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Extending n-convex functions

by

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Abstract. We are given data $\alpha_1, \ldots, \alpha_m$ and a set of points $E = \{x_1, \ldots, x_m\}$. We address the question of conditions ensuring the existence of a function f satisfying the interpolation conditions $f(x_i) = \alpha_i$, $i = 1, \ldots, m$, that is also n-convex on a set properly containing E. We consider both one-point extensions of E, and extensions to all of $\mathbb{R}$. We also determine bounds on the n-convex functions satisfying the above interpolation conditions.

1. Introduction. A function f defined on a set E in $\mathbb{R}$ is said to be *n-convex* on E if for every choice of $n+1$ distinct points $\{x_i\}_{i=1}^{n+1}$ in E the $n+1$st divided difference on these points is nonnegative. This divided difference may be formally defined by

$$[x_1, \ldots, x_{n+1}; f] := \frac{\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & \ddots & \vdots \\ x_1^{n-1} & \cdots & x_{n+1}^{n-1} \\ f(x_1) & \cdots & f(x_{n+1}) \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & \ddots & \vdots \\ x_1^{n-1} & \cdots & x_{n+1}^{n-1} \\ x_1^n & \cdots & x_{n+1}^n \end{vmatrix}}. \tag{1.1}$$

Alternatively we can set

$$[x; f] = f(x),$$

and

$$[x_1, \ldots, x_{k+1}; f] = \frac{[x_2, \ldots, x_{k+1}; f] - [x_1, \ldots, x_k; f]}{x_{k+1} - x_1},$$

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$k = 1, 2, \ldots$, and obtain $[x_1, \ldots, x_{n+1}; f]$. As the denominator in (1.1) is the Vandermonde determinant and equals $\prod_{1\le i<j\le n+1}(x_j - x_i)$, a function f is n-convex on E if and only if

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+1} \\ \vdots & \ddots & \vdots \\ x_1^{n-1} & \cdots & x_{n+1}^{n-1} \\ f(x_1) & \cdots & f(x_{n+1}) \end{vmatrix} \ge 0 \qquad \text{for all } x_1 < \cdots < x_{n+1} \text{ in } E.$$

From this definition we see that a function is 0-convex if it is nonnegative, 1-convex if it is nondecreasing, and 2-convex if it is convex in the usual sense. n-Convexity for $n \ge 3$ on an interval was first considered by Eberhard Hopf [10] in his dissertation, and was rather extensively developed by Popoviciu in his thesis [17] and in his monograph [19].

The central questions we will address in this paper are the following. Assume we are given data $\alpha_1, \ldots, \alpha_m$, a finite set of points $E = \{x_1, \ldots, x_m\}$, and an n-convex function f on E satisfying $f(x_i) = \alpha_i$, $i = 1, \ldots, m$. What conditions ensure that f can be extended to an n-convex function on a given set that properly contains E? In this paper we consider in detail one-point extensions of E, and extensions to all of $\mathbb{R}$. For f to be n-convex on $E = \{x_1, \ldots, x_m\}$ it is obviously necessary that

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & \ddots & \vdots \\ x_{i_1}^{n-1} & \cdots & x_{i_{n+1}}^{n-1} \\ \alpha_{i_1} & \cdots & \alpha_{i_{n+1}} \end{vmatrix} \ge 0 \qquad \text{for all } i_1 < \cdots < i_{n+1} \text{ in } \{1, \ldots, m\}. \tag{1.2}$$

These, and in fact fewer, necessary conditions also suffice for the existence of extensions of n-convex f to all of $\mathbb{R}$, when $n = 0, 1, 2$, and for all n if $m \le n+2$. However, for $n \ge 3$ and $m \ge n+3$ these conditions are not sufficient for either of the above-mentioned extension problems. This problem was first considered by Popoviciu in [18]. He fully solved it for one-point extensions of E. We will explain his result in Sections 3 and 4. Popoviciu did not solve the problem for extensions to all of $\mathbb{R}$. This latter problem will be solved here, but not to our complete satisfaction since the characterization we obtain depends upon the existence of Borel measures with certain properties and is not constructive (see Section 6). Previously, the only related problem that had been fully solved is the problem for data

$$f^{(i)}(a) = \alpha_i, \qquad f^{(i)}(b) = \beta_i, \qquad i = 0, 1, \ldots, n-1. \tag{1.3}$$

Conditions, albeit not easily verified, on the $\{\alpha_i\}_{i=0}^{n-1}$ and $\{\beta_i\}_{i=0}^{n-1}$ such that there exists an n-convex f on $[a, b]$ satisfying (1.3) were given by Kakeya

[11]. The other problem we will consider in this paper is that of determining bounds on n-convex functions f satisfying $f(x_i) = \alpha_i$, $i = 1, \ldots, m$. This is addressed in Section 7 where we generalize results of Burchard [6].

2. Properties of n-convex functions. In this section we survey some known, but seemingly not sufficiently well known, properties of n-convex functions.

We start with some equivalent definitions of n-convexity. A function f is nondecreasing (1-convex) on (a, b) if and only if for any constant function p satisfying $f(x_1) = p(x_1)$ we have $f(x) \geq p(x)$ for $x \in [x_1, b)$, while $f(x) \leq p(x)$ for $x \in (a, x_1]$, for all $x_1 \in (a, b)$. Similarly, f is convex (2-convex) on (a, b) if and only if for every $a < x_1 < x_2 < b$ the linear polynomial p for which $f(x_i) = p(x_i)$, $i = 1, 2$, satisfies $f(x) \geq p(x)$ for $x \in (a, x_1] \cup [x_2, b)$, and $f(x) \leq p(x)$ for $x \in [x_1, x_2]$. The exact similar result for n-convex functions was proved by Hopf [10] and by Popoviciu [17], and later by others. Assume we are given $a < x_1 < \cdots < x_n < b$. Let $P(x_1, \ldots, x_n; x)$ denote the unique polynomial of degree at most $n - 1$ that interpolates f at $x_1, \ldots, x_n$, and set $\phi(x) = (x - x_1) \cdots (x - x_n)$. With the use of the Vandermonde formula, it follows from (1.1) with $x_{n+1} = x \in \mathbb{R}$ that

$$f(x) - P(x_1, \ldots, x_n; x) = \phi(x)[x_1, \ldots, x_n, x; f]. \tag{2.1}$$

It now follows that f is n-convex on (a, b) if and only if for every choice of $x_0 = a < x_1 < \cdots < x_n < b = x_{n+1}$ we have

$$(-1)^{j+n}[f(x) - P(x_1, \ldots, x_n; x)] \geq 0$$

for all $x \in (x_j, x_{j+1})$, $j = 0, \ldots, n$. In fact, due to the arbitrariness of the choice of the x_j's it suffices for the above to hold only for any one fixed j. For example, f is convex (2-convex) if and only if for every $a < x_1 < x_2 < b$ the linear polynomial p interpolating f at x_1 and x_2 satisfies $f(x) \leq p(x)$ for all $x \in (x_1, x_2)$.

This leads to another equivalent definition of n-convexity. A function f is n-convex ($n \geq 3$) on (a, b) if and only if its derivative $f^{(n-2)}$ exists and is convex on (a, b). This fact was also first proven by Hopf [10, p. 24] and by Popoviciu [17, p. 48]; see also Boas and Widder [1]. Recall that convex functions enjoy various smoothness properties. A convex function defined on (a, b) is continuous and has both right and left derivatives f'_+ and f'_- at each point of (a, b). In addition, both these functions are nondecreasing and satisfy $f'_-(x) \leq f'_+(x)$ for all $x \in (a, b)$.

