf{c}_j, \qquad j = n - k + 1, \ldots, n, \end{aligned} \tag{3.2}$$

for some $0 \le \lambda_j < 1$, $j = n - k + 1, \ldots, n$. Thus in matrix terms $\mathbf{d} = L^k\mathbf{c}$ where $L^k = (L_{ij}^k)_{i,j=1}^n$ is a *nonsingular lower triangular one-banded row stochastic matrix* with $L_{i,i-1}^k = 0$, $i = 2, \ldots, n - k$, for $k = 1, \ldots, n - 2$. Similarly, *cutting $k$ corners from the left* has the form $\mathbf{d} = U^k\mathbf{c}$ where $\mathbf{d}$ is given by

$$\begin{aligned} &\mathbf{d}_j = \mu_j \mathbf{c}_j + (1 - \mu_j)\mathbf{c}_{j+1}, \qquad j = 1, \ldots, k, \\ &\mathbf{d}_j = \mathbf{c}_j, \qquad j = k + 1, \ldots, n, \end{aligned} \tag{3.3}$$

and $0 < \mu_j \le 1$, $j = 1, \ldots, k$.

In each case above the corner-cutting matrices are square. In contrast *corner cutting from both ends* increases the number of control points by one. This procedure is defined by the equations

$$\begin{aligned} \mathbf{d}_1 &= \mathbf{c}_1, \\ \mathbf{d}_j &= \nu_j \mathbf{c}_{j-1} + (1 - \nu_j)\mathbf{c}_j, \qquad j = 2, \ldots, n, \\ \mathbf{d}_{n+1} &= \mathbf{c}_n, \end{aligned} \tag{3.4}$$

where $0 < \nu_j < 1$, $j = 2, \ldots, n$. Thus in this case $\mathbf{d} = B^n\mathbf{c}$ where $B^n$ is an $(n + 1) \times n$ matrix with two nonzero "diagonals." Our goal is to show that any rectangular TP matrix can be essentially decomposed into these basic factors.

**Theorem 3.1.** *Let $A$ be an $m \times n$, $m \ge n$, TP matrix. Then it can be factored as*

$$A = DB^m \cdots B^{n+1}L^1 \cdots L^{n-1}U^1 \cdots U^{n-1} = DBLU, \tag{3.5}$$

*where $D$ is a nonnegative $m \times m$ diagonal matrix and the other factors have the following properties. The matrices $L^k = (L^k_{ij})^n_{i,j=1}$, $k = 1, \ldots, n-1$, are lower triangular $n \times n$ one-banded stochastic matrices and $L^k_{i,i-1} = 0$, $i = 2, \ldots, n-k$ for $k \in \{1, \ldots, n-2\}$. The matrices $U^k = (U^k_{ij})^n_{i,j=1}$, $k = 1, \ldots, n-1$, are upper triangular $n \times n$ one-banded stochastic matrices and $U^k_{i-1,i} = 0$, $i = k+2, \ldots, n$, for $k \in \{1, \ldots, n-2\}$. For $k \in \{n+1, \ldots, m\}$, the matrices $B^k = (B^k_{ij})^k_{i=1}{}^{k-1}_{j=1}$ are $k \times k-1$ stochastic matrices with $B^k_{ij} = 0$, if $i \neq j$ or $i \neq j+1$.*

*Remark 3.1.* As shall be shown in the proof of Theorem 3.1, the factorization (3.5) is just one of many such factorizations. In fact, it is analytically one of the more involved. We have chosen it because of its geometric interpretation and connection with (3.2)-(3.4).

Before proving Theorem 3.1, we consider various consequences. Note that by construction $B_{ij} = 0$ for $i < j$ and $i > j + m - n$.

**Proposition 3.1.** *Assume $A$ is an $m \times n$ ($m \geq n$) TP matrix, and*

$$A = DB^m \cdots B^{n+1}L^1 \cdots L^{n-1}U^1 \cdots U^{n-1} = DBLU \tag{3.6}$$

*as in Theorem 3.1.*

(i) *$\sum_{j=1}^n A_{ij} = D_{ii}$ for $i = 1, \ldots, m$. Thus, if $A$ is stochastic, then $D = I$.*
(ii) *If $A_{1j} = \delta_{1j}, j = 1, \ldots, n$, then $U^k_{12} = 0, k = 1, \ldots, n-1$. Thus, in particular, $U^1 = I$.*
(iii) *If $\operatorname{rank} A = n$ and $A_{mj} = \delta_{nj}, j = 1, \ldots, n$, then $L^k_{n,n-1} = 0, k = 1, \ldots, n-1$. Thus, in particular, $L^1 = I$.*
(iv) *If*

$$A\begin{pmatrix}1, \ldots, n\\ 1, \ldots, n\end{pmatrix}, A\begin{pmatrix}m-n+1, \ldots, m\\ 1, \ldots, n\end{pmatrix} > 0,$$

*and $A_{ij} = 0$ for $i < j$ and $i > j + m - n$, then necessarily $L = U = I$.*

**Proof.** (i) Since each of the $B^l$, $L^k$, and $U^k$ is stochastic, it is readily verified that $\sum_{j=1}^n A_{ij} = D_{ii}$, $i = 1, \ldots, m$.