If f is sufficiently smooth on $[a, b]$, then from Taylor's Theorem we have

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{(n-1)!} \int_a^b (x - t)_+^{n-1} f^{(n)}(t)\, dt,$$

where

$$(x-t)_+^{n-1} = \begin{cases} (x-t)^{n-1}, & x \geq t, \\ 0, & x < t. \end{cases}$$

Now assume f is n-convex on (a,b), $n \geq 2$. Thus the left and right derivatives $f_-^{(n-1)}$ and $f_+^{(n-1)}$ exist on (a,b). To each such f we associate a measure μ defined on (a,b) by

$$\mu([x,y]) = f_+^{(n-1)}(y) - f_-^{(n-1)}(x)$$

for $a < x \leq y < b$. This is a nonnegative Borel measure on (a,b). If $f_+^{(n-1)}(a)$ is finite then μ may be extended as a bounded (finite) measure to all of $[a,c]$, for all $c < b$. In this case f has the representation

$$f(x) = \sum_{k=0}^{n-1} \frac{f_+^{(k)}(a)(x-a)^k}{k!} + \frac{1}{(n-1)!} \int_a^b (x-t)_+^{n-1}\, d\mu(t)$$

for $x \in [a+, b)$. If we cannot extend μ to the endpoint a, then we will have this representation only on closed subintervals of (a,b). The converse also holds. If μ is a nonnegative Borel measure on $[a,b]$, then any f of the form

$$f(x) = P(x) + \frac{1}{(n-1)!} \int_a^b (x-t)_+^{n-1}\, d\mu(t) \tag{2.2}$$

is n-convex where P is an arbitrary polynomial of degree at most $n-1$. In particular, for every $t \in \mathbb{R}$, $(\cdot - t)_+^{n-1}$ is n-convex on $\mathbb{R}$. These results are essentially in Popoviciu [19]. They are given in detailed form in Karlin and Studden [13, Chap. XI], with generalizations, and may also be found in Bullen [5] and Brown [4]. The above results underscore an essential trait of n-convex functions on intervals, namely that they are the appropriate closure of functions with nonnegative nth derivatives.

We state here for convenience a technical result we will later need, related to (1.1) and (1.2), and of independent interest. Assume we are given ordered points $x_1 < \cdots < x_m$, $m \geq n+1$. Then for

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_{i_1} & \cdots & x_{i_{n+1}} \\ \vdots & \ddots & \vdots \\ x_{i_1}^{n-1} & \cdots & x_{i_{n+1}}^{n-1} \\ f(x_{i_1}) & \cdots & f(x_{i_{n+1}}) \end{vmatrix} \geq 0$$

to hold for all $i_1 < \cdots < i_{n+1}$ in $\{1,\ldots,m\}$ it suffices that it hold only for the consecutive indices $\{j,\ldots,j+n\}$, $j = 1,\ldots,m-n$. Thus to determine whether f is n-convex on a finite ordered set of points $\{x_1,\ldots,x_m\}$ it suffices to verify that it is n-convex only on each $n+1$ consecutive points. These results follow from the fact (see Popoviciu [17, p. 7]) that

$$[x_{i_1}, \ldots, x_{i_{n+1}}; f] = \sum_{j=1}^{m-n} A_j [x_j, \ldots, x_{j+n}; f], \tag{2.3}$$

where the $\{A_j\}_{j=1}^{m-n}$ are nonnegative values that sum to one. The $\{A_j\}_{j=1}^{m-n}$ do not depend upon the specific function f.

3. Elementary extensions. We are interested in the question of when an n-convex function defined on E can be extended to an n-convex function defined on some larger set E'. We will always assume that $E = \{x_1, \ldots, x_m\}$, i.e., E is a finite point set in $\mathbb{R}$. We will deal with either $E' = E \cup \xi$, i.e., one point extensions of E, or $E' = \mathbb{R}$. This problem was first considered in Popoviciu [18].

The first thing to be noted is that for any finite point set E, any 0-convex (nonnegative), 1-convex (nondecreasing) or 2-convex (convex) function on E can always be extended to a 0-convex, 1-convex or 2-convex function, respectively, on all of $\mathbb{R}$. For 0-convex and 1-convex functions this is obvious. It is also simple for 2-convex functions, but we will nonetheless detail this case (see also Galvani [9]).

Assume f is 2-convex on $E = \{x_i\}_{i=1}^m$, where $x_1 < \cdots < x_m$. From (2.3), f is 2-convex on E if and only if

$$[x_j, x_{j+1}, x_{j+2}; f] \geq 0, \qquad j = 1, \ldots, m-2.$$

This, in turn, is equivalent to

$$\frac{f(x_{j+2}) - f(x_{j+1})}{x_{j+2} - x_{j+1}} \geq \frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j}, \qquad j = 1, \ldots, m-2.$$

Thus f can be extended to be 2-convex on $E' = \{x_i\}_{i=1}^m \cup \xi$ for $\xi \in (x_k, x_{k+1})$ if and only if we can define f at ξ so that

$$\frac{f(x_{k+2}) - f(x_{k+1})}{x_{k+2} - x_{k+1}} \geq \frac{f(x_{k+1}) - f(\xi)}{x_{k+1} - \xi} \geq \frac{f(\xi) - f(x_k)}{\xi - x_k} \geq \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}},$$

whenever these inequalities make sense. (That is, for $\xi < x_1$ and for $x_m < \xi$ there is one inequality. For $\xi \in (x_1, x_2)$ or $\xi \in (x_{m-1}, x_m)$ only two inequalities apply.)

That this can be done is easily seen geometrically. Let $P_j(x)$ denote the straight line (linear polynomial) interpolating f at x_j and x_{j+1}, $j = 1, \ldots, m-1$. Assume $k \in \{2, \ldots, m-2\}$. In (x_k, x_{k+1}) the line $P_k(x)$ lies above both $P_{k-1}(x)$ and $P_{k+1}(x)$. For any value of $f(\xi)$ satisfying

$$\max\{P_{k-1}(\xi), P_{k+1}(\xi)\} \leq f(\xi) \leq P_k(\xi)$$

it follows that f is 2-convex on $E' = \{x_i\}_{i=1}^m \cup \xi$. The other cases, i.e., $k \in \{0, 1, m-1, m\}$, are similarly handled.

f can also be extended in this same manner so that it is 2-convex on all of $\mathbb{R}$. For example, one might take f to equal P_k on (x_k, x_{k+1}) for $k = 1, \ldots, m-1$, and suitably define it on $(-\infty, x_1)$ and (x_m, ∞). This is just the obvious choice of taking the continuous, piecewise linear function obtained by joining the point $(x_k, f(x_k))$ to $(x_{k+1}, f(x_{k+1}))$, $k = 1, \ldots, m-1$.

Another interesting feature that follows from the above analysis is that any 2-convex extension of f from E is necessarily bounded above and below on $[x_1, x_m]$, and is also bounded below on $(a, x_1) \cup (x_m, b)$. Let $U(x) = P_k(x)$ on $[x_k, x_{k+1}]$, $k = 1, \ldots, m-1$. Set

$$L(x) = \begin{cases} P_1(x) & \text{for } x \in (-\infty, x_1], \\ P_2(x) & \text{for } x \in (x_1, x_2), \\ \max\{P_{k-1}(x), P_{k+1}(x)\} & \text{for } x \in [x_k, x_{k+1}],\ k = 2, \ldots, m-2, \\ P_{m-2}(x) & \text{for } x \in [x_{m-1}, x_m), \\ P_{m-1}(x) & \text{for } x \in [x_m, \infty). \end{cases}$$

(Note that L need not be continuous at x_1 or x_m.) Then any 2-convex extension f from E necessarily satisfies

$$\begin{aligned} f(x) \le U(x) &\quad \text{for all } x \in [x_1, x_m], \\ L(x) \le f(x) &\quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

Furthermore these bounds are tight in the sense that for any $\xi \in \mathbb{R}$ and α satisfying $L(\xi) \le \alpha \le U(\xi)$ (the upper bound is not present outside $[x_1, x_m]$) there exists a function f that is a 2-convex extension from E to all of $\mathbb{R}$ and satisfies $f(\xi) = \alpha$.

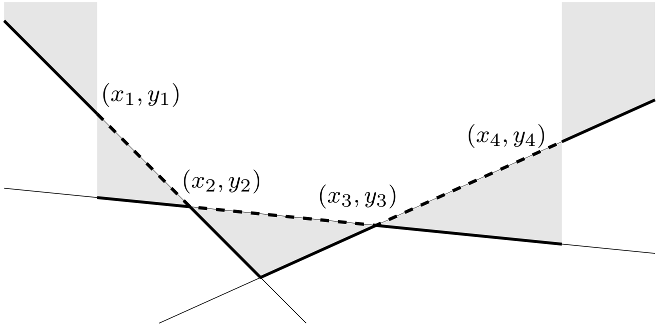

Fig. 1. $n = 2$ and $m = 4$. L is the full line, while U is the dashed line

These elementary results do not always extend to n-convex functions for $n \ge 3$, unless $m = n+1$ or $m = n+2$. We first explain the case $m = n+1$.

A polynomial Q of degree at most n is n-convex if and only if its leading coefficient (the coefficient of x^n) is nonnegative. In fact from (1.1), by the

Vandermonde formula, we immediately see that if $Q(x) = \sum_{j=0}^{n} a_j x^j$, then

$$[x_1, \dots, x_{n+1}; Q] = a_n \tag{3.1}$$

for any choice of $\{x_i\}_{i=1}^{n+1}$. Furthermore, if Q is the polynomial of degree at most n that interpolates f at $\{x_i\}_{i=1}^{n+1}$, then it follows that

$$[x_1, \dots, x_{n+1}; f] = [x_1, \dots, x_{n+1}; Q].$$

Thus, if f is n-convex on the $n+1$ points $\{x_1, \dots, x_{n+1}\}$ then so is the polynomial Q of degree at most n that interpolates f at these points. Then, due to (3.1), Q is n-convex on all of $\mathbb{R}$, and is therefore an n-convex extension of f defined on $E = \{x_1, \dots, x_{n+1}\}$ to all of $\mathbb{R}$.

Moreover, if f is n-convex on the $n+2$ points $\{x_1, \dots, x_{n+2}\}$ then there also exists an n-convex extension of f to all of $\mathbb{R}$. There are various methods of proving this result. It follows easily from what we will show in Section 6. However, here is a more elementary explanation. Let Q_1 denote the polynomial of degree at most n that interpolates f at $\{x_i\}_{i=1}^{n+1}$, and let Q_2 denote the polynomial of degree at most n that interpolates f at $\{x_i\}_{i=2}^{n+2}$. Since f is n-convex on $\{x_i\}_{i=1}^{n+2}$, we know that

$$Q_r(x) = A_r x^n + \cdots$$

for $r = 1, 2$, where $A_1, A_2 \ge 0$. As $Q_1(x_j) = Q_2(x_j)$, $j = 2, \dots, n+1$, and $Q_1 - Q_2$ is a polynomial of degree at most n, we have

$$(Q_2 - Q_1)(x) = (A_2 - A_1) \prod_{j=2}^{n+1} (x - x_j).$$

We first assume that $A_2 \ge A_1$. Thus

$$f(x_{n+2}) - Q_1(x_{n+2}) = Q_2(x_{n+2}) - Q_1(x_{n+2}) \ge 0.$$

Set

$$S(x) = Q_1(x) + C(x - x_{n+1})_+^{n-1} \quad \text{where} \quad C = \frac{f(x_{n+2}) - Q_1(x_{n+2})}{(x_{n+2} - x_{n+1})^{n-1}} \ge 0.$$

By construction $f(x_j) = S(x_j)$, $j = 1, \dots, n+2$. Furthermore S is n-convex on all of $\mathbb{R}$ since it is the sum of two n-convex functions. From symmetry considerations a similar result holds for $A_2 \le A_1$. It suffices to note that

$$(-1)^n (x_2 - x)_+^{n-1} = (x - x_2)_+^{n-1} - (x - x_2)^{n-1}.$$

4. Point extensions. Based on the results of the previous section we assume in what follows that $n \ge 3$ and $m \ge n+3$. Given $E = \{x_1, \dots, x_m\}$ with $x_1 < \cdots < x_m$, we will delineate necessary and sufficient conditions for an n-convex f defined on E to have an n-convex extension onto $E' = E \cup \xi$. Fix $\xi \in (x_k, x_{k+1})$, where $k \in \{0, 1, \dots, m\}$ (we set $x_0 = -\infty$ and $x_{m+1} = \infty$).

We know from (2.3) that f is n-convex on E if and only if

$$[x_j, \dots, x_{j+n}; f] \geq 0, \qquad j = 1, \dots, m-n.$$

Thus a necessary and sufficient condition for an n-convex f defined on E to have an n-convex extension to $E \cup \xi$ is

$$[x_j, \dots, x_{j+n-1}, \xi; f] \geq 0$$

for all relevant $j \in \{1, \dots, m-n+1\}$, i.e., for j satisfying

$$\max\{1, k-n+1\} \leq j \leq \min\{k+1, m-n+1\}. \tag{4.1}$$

It is difficult to deal directly with $[x_j, \dots, x_{j+n-1}, \xi; f] \geq 0$. Thus we will rework this inequality using (2.1). Let P_j denote the unique polynomial of degree at most $n-1$ that interpolates f at the n points $x_j, \dots, x_{j+n-1}$, and set $\phi_j(x) = (x - x_j) \cdots (x - x_{j+n-1})$. Then from (2.1) we have

$$f(x) - P_j(x) = \phi_j(x)[x_j, \dots, x_{j+n-1}, x; f]. \tag{4.2}$$

Since $\xi \in (x_k, x_{k+1})$ and $\mathrm{sgn}\{\phi_j(\xi)\} = (-1)^{j+k-n+1}$ for $k-n+1 \leq j \leq k+1$, we have that an n-convex f defined on E, as above, may be extended to an n-convex function on $E \cup \xi$ with $\xi \in (x_k, x_{k+1})$ if and only if

$$(-1)^{j+k-n+1}\left(f(\xi) - P_j(\xi)\right) \geq 0$$

for each j satisfying (4.1). Set

$$L(\xi) = \max\{P_j(\xi) : j+k-n+1 \text{ even}, \ j \text{ satisfies } (4.1)\},$$
$$U(\xi) = \min\{P_j(\xi) : j+k-n+1 \text{ odd}, \ j \text{ satisfies } (4.1)\}.$$

We summarize the above analysis as follows.