(ii) First note that since $U$ is upper triangular, we get

$$A_{11} = D_{11}(BL)_{11}U_{11}.$$

Furthermore, from (i), $A_{11} = D_{11} = 1$, and since $B$, $L$, and $U$ are stochastic, we also obtain $(BL)_{11} = U_{11} = 1$. Thus we conclude that $U_{1j} = \delta_{1j}, j = 1, \ldots, n$. Proceeding further we observe that

$$U_{11} = U^1_{11} \cdots U^{n-1}_{11}.$$

Again using the fact that each $U^k$ is stochastic, we obtain $U^k_{11} = 1, k = 1, \ldots, n-1$, and the result follows.

(iii) Since rank $A = n$, both $L$ and $U$ are nonsingular. Thus $U_{ii} > 0, i = 1, \ldots, n$. From (i) we conclude that $D_{mm} = 1$. Because $B$ is stochastic, $B_{mj} = \delta_{nj}, j = 1, \ldots, n$.

Consequently we obtain

$$\delta_{nj} = A_{mj} = \sum_{k=1}^{n} L_{nk} U_{kj}.$$

For $j = 1, \dots, n-1$,

$$0 = \sum_{k=1}^{n} L_{nk} U_{kj} \geq L_{nj} U_{jj}.$$

Thus $L_{nj} = 0$ for $j = 1, \dots, n-1$, which also implies that $L_{nn} = 1$. Now,

$$1 = L_{nn} = L_{nn}^{1} \cdots L_{nn}^{n-1}.$$

Since each $L^k$ is stochastic, we have $L_{nn}^k = 1$, $k = 1, \dots, n$, and the result follows.

(iv) Since

$$0 < A\begin{pmatrix} 1, \dots, n \\ 1, \dots, n \end{pmatrix} = (DBL)\begin{pmatrix} 1, \dots, n \\ 1, \dots, n \end{pmatrix} U\begin{pmatrix} 1, \dots, n \\ 1, \dots, n \end{pmatrix}$$

and $DBL$ is TP, we get $(DBL)_{ii} > 0$, $i = 1, \dots, n$. For $i < j$,

$$0 = A_{ij} \geq (DBL)_{ii} U_{ij}.$$

Thus $U_{ij} = 0$ for $i < j$, i.e., $U = I$. Now,

$$0 < A\begin{pmatrix} m-n+1, \dots, m \\ 1, \dots, n \end{pmatrix} = (DB)\begin{pmatrix} m-n+1, \dots, m \\ 1, \dots, n \end{pmatrix} L\begin{pmatrix} 1, \dots, n \\ 1, \dots, n \end{pmatrix}$$

and thus $(DB)_{i+m-n,i} > 0$, $i = 1, \dots, n$. For $1 \leq j < i \leq n$,

$$0 = A_{i+m-n,j} \geq (DB)_{i+m-n,i} L_{ij}.$$

Thus $L_{ij} = 0$ for $j < i$, i.e., $L = I$. ■

Proposition 3.1 leads us to the following result concerning matrices considered in Theorem 2.1.

**Corollary 3.1.** *Let $A_0$, $A_1$ be nonsingular $n \times n$ stochastic matrices such that*

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix}$$

*is TP. Suppose further that the first row of $A_0$ is $(1, 0, \dots, 0)$, the last row of $A_1$ is $(0, \dots, 0, 1)$, and the last row of $A_0$ and the first row of $A_1$ are the same. Then*

$$A = B^{2n} \cdots B^{n+1} L^2 \cdots L^{n-1} U^2 \cdots U^{n-1},$$

*where the $B^l$, $L^k$, and $U^k$ are as in Theorem 3.1. Furthermore, $U_{12}^k = L_{n,n-1}^k = 0$, $k = 2, \dots, n-1$, while $B_{ii}^{2n} = 1$, $i = 1, \dots, n$, $B_{i,i-1}^{2n} = 1$, $i = n+1, \dots, 2n$ (and $B_{ij}^{2n}$ is zero elsewhere).*

**Proof.** Let $\tilde{A}$ be the $2n - 1 \times n$ matrix obtained from $A$ by deleting the $n$th or

$(n+1)$st row (which are the same). Apply Theorem 3.1 and Proposition 3.2(i)–(iii) to $\tilde{A}$. For $B^{2n}$ as above, $A = B^{2n}\tilde{A}$. ∎

In proving Theorem 3.1, we first recall how to factor $r \times r$ *strictly* totally positive (STP) matrices. In doing so we review certain results from [3], [5], and [8].