THEOREM 4.1 (Popoviciu [18, p. 78]). *Assume f is n-convex on $\{x_i\}_{i=1}^m$ where $-\infty = x_0 < x_1 < \cdots < x_m < x_{m+1} = \infty$. Let $\xi \in (x_k, x_{k+1})$. Then a necessary and sufficient condition for f to have an n-convex extension to $\{x_i\}_{i=1}^m \cup \xi$ is that $L(\xi) \leq U(\xi)$ where $L(\xi)$ and $U(\xi)$ are defined as above. f is n-convex on $\{x_i\}_{i=1}^m \cup \xi$ if and only if $f(\xi) \in [L(\xi), U(\xi)]$.*

For $j \in \{1, \dots, m-n\}$, $P_{j+1}(x) - P_j(x)$ is a polynomial of degree at most $n-1$ that vanishes at $x_{j+1}, \dots, x_{j+n-1}$. Thus

$$P_{j+1}(x) - P_j(x) = C_j \prod_{i=j+1}^{j+n-1} (x - x_i).$$

From (4.2) we see that $f(x_{j+n}) - P_j(x_{j+n}) \geq 0$. Furthermore $f(x_{j+n}) = P_{j+1}(x_{j+n})$. Thus $C_j \geq 0$, implying that the inequality

$$(-1)^{j+k-n+1}\left(P_{j+1}(\xi) - P_j(\xi)\right) \geq 0 \tag{4.3}$$

holds for the relevant j.

In the case $n = 2$, (4.1) reads

$$\max\{1, k-1\} \le j \le \min\{k+1, m-1\}.$$

Thus for $2 \le k \le m-2$ the set of j satisfying these inequalities is $j = k-1, k, k+1$. In this case

$$L(\xi) = \max_{j=k-1,k+1} P_j(\xi), \qquad U(x) = P_k(\xi).$$

From (4.3) it follows that $P_k(\xi) \ge P_{k-1}(\xi)$ and $P_k(\xi) \ge P_{k+1}(\xi)$. Thus we always have $L(\xi) \le U(\xi)$. This is the same analysis as done in Section 3 from a slightly different perspective.

In the case $n = 3$, (4.1) reads

$$\max\{1, k-2\} \le j \le \min\{k+1, m-2\}.$$

Thus for $3 \le k \le m-3$ the set of j satisfying the above is $j = k-2, k-1, k, k+1$. In this case

$$L(\xi) = \max_{j=k-2,k} P_j(\xi), \qquad U(\xi) = \min_{j=k-1,k+1} P_j(\xi).$$

From (4.3) we also have $(-1)^{j+k-2}(P_{j+1}(\xi) - P_j(\xi)) \ge 0$ for all relevant j, implying that

$$P_{k-1}(\xi) \ge P_{k-2}(\xi), \qquad P_{k-1}(\xi) \ge P_k(\xi), \qquad P_{k+1}(\xi) \ge P_k(\xi).$$

This does not imply, offhand, that $L(\xi) \le U(\xi)$. Namely, we are missing the inequality $P_{k+1}(\xi) \ge P_{k-2}(\xi)$. In fact this inequality need not hold, as we show below. For $n = 3$ (and therefore for all $n \ge 3$) there exist n-convex functions f defined on $\{x_i\}_{i=1}^m$, and points ξ such that there is no n-convex extension of f to $\{x_i\}_{i=1}^m \cup \xi$.

It is readily verified that for $\xi \le x_3$ and $\xi \ge x_{m-2}$, i.e., $k \in \{0, 1, 2, m-2, m-1, m\}$, we always have $L(\xi) \le U(\xi)$, and thus every n-convex function defined on the ordered points $\{x_i\}_{i=1}^m$ has an n-convex extension to $\{x_i\}_{i=1}^m \cup \xi$ for all $\xi < x_3$ and for all $\xi > x_{m-2}$.

Example 4.2. Here is an example of a 3-convex function f on E and a point ξ with no 3-convex extension of f to $E \cup \{\xi\}$. Let $n = 3$, $m = 6$, and $x_i = i$, $i = 1, \dots, 6$. Set $f(x_1) = 0$, $f(x_2) = 0$, $f(x_3) = 0$, $f(x_4) = 0$, $f(x_5) = 2$, and $f(x_6) = 6$. It is easily verified that f is 3-convex on $\{x_1, \dots, x_6\}$, as

$$[x_1, x_2, x_3, x_4; f] = 0, \qquad [x_2, x_3, x_4, x_5; f] = 1/3, \qquad [x_3, x_4, x_5, x_6; f] = 0.$$

Furthermore

$$P_1(x) = P_2(x) = 0, \qquad P_3(x) = P_4(x) = (x-3)(x-4).$$

Thus for any $\xi \in (x_3, x_4) = (3, 4)$,

$$L(\xi) = \max\{0, (\xi-3)(\xi-4)\} = 0,$$
$$U(\xi) = \min\{0, (\xi-3)(\xi-4)\} = (\xi-3)(\xi-4),$$

implying $L(\xi) > U(\xi)$. Therefore f has no extension as a 3-convex function to any point in $(3,4)$.

Example 4.3. Here is an example of a function f that is 3-convex on $E = \{x_i\}_{i=1}^8$, can be extended to be 3-convex on $E \cup \xi$ for every $\xi \in \mathbb{R}$, but cannot be extended to be 3-convex on all of $[x_1, x_8]$. In other words, *while extension at all individual points is necessary for extension to a full interval, it is not sufficient.*

Set $x_i = i$ for $i = 1, \ldots, 8$. Let $f(1) = f(2) = f(3) = 0$, implying $P_1(x) = 0$. We define, for $j = 2, \ldots, 6$,

$$P_j(x) = P_{j-1}(x) + a_j(x-j)(x-j-1)$$

where $a_j \geq 0$. In this way we define f at x_{j+2}, i.e., set $f(x_{j+2}) = P_j(x_{j+2})$. As $f(x_j) = P_j(x_j)$ and $f(x_{j+1}) = P_j(x_{j+1})$, it follows from (2.1) that f is 3-convex on the x_i. Set $a_2 = 1$, $a_3 = 6$, $a_4 = 1$, $a_5 = 6$, and $a_6 = 1$. (This implies that $f(4) = 2$, $f(5) = 18$, $f(6) = 50$, $f(7) = 110$, and $f(8) = 200$.) As we have seen, to verify that $L(\xi) \leq U(\xi)$ in this case of $n = 3$ it is both necessary and sufficient to prove that

$$P_{k+1}(x) \geq P_{k-2}(x) \quad \text{for all } x \in (x_k, x_{k+1}),\ k = 3,4,5.$$

We verify this as follows. For $k = 3$, we need $P_4(x) - P_1(x) \geq 0$ on $(3,4)$. Now

$$P_4(x) - P_1(x) = (x-4)(x-5) + 6(x-3)(x-4) + (x-2)(x-3) = 2(2x-7)^2 \geq 0.$$

For $k = 4$, we need $P_5(x) - P_2(x) \geq 0$ on $(4,5)$. Now

$$\begin{aligned} P_5(x) - P_2(x) &= 6(x-5)(x-6) + (x-4)(x-5) + 6(x-3)(x-4) \\ &= 6(x-5)(x-6) + (7x-23)(x-4) > 0. \end{aligned}$$

For $k = 5$, we need $P_6(x) - P_3(x) \geq 0$ on $(5,6)$. Now

$$P_6(x) - P_3(x) = (x-6)(x-7) + 6(x-5)(x-6) + (x-4)(x-5) = 2(2x-11)^2 \geq 0.$$

Thus f may be extended to be 3-convex on $\{x_i\}_{i=1}^8 \cup \xi$ for any $\xi \in \mathbb{R}$.

However, as we now show, f cannot be extended to be 3-convex on the full interval. Assume f is 3-convex on $[1,8]$. Then on $[3,4]$,

$$L(x) = \max\{P_1(x), P_3(x)\} \leq f(x) \leq \min\{P_2(x), P_4(x)\} = U(x).$$

This implies that we must have $f(7/2) = P_1(7/2) = P_4(7/2) (= 0)$. As f agrees with P_1 (a polynomial of degree at most 2) at the four points $1, 2, 3$, and $7/2$, the 3-convex function f must agree with P_1 on all of $[1, 7/2]$. (This can be shown in a number of ways. For example, the divided difference at these four points is zero, and thus from (2.3) must be zero for any four points in $[1, 7/2]$.) Similarly f agrees with P_4 at $7/2$ by the above analysis and at $4, 5$, and 6 by definition. Thus f must agree with P_4 on all of $[7/2, 6]$. Now on $[5,6]$,

$$\max\{P_3(x),P_5(x)\}\le f(x)\le \min\{P_4(x),P_6(x)\}.$$

Thus we must have $f(11/2)=P_3(11/2)=P_6(11/2)$. As f agrees with P_3 at $3,4,5$, and now also at $11/2$, it must agree with P_3 on all of $[3,11/2]$. But now we have reached a contradiction, as $P_1\ne P_3$ on $[3,7/2]$, or $P_4\ne P_3$ on $[7/2,6]$.

5. n-Convexity and moment conditions. If f is n-convex on an open set containing $[c,d]$, then according to (2.2) we have

$$f(x)=P(x)+\frac{1}{(n-1)!}\int_c^d (x-t)_+^{n-1}\,d\mu(t) \tag{5.1}$$

where P is a polynomial of degree at most $n-1$ (in fact it is the Taylor polynomial of f at c) and μ is a nonnegative bounded Borel measure with support in $[c,d]$.

Let

$$B_j(t)=[x_j,\dots,x_{j+n};(\cdot-t)_+^{n-1}],\qquad j=1,\dots,m-n.$$

These B_j are called *B-splines* of degree $n-1$ with the (simple) knots $x_j,\dots,x_{j+n}$. It is well known that each B_j is strictly positive on (x_j,x_{j+n}), and vanishes identically outside (x_j,x_{j+n}). Furthermore, the B_j are normalized so that

$$\int_{x_j}^{x_{j+n}} B_j(t)\,dt=\frac{1}{n},\qquad j=1,\dots,m-n.$$

B-splines were introduced in Popoviciu [18] in connection with n-convexity. This seems to have been the first consideration of B-splines on nonequidistant knots. See de Boor and Pinkus [3] for a history and explanation thereof.

Recall (from (3.1)) that $[x_j,\dots,x_{j+n};P]=0$ for every polynomial P of degree at most $n-1$. Set $\alpha_j=f(x_j)$, $j=1,\dots,m$, and

$$c_j=[x_j,\dots,x_{j+n};f],\qquad j=1,\dots,m-n.$$

It follows from (5.1) that if there exists an n-convex extension of f on (a,b) containing E, then

$$c_j=\frac{1}{(n-1)!}\int_{x_1}^{x_m} B_j(t)\,d\mu(t),\qquad j=1,\dots,m-n, \tag{5.2}$$

where μ is a nonnegative bounded Borel measure. The converse is also true. If there exists a nonnegative Borel measure μ satisfying (5.2), then for any polynomial P of degree at most $n-1$ the function

$$g(x) = P(x) + \frac{1}{(n-1)!} \int_{x_1}^{x_m} (x-t)_+^{n-1}\, d\mu(t)$$

is n-convex and satisfies

$$[x_j, \dots, x_{j+n}; g] = c_j, \qquad j = 1, \dots, m-n.$$

Furthermore, if we choose the polynomial P so that $g(x_j) = \alpha_j$, $j = 1, \dots, n$, then from (1.1) we have $g(x_j) = \alpha_j$, $j = 1, \dots, m$, i.e., $g = f$ on E.

Thus we arrive at the following equivalence, which may be found in Popoviciu [18] (see also de Boor [2]).

PROPOSITION 5.1. *Let $a < x_1 < \cdots < x_m < b$. Then there exists an n-convex f on (a, b) satisfying $f(x_j) = \alpha_j$, $j = 1, \dots, m$, if and only if there exists a nonnegative bounded Borel measure μ on $[x_1, x_m]$ satisfying*

$$c_j = \frac{1}{(n-1)!} \int_{x_1}^{x_m} B_j(t)\, d\mu(t),$$

where $c_j = [x_j, \dots, x_{j+n}; f]$, $j = 1, \dots, m-n$, and $B_j(t) = [x_j, \dots, x_{j+n}; (\cdot - t)_+^{n-1}]$ is the B-spline of degree $n-1$ with simple knots $x_j, \dots, x_{j+n}$, $j = 1, \dots, m-n$.

What are necessary and sufficient conditions for $c_1, \dots, c_{m-n}$ to satisfy (5.2) for some nonnegative bounded Borel measure? The first characterization of such moments follows from a general known result in moment theory and convex analysis (via dual cones) which we state and prove.

PROPOSITION 5.2. *Let $\{g_j\}_{j=1}^r$ be functions in $C[c, d]$. Then*

$$c_j = \int_c^d g_j(t)\, d\mu(t), \qquad j = 1, \dots, r,$$

for some nonnegative Borel measure on $[c, d]$ if and only if

$$\sum_{j=1}^r a_j c_j \ge 0$$

for all $\{a_1, \dots, a_r\}$ satisfying $\sum_{j=1}^r a_j g_j(t) \ge 0$ for all $t \in [c, d]$.

Proof. We first recall a result concerning convex cones. Assume $\mathcal{A}$ is a convex cone in $\mathbb{R}^r$. Then the dual cone $\mathcal{A}^+$ is defined via

$$\mathcal{A}^+ = \{\mathbf{b} \in \mathbb{R}^r : (\mathbf{b}, \mathbf{a}) \ge 0 \text{ for all } \mathbf{a} \in \mathcal{A}\}.$$

It is a classical result that if $\mathcal{C}$ is a closed convex cone in $\mathbb{R}^r$, then $\mathcal{C} = \mathcal{C}^{++}$. The fact that $\mathcal{C} \subseteq \mathcal{C}^{++}$ follows by definition. Furthermore, if there exists a $\mathbf{b} \in \mathcal{C}^{++} \setminus \mathcal{C}$, then by the separation of hyperplanes ($\mathcal{C}$ and $\mathcal{C}^{++}$ are both closed) there exists an $\mathbf{a} \in \mathbb{R}^r$ satisfying $(\mathbf{b}, \mathbf{a}) < 0 \le (\mathbf{c}, \mathbf{a})$ for all $\mathbf{c} \in \mathcal{C}$. Thus $\mathbf{a} \in \mathcal{C}^+$. But since $(\mathbf{b}, \mathbf{a}) < 0$ we have $\mathbf{b} \notin \mathcal{C}^{++}$, which is a contradiction. Thus $\mathcal{C} = \mathcal{C}^{++}$.

Set

$$\mathcal{M} = \Big\{\mathbf{a} = (a_1, \ldots, a_r) : \sum_{j=1}^{r} a_j g_j(t) \ge 0 \text{ for all } t \in [c, d]\Big\},$$

$$\mathcal{P} = \Big\{\mathbf{c} = (c_1, \ldots, c_r) : \, c_j = \int_c^d g_j(t)\, d\mu(t), \; j = 1, \ldots, r, \; \text{for some } \mu \ge 0\Big\},$$

where μ in $\mathcal{P}$ varies over all nonnegative Borel measures on $[c, d]$.