Let $A$ be an $r \times r$ STP matrix, i.e., all minors of $A$ are strictly positive. Then, as is well known, $A$ can be written in the form

$$A = LDU, \tag{3.7}$$

where $L$ is a unit diagonal lower triangular matrix, $D$ is a strictly positive diagonal matrix, and $U$ is a unit diagonal upper triangular matrix. In fact $L$, $D$, and $U$ are explicitly given by

$$L_{ij} = \begin{cases} 0, & i < j, \\ A\begin{pmatrix} 1, \ldots, j-1, i \\ 1, \ldots, j-1, j \end{pmatrix} \Big/ A\begin{pmatrix} 1, \ldots, j \\ 1, \ldots, j \end{pmatrix}, & i \geq j \end{cases}$$

$$D_{ij} = \begin{cases} 0, & i \neq j, \\ A\begin{pmatrix} 1, \ldots, i \\ 1, \ldots, i \end{pmatrix} \Big/ A\begin{pmatrix} 1, \ldots, i-1 \\ 1, \ldots, i-1 \end{pmatrix}, & i = j \end{cases}$$

and

$$U_{ij} = \begin{cases} A\begin{pmatrix} 1, \ldots, i-1, i \\ 1, \ldots, i-1, j \end{pmatrix} \Big/ A\begin{pmatrix} 1, \ldots, i \\ 1, \ldots, i \end{pmatrix}, & i \leq j, \\ 0, & i > j. \end{cases}$$

As it turns out, if $A$ is STP then both $L$ and $U$ are TP. Even more, from Cryer [5] we know that $L$ and $U$ are what Cryer calls $\Delta$STP, i.e.,

$$L\begin{pmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{pmatrix} > 0$$

if and only if $i_1 \geq j_1, \ldots, i_k \geq j_k$, while

$$U\begin{pmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{pmatrix} > 0$$

if and only if $i_1 \leq j_1, \ldots, i_k \leq j_k$. This result is also a consequence of what we prove below.

Since $L$ and $U$ are triangular (unit diagonal) matrices, each can be factored in many ways as a product of $r-1$ unit diagonal one-banded matrices. One such factorization is given on p. 167 of [8].

We restrict our remarks to $L$. Parallel arguments apply to $U$.

**Proposition 3.2.** *Let $L$ be a unit diagonal $r \times r$ lower triangular $\Delta STP$ matrix. Then,*

$$L = \hat{L}^1 \cdots \hat{L}^{r-1} = \tilde{L}^{r-1} \cdots \tilde{L}^1,$$

*where each $\hat{L}^k$, $\tilde{L}^k$ is a one-banded unit diagonal lower triangular matrix such that*

$$\begin{aligned}&\hat{L}^k_{i,i-1} = 0, \qquad i = 2,\ldots,r-k, \quad k = 1,\ldots,r-2,\\ &\hat{L}^k_{i,i-1} > 0, \qquad i = r-k+1,\ldots,r, \quad k = 1,\ldots,r-1,\end{aligned} \tag{3.8}$$

*and*

$$\begin{aligned}&\tilde{L}^k_{i,i-1} > 0, \qquad i = 2,\ldots,k+1, \quad k = 1,\ldots,r-1,\\ &\tilde{L}^k_{i,i-1} = 0, \qquad i = k+2,\ldots,r, \quad k = 1,\ldots,r-2.\end{aligned} \tag{3.9}$$

**Proof.** At the first stage of the factorization process we eliminate (make zero) the $(r, 1)$ element of $L$ by using either the $(r-1)$st row or the 2nd column of $L$. In this way we express $L$ either as

$$L = \hat{L}^1 L^*,$$

where

$$\hat{L}^1 = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \vdots \\ 0 & \cdots & 0 & x & 1 \end{bmatrix}, \qquad L^* = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ x & 1 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ x & \cdots & x & 1 & \vdots \\ 0 & x & \cdots & x & 1 \end{bmatrix}$$

(by using the $(r-1)$st row), or as

$$L = L^{**}\tilde{L}^1,$$

where

$$L^{**} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ x & 1 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ x & \cdots & x & 1 & \vdots \\ 0 & x & \cdots & x & 1 \end{bmatrix}, \qquad \tilde{L}^1 = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ x & 1 & \cdots & \cdots & \vdots \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

(by using the second column). This process can be repeated. At every stage we use either row or column elimination to eliminate successive off-diagonals, starting from the left. For our purposes here, we use either row elimination or column elimination throughout. In this way, by *row elimination* we get

$$L = \hat{L}^1 \cdots \hat{L}^{r-1},$$

where $\hat{L}^k$ is a unit diagonal one-banded lower triangular matrix satisfying

$$\hat{L}^k_{i,i-1} = 0, \qquad i = 2,\ldots,r-k, \quad k = 1,\ldots,r-2.$$

Using *column elimination* we obtain

$$L = \tilde{L}^{r-1} \cdots \tilde{L}^1,$$

where $\tilde{L}^k$ is a unit diagonal one-banded lower triangular matrix satisfying

$$\tilde{L}^k_{i,i-1} = 0, \qquad i = k+2, \ldots, r, \quad k = 1, \ldots, r-2.$$

It remains to show that all possible nonzero elements of $\hat{L}^k$ and $\tilde{L}^k$ are in fact positive. It actually suffices to assume that

$$L\begin{pmatrix} l+1, \ldots, l+k \\ 1, \ldots, k \end{pmatrix} > 0 \qquad l = 1, \ldots, r-k, \quad k = 1, \ldots, r-1.$$