Both $\mathcal{M}$ and $\mathcal{P}$ are closed convex cones. Furthermore, if $\mathbf{a} \in \mathcal{M}$ and $\mathbf{c} \in \mathcal{P}$, then

$$\sum_{j=1}^{r} a_j c_j = \int_c^d \sum_{j=1}^{r} a_j g_j(t)\, d\mu(t) \ge 0$$

as $\sum a_j g_j$ and μ are nonnegative, and therefore $\mathcal{M} \subseteq \mathcal{P}^+$. On the other hand, if $\mathbf{d} \in \mathcal{P}^+$ then $\sum_{j=1}^{r} d_j c_j \ge 0$ for all $\mathbf{c} \in \mathcal{P}$. Thus

$$\int_c^d \sum_{j=1}^{r} d_j g_j(t)\, d\mu(t) \ge 0$$

for every nonnegative measure μ on $[c, d]$, including all point measures. Therefore, $\sum_{j=1}^{r} d_j g_j(t) \ge 0$ for all $t \in [c, d]$, implying $\mathbf{d} \in \mathcal{M}$. Hence $\mathcal{P}^+ \subseteq \mathcal{M}$, and we have the equality $\mathcal{P}^+ = \mathcal{M}$.

Applying the result of the first paragraph of the proof, it follows that $\mathcal{P} = \mathcal{M}^+$, which is exactly the statement of the proposition. ∎

Combining Propositions 5.1 and 5.2, we see that a necessary and sufficient condition for the existence of an n-convex extension of f from E to any interval containing E is that there exists a nonnegative Borel measure μ on $[x_1, x_m]$ satisfying (5.2), which in turn, by Proposition 5.2, is equivalent to

$$\sum_{j=1}^{m-n} a_j c_j \ge 0$$

for all $\{a_1, \ldots, a_{m-n}\}$ satisfying $\sum_{j=1}^{m-n} a_j B_j(t) \ge 0$ for all $t \in [x_1, x_m]$. This result may be found in Popoviciu [18, p. 90].

This last result is neither insightful nor very useful. Nonetheless for $n = 2$ these equivalent forms easily show what we have already verified in Section 3. For $n = 2$, B_j has its support on (x_j, x_{j+2}). It is continuous, linear on each of $[x_j, x_{j+1}]$ and $[x_{j+1}, x_{j+2}]$, satisfies

$$B_j(x_k) = \frac{1}{x_{j+2} - x_j}\,\delta_{j,k-1}, \qquad j = 1, \ldots, m-2,\ k = 1, \ldots, m,$$

and

$$\int_{x_j}^{x_{j+2}} B_j(t)\,dt = \frac{1}{2}.$$

To have

$$c_j = \int_{x_j}^{x_{j+2}} B_j(t)\,d\mu(t), \qquad j = 1, \ldots, m-2,$$

for any nonnegative measure μ it is both necessary and sufficient that the c_j be nonnegative. Indeed, this is clearly necessary. It is also sufficient since x_{j+1} lies in the support of B_j, but of no other B_k, $k \neq j$. Let δ_x denote the Dirac functional at x. Then the nonnegative Borel measure

$$\mu = \sum_{j=1}^{m-2} (x_{j+2} - x_j) c_j \delta_{x_{j+1}}$$

for any nonnegative $c_1, \ldots, c_{m-2}$ satisfies

$$c_j = \int_{x_1}^{x_m} B_j(t)\,d\mu(t), \qquad j = 1, \ldots, m-2.$$

Thus in this case the associated closed convex cone $\mathcal{P}$ is $\mathbb{R}_+^{m-2}$. The same can be obtained as follows. If $\sum_{j=1}^{m-2} a_j B_j(t) \geq 0$ for all $t \in [x_1, x_m]$, then setting $t = x_{j+1}$ we have $a_j \geq 0$. Thus the set

$$\mathcal{M} = \Big\{\mathbf{a} : \sum_{j=1}^{m-2} a_j B_j(t) \geq 0 \text{ for all } t \in [x_1, x_m]\Big\}$$

is exactly $\mathbb{R}_+^{m-2}$, and from Proposition 5.2 we again obtain $\mathcal{P} = \mathbb{R}_+^{m-2}$.

Similarly, assume $m = n + 2$. In this case we have only two B-splines, namely B_1 and B_2 with support (x_1, x_{n+1}) and (x_2, x_{n+2}), respectively. For any given $c_1, c_2 \geq 0$ we can set $\mu = a\delta_{y_1} + b\delta_{y_2}$ where $y_1 \in (x_1, x_2]$ and $y_2 \in [x_{n+1}, x_{n+2})$, and obtain $a, b \geq 0$ satisfying (5.2). (The case $m = n + 1$ is even simpler.)

For $n \geq 3$, we can immediately see that the situation is more complicated. Assume that $n = 3$. As B_j has support (x_j, x_{j+3}) it follows that if $c_j = 0$ then the nonnegative measure μ has no support in (x_j, x_{j+3}). Thus, for example, if $c_1 = c_3 = 0$ then μ has no support in (x_1, x_6) and if $c_2 > 0$ there exists no 3-convex f satisfying the interpolation data; see Example 4.2. (Even if $c_1, c_3 > 0$, it follows that the “admissible” c_2 are bounded above.)

For later convenience we renormalize the moments in the cone of moments $\mathcal{P}$ via

$$c_j = \frac{1}{(n-1)!} \int\limits_{x_1}^{x_m} B_j(t)\, d\mu(t), \qquad j = 1, \ldots, m-n.$$

Here μ varies over all nonnegative bounded Borel measures on $[x_1, x_m]$. This cone is often called the *moment space induced by the* $\{B_j\}_{j=1}^{m-n}$.

For $n = 2$, as we have seen, this cone is simply $\mathbb{R}_+^{m-2}$. Its extreme rays are the multiples of the $m-2$ unit vectors. These vectors may be obtained by choosing the measures

$$\mu_k = \alpha\delta_{x_{k+1}}, \qquad \alpha \geq 0,\ k = 1, \ldots, m-2,$$

since

$$B_j(x_k) \begin{cases} = 0, & j \neq k+1, \\ > 0, & j = k+1, \end{cases}$$

for $j = 1, \ldots, m-2$ and $k = 1, \ldots, m$.

However for $n \geq 3$ and $m \geq 6$ there are an infinity of extreme rays in $\mathcal{P}$. Obviously each $\alpha(B_1(\xi), \ldots, B_{m-n}(\xi))$, $\alpha \geq 0$, is a boundary ray of $\mathcal{P}$ for each $\xi \in (x_1, x_m)$. We also have

Proposition 5.3. *For each* $\xi \in (x_3, x_{m-2})$ *the vector* $\alpha(B_1(\xi), \ldots, B_{m-n}(\xi))$, $\alpha \geq 0$, *is an extreme ray of* $\mathcal{P}$.

Proof. Assume not. Then there exist $\{\xi_1, \ldots, \xi_r\}$ in (x_1, x_m), and $\gamma_k > 0$, $k = 1, \ldots, r$, r finite, such that

$$\sum_{k=1}^{r} \gamma_k B_j(\xi_k) = B_j(\xi), \qquad j = 1, \ldots, m-n, \tag{5.3}$$

with $\xi \notin \{\xi_1, \ldots, \xi_r\}$. Assume $\xi \in (x_s, x_{s+1})$ for some $s \in \{3, \ldots, m-3\}$. We must handle somewhat differently the cases $n+1 \leq s \leq m-n-1$, $s \leq n$ and $s \geq m-n$. In what follows we only detail the case $n+1 \leq s \leq m-n-1$. (If $\xi = x_j$ for some $j = 4, \ldots, m-3$, similar arguments apply.) For each $s+1 \leq j \leq m-n$ there exists the B-spline B_j satisfying $B_j(\xi) = 0$ and strictly positive on (x_j, x_{j+n}). Thus $\xi_k \leq x_{s+1}$ for all k. Similarly, since $s \geq n+1$, we have $\xi_k \geq x_s$ for all k. Hence $\xi_k \in [x_s, x_{s+1}]$ for $k = 1, \ldots, r$.