Because of the zero elements in $\hat{L}^k$, $k = 1, \ldots, r-1$, we readily see that

$$0 < L_{j+1,1} = \hat{L}^{r-j}_{j+1,j} \cdots \hat{L}^{r-1}_{2,1}, \qquad j = 1, \ldots, r-1.$$

Therefore, starting with $j = 1$ and proceeding successively we conclude that

$$\hat{L}^{r-1}_{2,1}, \hat{L}^{r-2}_{3,2}, \ldots, \hat{L}^{1}_{r,r-1} > 0.$$

Thus we have shown that the element of each $\hat{L}^k$, $k = 1, \ldots, r-1$, on the secondary diagonal after the last zero, is positive. Now we proceed one layer down the secondary diagonal by considering $2 \times 2$ minors of $L$. Since

$$0 < L\begin{pmatrix} l+1, l+2 \\ 1, 2 \end{pmatrix} = \hat{L}^{r-l}\begin{pmatrix} l+1, l+2 \\ l, l+1 \end{pmatrix} \cdots \hat{L}^{r-1}\begin{pmatrix} 2, 3 \\ 1, 2 \end{pmatrix}, \qquad l = 1, 2, \ldots, r-2,$$

we get

$$\hat{L}^{r-1}\begin{pmatrix} 2, 3 \\ 1, 2 \end{pmatrix}, \ldots, \hat{L}^{2}\begin{pmatrix} r-1, r \\ r-2, r-1 \end{pmatrix} > 0.$$

Based on what we have already proved and since

$$\hat{L}^{r-l}\begin{pmatrix} l+1, l+2 \\ l, l+1 \end{pmatrix} = \hat{L}^{r-l}_{l+1,l}\hat{L}^{r-l}_{l+2,l+1},$$

we conclude that

$$\hat{L}^{r-1}_{3,2}, \ldots, \hat{L}^{2}_{r,r-1} > 0.$$

We continue in this manner. Finally, to show that $\hat{L}^{r-1}_{r,r-1} > 0$ we use the $r-1 \times r-1$ minor of $L$ and

$$0 < L\begin{pmatrix} 2, \ldots, r \\ 1, \ldots, r-1 \end{pmatrix} = \hat{L}^{r-1}\begin{pmatrix} 2, \ldots, r \\ 1, \ldots, r-1 \end{pmatrix} = \hat{L}^{r-1}_{2,1} \cdots \hat{L}^{r-1}_{r,r-1}.$$

These facts also follow from explicit formulas for the entries of the factors $\hat{L}^1, \ldots, \hat{L}^{r-1}$ in terms of minors of $L$. However, as this is not important to use here we do not elaborate on this point.

These same arguments applied to the factorization

$$L = \tilde{L}^{r-1} \cdots \tilde{L}^1$$

give us

$$\tilde{L}^k_{i,i-1} > 0, \qquad i = 2, \ldots, k+1,$$

for all $k \in \{1, \ldots, r-1\}$. ∎

For easy reference we state the analogous factorizations of $U$. These easily follow by parallel arguments, or more simply by applying Proposition 3.2 to the lower triangular matrix $U^T$.

**Corollary 3.2.** *Let $U$ be a unit diagonal $r \times r$ upper triangular $\Delta STP$ matrix. Then*

$$U = \hat{U}^{r-1} \cdots \hat{U}^1 = \tilde{U}^1 \cdots \tilde{U}^{r-1},$$

*where each $\hat{U}^k$, $\tilde{U}^k$ is a one-banded unit diagonal upper triangular matrix, and*

$$\begin{aligned} &\hat{U}^k_{i-1,i} = 0, \quad && i = 2, \ldots, r-k, \quad k = 1, \ldots, r-2, \\ &\hat{U}^k_{i-1,i} > 0, \quad && i = r-k+1, \ldots, r, \quad k = 1, \ldots, r-1, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\tilde{U}^k_{i-1,i} > 0, \quad && i = 2, \ldots, k+1, \quad k = 1, \ldots, r-1, \\ &\tilde{U}^k_{i-1,i} = 0, \quad && i = k+2, \ldots, r, \quad k = 1, \ldots, r-2. \end{aligned} \tag{3.11}$$

If $A$ is merely TP, then there exists, for each $\varepsilon > 0$, an $r \times r$ STP matrix $A_\varepsilon$ such that

$$\lim_{\varepsilon \to 0^+} A_\varepsilon = A.$$

We can factor $A_\varepsilon$ as above in (3.7)

$$A_\varepsilon = L_\varepsilon D_\varepsilon U_\varepsilon.$$

By premultiplying $U_\varepsilon$ by a diagonal matrix and postmultiplying $L_\varepsilon$ by a diagonal matrix we can assume that $L_\varepsilon$ is $\Delta$STP with column sums one, $U_\varepsilon$ is $\Delta$STP with row sums one, and $D_\varepsilon$ is a positive diagonal matrix. It therefore follows that

$$\sum_{i,j=1}^{r} (A_\varepsilon)_{ij} = \sum_{i=1}^{r} (D_\varepsilon)_{ii}.$$

Since all elements of $L_\varepsilon$, $D_\varepsilon$, and $U_\varepsilon$ are now uniformly bounded, we can extract convergent subsequences to obtain

$$A = LDU.$$

The factors $L$, $D$, and $U$ are now only assured of being TP, as zero elements may result from the limiting process. The same argument applies to the factorization of $A_\varepsilon$ into one-banded factors. Thus in the limit we can write any $r \times r$ TP matrix as

$$A = \hat{L}^1 \cdots \hat{L}^{r-1} D \hat{U}^{r-1} \cdots \hat{U}^1$$

or

$$A = \tilde{L}^{r-1} \cdots \tilde{L}^1 D \tilde{U}^1 \cdots \tilde{U}^{r-1}$$

or

$$A = \tilde{L}^{r-1} \cdots \tilde{L}^1 D \hat{U}^{r-1} \cdots \hat{U}^1$$

or

$$A = \hat{L}^1 \cdots \hat{L}^{r-1} D \tilde{U}^1 \cdots \tilde{U}^{r-1}.$$

Here the $\tilde{U}^k$, $\hat{U}^k$ are row stochastic, while the $\tilde{L}^k$, $\hat{L}^k$ are column stochastic.

Four additional factorizations are possible by considering the inverse of $A$. Specifically, if $A$ is STP then so is $EA^{-1}E$ where $E = \mathrm{diag}(1, -1, \ldots, (-1)^{n-1})$. Thus we may factor $A^{-1}$ as

$$A^{-1} = ELDUE,$$

where $L$ and $U$ are $\Delta$STP and therefore $A = (EU^{-1}E)D^{-1}(EL^{-1}E)$. Now, both $EU^{-1}E$ and $EL^{-1}E$ can each be factored as above in two ways as products of one-banded factors. Note that what is obtained is a $UDL$ factorization. For a matrix $A$ which is only TP the limiting argument used above also applies to these four factorizations.

With this background in place, we now turn to the proof of Theorem 3.1.

**Proof of Theorem 3.1.** Assume $A = (A_{ij})_{i=1}^{m}{}_{j=1}^{n}$, is an $m \times n$ $(m \geq n)$ TP matrix. Let $\tilde{A} = (\tilde{A}_{ij})_{i,j=1}^{m}$, where

$$\tilde{A}_{ij} = \begin{cases} A_{ij}, & j \leq n, \\ 0, & j > n. \end{cases}$$

Let $\tilde{A}_\varepsilon$ be an $m \times m$ STP matrix such that $\lim_{\varepsilon \to 0^+} \tilde{A}_\varepsilon = \tilde{A}$. We factor $\tilde{A}_\varepsilon$ in the form

$$\tilde{A}_\varepsilon = \tilde{L}_\varepsilon \tilde{U}_\varepsilon, \tag{3.12}$$

where $\tilde{L}_\varepsilon$ is a unit diagonal lower triangular $\Delta$STP matrix and $\tilde{U}_\varepsilon$ is an upper triangular $\Delta$STP matrix. Let $L_\varepsilon$ be the submatrix of $\tilde{L}_\varepsilon$ composed of its $m$ rows and first $n$ columns. Let $U_\varepsilon$ be the matrix obtained from the first $n$ rows and columns of $\tilde{U}_\varepsilon$, while $A_\varepsilon$ is the matrix obtained from the $m$ rows and first $n$ columns of $\tilde{A}_\varepsilon$. We claim that $A_\varepsilon = L_\varepsilon U_\varepsilon$. To see this we recall that $(\tilde{U}_\varepsilon)_{kj} = 0$ if $k > j$. Thus for $j \leq n$,

$$(A_\varepsilon)_{ij} = (\tilde{A}_\varepsilon)_{ij} = \sum_{k=1}^{m} (\tilde{L}_\varepsilon)_{ik}(\tilde{U}_\varepsilon)_{kj} = \sum_{k=1}^{n} (\tilde{L}_\varepsilon)_{ik}(\tilde{U}_\varepsilon)_{kj} = \sum_{k=1}^{n} (L_\varepsilon)_{ik}(U_\varepsilon)_{kj}.$$

$U_\varepsilon$ is an upper triangular $\Delta$STP $n \times n$ matrix and can be factored in the two ways previously mentioned. We now work with $L_\varepsilon$.

From our previous analysis,

$$\tilde{L}_\varepsilon = \tilde{L}_\varepsilon^{m-1} \cdots \tilde{L}_\varepsilon^{1},$$

where each $\tilde{L}_\varepsilon^k$ is a unit diagonal one-banded $m \times m$ lower triangular matrix satisfying (3.9). We let $L_\varepsilon^k$ be the restriction of $\tilde{L}_\varepsilon^k$ to its first $n$ rows and columns, for $k = 1, \ldots, n-1$. Similarly, $B_\varepsilon^k$, $k = n+1, \ldots, m$, is the restriction of $\tilde{L}_\varepsilon^{k-1}$ to its first $k$ rows and $k-1$ columns. Thus $(B_\varepsilon^k)_{ii}$, $(B_\varepsilon^k)_{i+1,i} > 0$, while $(B_\varepsilon^k)_{ij} = 0$ for $i \notin \{j, j+1\}$. Let us show that

$$L_\varepsilon = B_\varepsilon^m \cdots B_\varepsilon^{n+1} L_\varepsilon^{n-1} \cdots L_\varepsilon^{1}. \tag{3.13}$$