The B_j with subscripts $j \in \{s-n+1, \ldots, s\}$ are those that do not vanish on (x_s, x_{s+1}). For $j \notin \{s-n+1, \ldots, s\}$ the equation (5.3) always holds. Let $\{\xi_1, \ldots, \xi_r\} \cup \{\xi\} = \{\eta_1, \ldots, \eta_{r+1}\}$ where $x_s \leq \eta_1 < \cdots < \eta_{r+1} \leq x_{s+1}$. By the Schoenberg–Whitney Theorem (see Schoenberg and Whitney [20]), the matrix

$$(B_j(\eta_k))_{j=s-n+1}^{s}{}_{k=1}^{r+1}$$

is an $n \times (r+1)$ matrix of rank $\min\{n, r+1\}$ since $B_{s-n+j}(\eta_j) > 0$, $j = 1, \ldots, \min\{n, r+1\}$. As

$$\sum_{k=1}^{r+1} \beta_k B_j(\eta_k) = 0, \qquad j = s-n+1, \ldots, s,$$

where $\beta_k = \gamma_l$ if $\eta_k = \xi_l$, and $\beta_k = -1$ if $\eta_k = \xi$, it follows that $n < r+1$. Furthermore, from Karlin [12, p. 230], as this matrix is totally positive (TP) and of full rank we have

$$S^+(\beta_1, \ldots, \beta_{r+1}) \geq n,$$

where S^+ is the number of (weak) sign changes. As $\beta_k > 0$ for all but one index, we have

$$S^+(\beta_1, \ldots, \beta_{r+1}) \leq 2.$$

But $n \geq 3$ and this contradiction proves the proposition. ∎

Having an infinite number of extreme rays makes it exceedingly difficult to determine easily verified criteria for when a specific vector belongs to $\mathcal{P}$.

The B-splines $\{B_j\}_{j=1}^{m-n}$ enjoy various well studied properties. One of these properties is that their span forms a weak Chebyshev (WT) system. That is, no linear combination of these B_j has more than $m-n-1$ strict sign changes. Both Chebyshev (T) and weak Chebyshev systems, and the moment spaces induced by them, have been studied in detail (see Karlin and Studden [13], Krein and Nudel'man [15], and Krein [14]).

One result from Micchelli and Pinkus [16] is particularly useful in our context. We first define a nonnegative Borel measure μ to be *positive* relative to a WT-system M if

$$\int m(t)\, d\mu(t) > 0$$

for every nonnegative nontrivial $m \in M$.

Theorem 5.4 (Micchelli and Pinkus [16]). *Let M be a WT-system of dimension $2r$ on $[c, d]$, containing a function strictly positive on (c, d). Then for every positive measure μ relative to M there exist $c < \xi_1 < \cdots < \xi_r < d$ and $\lambda_1, \ldots, \lambda_r > 0$ such that*

$$\int_c^d m(t)\, d\mu(t) = \sum_{k=1}^{r} \lambda_k m(\xi_k) \qquad \textit{for all } m \in M.$$

In other words, there is a discrete nonnegative measure with exactly r points of increase that provides a representation for the same moments as does the measure μ. Such a discrete measure is called a *lower principal representation* and it has various extremal properties.

Specializing this to B-splines we have some additional results. Assume we are given the B-splines $B_1, \ldots, B_{2r}$ based on the knots $x_1 < \cdots < x_{n+2r}$.

Given a positive measure μ on $[x_1, x_{n+2r}]$ relative to $\mathcal{B} = \text{span}\{B_1, \dots, B_{2r}\}$ we have from Theorem 5.4 the existence of $\xi_1 < \dots < \xi_r$ in (x_1, x_{n+2r}) and $\lambda_1, \dots, \lambda_r > 0$ satisfying

$$\int_{x_1}^{x_{n+2r}} B(t)\,d\mu(t) = \sum_{k=1}^{r} \lambda_k B(\xi_k), \qquad B \in \mathcal{B}, \tag{5.4}$$

since $\sum_{j=1}^{2r} B_j(t) > 0$ for all $t \in (x_1, x_{n+2r})$.

We say that g is in the *convexity cone* of the WT-system M of dimension n if $\widetilde{M} = \text{span}\{M, g\}$ is a WT-system, and if $h \in \widetilde{M}$ has n strict changes of sign and is nonnegative to the right of its largest change of sign, then

$$h = a_0 g + m$$

where $m \in M$ and $a_0 > 0$. (There is an equivalent determinantal condition.)

Regarding (5.4) we prove the following.

Proposition 5.5. *Under the above conditions we have*:

(a) $x_{2k} < \xi_k < x_{n+2k-1}, k = 1, \dots, r$.
(b) *The* $\{\xi_k\}_{k=1}^r$ *and* $\{\lambda_k\}_{k=1}^r$ *satisfying* (5.4) *are unique.*
(c) *For every nonnegative Borel measure* ν *satisfying*

$$\int_{x_1}^{x_{n+2r}} B(t)\,d\nu(t) = \int_{x_1}^{x_{n+2r}} B(t)\,d\mu(t), \qquad B \in \mathcal{B},$$

we have

$$\int_{x_1}^{x_{n+2r}} g(t)d\nu(t) \ge \sum_{k=1}^{r} \lambda_k g(\xi_k)$$

for every g *in the convexity cone of* $\text{span}\{B_1, \dots, B_{2r}\}$.

(d) *The constant function identically equal to* 1 *is in the convexity cone of* $\text{span}\{B_1, \dots, B_{2r}\}$ *and thus*

$$\int_{x_1}^{x_{n+2r}} d\nu(t) \ge \sum_{k=1}^{r} \lambda_k$$

for every nonnegative Borel measure ν *as in* (c).

Proof. (a) Assume

$$\xi_k \le x_{2k} \tag{5.5}$$

for some $k \in \{1, \dots, r\}$. Now, $\text{span}\{B_{2k}, \dots, B_{2r}\}$ constitutes a WT-system of dimension $2(r-k)+1$. Thus there exists a nontrivial linear combination of these B-splines that is nonnegative on $[x_1, x_{n+2r}]$ and vanishes at $\xi_{k+1}, \dots, \xi_r$. This nontrivial linear combination vanishes identically on $[x_1, x_{2k}]$, and thus by (5.5) at each of the $\xi_1, \dots, \xi_k$. The right-hand side of

(5.4) is therefore equal to zero. But since μ is a "positive" measure relative to $\mathcal{B}$ the left-hand side is strictly positive. This is a contradiction. The inequality $\xi_k < x_{n+2k-1}$ is proved similarly.

(b) Assume there are two such representations, i.e.,

$$\int_{x_1}^{x_{n+2r}} B(t)\,d\mu(t) = \sum_{k=1}^{r} \lambda_k B(\xi_k) = \sum_{k=1}^{r} \mu_k B(\varrho_k), \qquad B \in \mathcal{B}, \tag{5.6}$$

where $\lambda_k, \mu_k > 0$ (it is easily proven that the coefficients must be strictly positive) and $x_1 < \xi_1 < \cdots < \xi_r < x_{n+2r}$, $x_1 < \varrho_1 < \cdots < \varrho_r < x_{n+2r}$. From (a) we have $x_{2k} < \xi_k < x_{n+2k-1}$ and $x_{2k} < \varrho_k < x_{n+2k-1}$ for $k = 1, \ldots, r$. For ease of exposition assume that all the ξ_k and ϱ_k are distinct. Let

$$\{\eta_k\}_{k=1}^{2r} = \{\xi_k\}_{k=1}^{r} \cup \{\varrho_k\}_{k=1}^{r}$$

arranged in increasing order of magnitude. Then from (a) it may be seen that

$$x_k < \eta_k < x_{k+n}, \qquad k = 1, \ldots, 2r.$$

But these are exactly the Schoenberg–Whitney conditions which imply that we can uniquely interpolate to any values at $\{\eta_k\}_{k=1}^{2r}$ from $\mathcal{B}$. Thus, for example, there exists a nontrivial $B \in \mathcal{B}$ vanishing at all the η_k but one. This is easily seen to contradict (5.6).