Since $\tilde{L}_\varepsilon = \tilde{L}_\varepsilon^{m-1} \cdots \tilde{L}_\varepsilon^{1}$ we have for $j \leq n$

$$(L_\varepsilon)_{ij} = (\tilde{L}_\varepsilon)_{ij} = \sum_{k_1, \ldots, k_{m-2}} (\tilde{L}_\varepsilon^{m-1})_{ik_1}(\tilde{L}_\varepsilon^{m-2})_{k_1k_2} \cdots (\tilde{L}_\varepsilon^{1})_{k_{m-2},j}.$$

Suppose $k_1, k_2, \ldots, k_{m-2}$ gives a nonzero product in the above sum. Since $(\tilde{L}_\varepsilon^{n-1} \cdots \tilde{L}_\varepsilon^1)_{ij} = 0$ if $i > j$ and $i > n$, it follows that $k_{m-n} \leq n$. Furthermore, since each above factor is nonzero, we have $k_i - 1 \leq k_{i+1} \leq k_i$. Thus, in particular, $k_{m-n}, \ldots, k_{m-2}$ are all less than $n$, and $k_{m-n-1} \leq n + 1$, $k_{m-n-2} \leq n + 2, \ldots, k_1 \leq m - 1$. Thus, by our definition, (3.13) follows. Combining this conclusion with our previous analysis, we get the factorization

$$A_\varepsilon = B_\varepsilon^m \cdots B_\varepsilon^{n+1} \hat{L}_\varepsilon D_\varepsilon U_\varepsilon,$$

where each $B_\varepsilon^k$ is as defined above (with unit upper diagonal), $\hat{L}_\varepsilon$ is an $n \times n$ ΔSTP lower triangular unit diagonal matrix, $D_\varepsilon$ is a positive diagonal $n \times n$ matrix, and $U_\varepsilon$ is an $n \times n$ ΔSTP upper triangular unit diagonal matrix.

We obtain the factorization

$$A_\varepsilon = B_\varepsilon^m \cdots B_\varepsilon^{n+1} \hat{L}_\varepsilon^1 \cdots \hat{L}_\varepsilon^{n-1} D_\varepsilon \tilde{U}_\varepsilon^1 \cdots \tilde{U}_\varepsilon^{n-1}$$

by applying (3.8) to $\hat{L}_\varepsilon$, and (3.11) to $U_\varepsilon$. Note that $(\hat{L}_\varepsilon^k)_{i,i-1} = 0$, $i = 2, \ldots, n - k$, for $k \in \{1, \ldots, n - 2\}$, while $(\tilde{U}_\varepsilon^k)_{i-1,i} = 0$, $i = k + 2, \ldots, n$ for $k \in \{1, \ldots, n - 2\}$, as is desired. Since the row sums of all these factors are nonzero, we can rewrite $A_\varepsilon$ as

$$A_\varepsilon = \bar{D}_\varepsilon \bar{B}_\varepsilon^m \cdots \bar{B}_\varepsilon^{n+1} \bar{L}_\varepsilon^1 \cdots \bar{L}_\varepsilon^{n-1} \bar{U}_\varepsilon^1 \cdots \bar{U}_\varepsilon^{n-1}, \tag{3.14}$$

where now the $\bar{B}_\varepsilon^k$, $\bar{L}_\varepsilon^k$, and $\bar{U}_\varepsilon^k$ are all stochastic. This is easily done by pre- and postmultiplying by diagonal matrices. Note that $\sum_{j=1}^n (A_\varepsilon)_{ij} = (\bar{D}_\varepsilon)_{ii}$, $i = 1, \ldots, m$, so that all entries in each of the factors are uniformly bounded. We now pass to the limit ($\varepsilon \to 0^+$) in (3.14) through a subsequence and verify (3.5). ∎

*Remark 3.2.* One of the other factorizations of $A$ satisfying the conditions of Theorem 3.1 is worth mentioning. Such $A$ may also be factored in the form

$$A = DL^1 \cdots L^{n-1} B^m \cdots B^{n+1} U^1 \cdots U^{n-1},$$

where $D$ and the $U^k$ are as in Theorem 3.1, the $L^k$ are $m \times m$ lower triangular one-banded stochastic matrices with

$$L_{i,i-1}^k = 0, \qquad i = 2, \ldots, m - k,$$

while the $B^k$ are $k \times k - 1$ stochastic matrices with $B_{ij}^k = 0$ if $i \neq j$ or $i \neq j + 1$. If $A$ is as in Theorem 2.1, then $D = L^1 = U^1 = I$.

## 4. Reparametrized Bernstein Polynomials

We end this paper with a comment concerning some specific corner-cutting strategies which provide concrete variations on de Casteljau's method mentioned in the introduction. Beginning with an initial control polygon $\mathbf{c}^0 = (\mathbf{c}_0^0, \ldots, \mathbf{c}_m^0)$, we form the weighted averages

$$\mathbf{c}_r^l = (1 - x)\mathbf{c}_r^{l-1} + x\mathbf{c}_{r+1}^{l-1}, \qquad r = 0, 1, \ldots, m - l, \quad l = 1, \ldots, m, \tag{4.1}$$

where $x$ is any number chosen in the interval $(0, 1)$. The case $x = \frac{1}{2}$ is de Casteljau's method (1.1). In the general case the $(m + 1) \times (m + 1)$ matrices for the corresponding MSS may be identified as

$$(A_0(x))_{ij} := \binom{i}{j}(1-x)^{i-j}x^j, \qquad i,j = 0, 1, \ldots, m,$$

where

$$\binom{i}{j} = 0 \qquad \text{if} \quad j > i,$$

and

$$A_1(x) := P_m A_0(1-x) P_m, \tag{4.2}$$

where $P_m$ is the permutation matrix defined by $(P_m)_{ij} := \delta_{i,m-j}$, $i, j = 0, \ldots, m$. (Note that these reduce to (1.6) and (1.7) when $x = \frac{1}{2}$.) Thus we have

$$(\mathbf{c}_0^0, \ldots, \mathbf{c}_0^m) = A_0(x)\mathbf{c}^0$$

and

$$(\mathbf{c}_0^m, \ldots, \mathbf{c}_m^0) = A_1(x)\mathbf{c}^0.$$

To identify the limiting curve we introduce the vector

$$\boldsymbol{\mu}(\gamma) := (\gamma_1^m, \gamma_1^{m-1}\gamma_2, \ldots, \gamma_1\gamma_2^{m-1}, \gamma_2^m)$$

in $\mathbf{R}^{m+1}$, where $\gamma := (\gamma_1, \gamma_2)$ is an arbitrary real vector in $\mathbf{R}^2$. Note that $\{\boldsymbol{\mu}(\gamma) : \gamma \in \mathbf{R}^2\}$ span $\mathbf{R}^{m+1}$. It follows directly that

$$A_0(x)\boldsymbol{\mu}(\gamma) = \boldsymbol{\mu}(T_0(x)\gamma),$$

where

$$T_0(x) := \begin{pmatrix} 1 & 0 \\ 1-x & x \end{pmatrix}$$

and similarly by (4.2),

$$A_1(x)\boldsymbol{\mu}(\gamma) = \boldsymbol{\mu}(T_1(x)\gamma),$$

where

$$T_1(x) := P_1 T_0(1-x) P_1.$$

The matrices $T_0(x)$ and $T_1(x)$ are stochastic and have a positive column. Hence by Theorem 1.1 there is a continuous fundamental curve $\Phi(\cdot|x)$: $[0, 1] \to \mathbf{R}^2$ defined by the MSS determined by $T_0$ and $T_1$,

$$\lim_{k\to\infty} T_{\varepsilon_k} \cdots T_{\varepsilon_1}\gamma = (\gamma, \Phi(t|x))\mathbf{e}_2, \qquad t = \sum_{j=1}^{\infty} \varepsilon_j 2^{-j}, \qquad \mathbf{e}_m := (1, \ldots, 1)^T \in \mathbf{R}^m. \tag{4.3}$$

Note that for $x = \frac{1}{2}$ the functional equation for $\Phi(\cdot|\frac{1}{2})$ shows that

$$\Phi(t|\tfrac{1}{2}) = (1-t, t).$$

Let us denote by $\Psi(\cdot|x)$ the limiting curve for the MSS based on the matrices $A_0(x)$

and $A_1(x)$. (Theorem 1.1 also assures the existence of $\Psi(\cdot|x)$ as a continuous curve.) Then we have from (4.3)

$$\begin{aligned}(\boldsymbol{\mu}(\gamma), \Psi(t|x))\mathbf{e}_m &= \lim_{k\to\infty} A_{\varepsilon_k}(x)\cdots A_{\varepsilon_1}(x)\boldsymbol{\mu}(\gamma)\\ &= \lim_{k\to\infty} \boldsymbol{\mu}(T_{\varepsilon_k}(x)\cdots T_{\varepsilon_1}(x)\gamma)\\ &= \boldsymbol{\mu}((\gamma, \Phi(t|x))\mathbf{e}_2) = (\gamma, \Phi(t|x))^m\mathbf{e}_m.\end{aligned}$$

Consequently, by the Binomial Theorem we get

$$\Psi(t|x) = \Psi^b(\Phi(t|x)),$$

where

$$\Psi^b(\gamma_1, \gamma_2) = \left(\binom{m}{0}\gamma_1^m, \binom{m}{1}\gamma_1^{m-1}\gamma_2, \ldots, \binom{m}{m}\gamma_2^m\right)$$

is the Bernstein polynomial curve represented in terms of homogeneous coordinates.

This example has wider implications beyond the geometrically apparent corner-cutting procedure (4.1). We have in mind the following: for any $\gamma \in \mathbf{R}^{s+1}$ we set $\gamma^\alpha := \gamma_1^{\alpha_1}\cdots\gamma_{s+1}^{\alpha_{s+1}}$, $|\alpha| := \alpha_1 + \cdots + \alpha_{s+1}$, $\alpha = (\alpha_1, \ldots, \alpha_{s+1}) \in \mathbf{Z}_+^{s+1}$, and form the vector

$$\boldsymbol{\mu}(\gamma) := (\gamma^\alpha : |\alpha| = m) \in \mathbf{R}^N, \qquad N = \binom{m+s}{s}.$$

For every $(s+1) \times (s+1)$ matrix $T$ we define an $N \times N$ matrix by the equation

$$\boldsymbol{\mu}(T\gamma) = A\boldsymbol{\mu}(\gamma). \tag{4.4}$$

Note that the coordinates of $\boldsymbol{\mu}(\gamma)$ span all homogeneous polynomials of degree $m$ on $\mathbf{R}^{s+1}$. Since each coordinate of $\boldsymbol{\mu}(T\gamma)$ is a homogeneous polynomial, the matrix $A$ is well defined by (4.4).

There is an elegant interpretation of the process of passing from the matrix $T$ to the matrix $A$ by using the notion of the permanent of a matrix. For every $\alpha$, $\beta \in \mathbf{Z}_+^{s+1}$ with $|\alpha| = |\beta| = m$ we form the $m \times m$ matrix $T(\alpha, \beta)$ by repeating $\alpha_i$, $\beta_j$ times the $i$th row and $j$th column of $T$, respectively. Then $A = (\beta!^{-1} \operatorname{per} T(\alpha, \beta) : |\alpha| = |\beta| = m,\ \alpha, \beta \in \mathbf{Z}_+^{s+1})$, that is, for every $\gamma \in \mathbf{R}^{s+1}$

$$(T\gamma)^\alpha = \sum_{|\beta| = m} \frac{\gamma^\beta}{\beta!} \operatorname{per} T(\alpha, \beta), \qquad |\alpha| = m, \quad \alpha \in \mathbf{Z}_+^{s+1}, \quad \beta! := \beta_1! \cdots \beta_{s+1}!.$$

This formula follows from the well-known formula which has some relation to the MacMahon Master theorem (see [1]).

**Theorem 4.1.** *For any $k \times l$ matrix $C$*

$$(C\mathbf{x})_1 \cdots (C\mathbf{x})_k = \sum_{\substack{|\beta| = k\\ \beta \in \mathbf{Z}_+^l}} \frac{\mathbf{x}^\beta}{\beta!} \operatorname{per} C(\beta), \qquad \mathbf{x} \in \mathbf{R}^l,$$

*where $C(\beta)$ is the $k \times k$ matrix obtained by taking $\beta_i$ copies of the* ith *column of $C$.*

Pick any two $(s+1) \times (s+1)$ stochastic matrices $T_0$ and $T_1$ which satisfy the hypothesis of Theorem 2.1 and suppose $\Phi: [0, 1] \to \mathbf{R}^{s+1}$ is its corresponding continuous fundamental curve. Let $A_0, A_1$ be the corresponding $N \times N$ matrices defined by (4.4) and suppose that

$$\Psi_s^b(\lambda) = \left(\binom{m}{\alpha}\lambda^\alpha : \alpha \in \mathbf{Z}_+^{s+1}, |\alpha| = m\right)$$

is the multivariate Bernstein polynomials in homogeneous coordinates $\lambda = (\lambda_1, \ldots, \lambda_{s+1})$. Then the Binomial Theorem in $s+1$ variables takes the form

$$(\gamma, \lambda)^m = (\boldsymbol{\mu}(\gamma), \Psi_s^b(\lambda)).$$

Thus, for every $\gamma \in \mathbf{R}^{s+1}$ and $t = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k} \in [0, 1]$, we have

$$\begin{aligned}\lim_{k\to\infty} A_{\varepsilon_k} \cdots A_{\varepsilon_1}\boldsymbol{\mu}(\gamma) &= \lim_{k\to\infty} \boldsymbol{\mu}(T_{\varepsilon_k} \cdots T_{\varepsilon_1}\gamma)\\ &= \boldsymbol{\mu}((\gamma, \Phi(t))\mathbf{e}_{s+1})\\ &= (\gamma, \Phi(t))^m \mathbf{e}_N\\ &= (\boldsymbol{\mu}(\gamma), \Psi_s^b(\Phi(t)))\mathbf{e}_N.\end{aligned}$$

Consequently, the MSS based on the matrices $A_0, A_1$ converge to the curve $\Psi_s^b(\Phi(t))$. The fundamental curve $\Phi$ for MSS based on $T_0$ and $T_1$ furnishes a continuous imbedding of $[0, 1]$ into the standard simplex $\{\lambda: \lambda = (\lambda_1, \ldots, \lambda_{s+1}),$ $\lambda_i \geq 0, \sum_{i=1}^{s+1} \lambda_i = 1\}$, and $\Phi(t)$ provide barycentric coordinates at which to evaluate the multivariate Bernstein polynomials.

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