(c) This inequality is a well known result called the Markov–Krein inequality (see Karlin and Studden [13, p. 80], Krein and Nudel'man [15, p. 109], and more explicitly in this case Micchelli and Pinkus [16, Cor. 2.1]).

One proof is the following. Assume that g is in the convexity cone of $\{B_1, \ldots, B_{2r}\}$. There exists a nontrivial linear combination

$$h(t) = d_0 g(t) + \sum_{k=1}^{2r} d_j B_j(t)$$

that is nonnegative and vanishes on $\{\xi_k\}_{k=1}^{r}$. If $d_0 = 0$, then

$$B(t) = \sum_{k=1}^{2r} d_j B_j(t)$$

is nontrivial, nonnegative and vanishes on $\{\xi_k\}_{k=1}^{r}$, contradicting the positivity of μ. It may also be easily shown that $d_0 > 0$. Normalize so that $d_0 = 1$. Let ν be any nonnegative measure satisfying

$$\int_{x_1}^{x_{n+2r}} B(t)\,d\nu(t) = \int_{x_1}^{x_{n+2r}} B(t)\,d\mu(t) = \sum_{k=1}^{r} \lambda_k B(\xi_k), \qquad B \in \mathcal{B}.$$

Thus

$$0 \le \int_{x_1}^{x_{n+2r}} h(t)\,d\nu(t) = \int_{x_1}^{x_{n+2r}} h(t)\,d\nu(t) - \sum_{k=1}^{r} \lambda_k h(\xi_k),$$

that immediately reduces to

$$\int_{x_1}^{x_{n+2r}} g(t)\,d\nu(t) \ge \sum_{k=1}^{r} \lambda_k g(\xi_k).$$

(d) Assume $\operatorname{span}\{B_1, \ldots, B_{2r}, 1\}$ is not a WT-system. There then exists a linear combination h of $B_1, \ldots, B_{2r}, 1$ with at least $2r + 1$ strict sign changes. Set

$$h(t) = b_0 + \sum_{j=1}^{2r} b_j B_j(t).$$

We must have $b_0 \neq 0$ since $\operatorname{span}\{B_1, \ldots, B_{2r}\}$ is a WT-system of dimension $2r$. As $h(t) = b_0$ for $t \le x_1$ and $t \ge x_{n+2r}$, it follows that h has at least $2r + 2$ strict sign changes. Thus h' has at least $2r + 1$ strict sign changes. However,

$$h'(t) = \sum_{j=1}^{2r} b_j B_j'(t) = \sum_{j=1}^{2r+1} d_j \widetilde{B}_j(t),$$

where $\widetilde{B}_j$ is the B-spline of degree $n-2$ supported on the knots $x_j < \cdots < x_{j+n-1}$. As $\operatorname{span}\{\widetilde{B}_1, \ldots, \widetilde{B}_{2r+1}\}$ forms a WT-system of dimension $2r + 1$, the nontrivial function h' has at most $2r$ strict sign changes, a contradiction.

Thus $\operatorname{span}\{B_1, \ldots, B_{2r}, 1\}$ is a WT-system of dimension $2r+1$, and there exists a nontrivial linear combination

$$h(t) = a_0 + \sum_{j=1}^{2r} a_j B_j(t)$$

that has $2r$ sign changes and is nonnegative to the right of its largest sign change. We cannot have $a_0 = 0$ as $\operatorname{span}\{B_1, \ldots, B_{2r}\}$ is a WT-system. As $h(x_{n+2r}) = a_0 \neq 0$, we must have $a_0 > 0$. This proves (d). ∎

It should be emphasized that the above proof of (a) and (b) implies that if

$$\int_{x_1}^{x_{n+2r}} B(t)\,d\mu(t) = \sum_{k=1}^{s} \mu_k B(\eta_k), \qquad B \in \mathcal{B},$$

for some $s \le r$ and ordered $\{\eta_k\}$, then necessarily $s = r$, $\mu_k = \lambda_k$ and $\eta_k = \xi_k$.

Let us now reformulate the consequences of Theorem 5.4 and Proposition 5.5 in the language of our original interpolation problem.

Assume f is n-convex on (a,b) and let $\alpha_i = f(x_i)$, $i = 1,\ldots,n+2r$, where

$$x_0 = a < x_1 < \cdots < x_{n+2r} < b = x_{n+2r-1}.$$

(It is convenient, see Theorem 5.4, to assume $m = n+2r$.) Let

$$c_j = [x_j,\ldots,x_{j+n}; f], \qquad j = 1,\ldots,2r.$$

Then

$$c_j = \frac{1}{(n-1)!} \int_{x_1}^{x_{n+2r}} B_j(t)\, d\mu(t), \qquad j = 1,\ldots,2r,$$

for some nonnegative bounded Borel measure μ. For the moment assume that $\mathbf{c} = (c_1,\ldots,c_{2r})$ is in the interior of $\mathcal{P}$. (If $\mathbf{c} \in \mathcal{P}$, then $\mathbf{c} + \boldsymbol{\varepsilon} \in \operatorname{int} \mathcal{P}$ where $\boldsymbol{\varepsilon} = (\varepsilon,\ldots,\varepsilon)$ for any $\varepsilon > 0$.) Then μ is positive relative to $\operatorname{span}\{B_1,\ldots,B_{2r}\}$. From Theorem 5.4 and Proposition 5.5 we have

$$c_j = \sum_{k=1}^{r} \lambda_k B_j(\xi_k), \qquad j = 1,\ldots,2r,$$

where $x_{2k} < \xi_k < x_{n+2k-1}$, and $\lambda_k > 0$, $k = 1,\ldots,r$. Set

$$s^*(x) = p(x) + \sum_{k=1}^{r} \lambda_k (x-\xi_k)_+^{n-1},$$

where we choose the polynomial p of degree at most $n-1$ so that

$$s^*(x_i) = \alpha_i, \qquad i = 1,\ldots,n.$$

Then, as is readily verified,

$$s^*(x_i) = f(x_i) = \alpha_i, \qquad i = 1,\ldots,n+2r.$$

Note that s^* is n-convex on all $\mathbb{R}$ as it is a nonnegative combination of n-convex functions, namely a polynomial of degree at most $n-1$ and the functions $(x-\xi_k)_+^{n-1}$.

The important Proposition 5.5(c) can be applied to obtain the following.

Theorem 5.6. *Let f and s^* be as above. Then*

$$(-1)^{n+i}(f-s^*)(x) \ge 0 \quad \text{for } x \in (x_i, x_{i+1}),\ i = 0,1,\ldots,n+2r.$$

Proving Theorem 5.6 using Proposition 5.5(c) involves the identification of the convexity cone of $\{B_1,\ldots,B_{2r}\}$ and some other technical details. We have chosen to provide a more direct elementary proof that provides some insight into why the result is valid.

Proof. We first prove this result assuming that $f - s^*$ does not vanish identically on any interval (x_i, x_{i+1}), $i = 1,\ldots,n+2r-1$. With this assumption, and $f - s^*$ vanishing at $\{x_i\}_{i=1}^{n+2r}$, it follows that $f' - s^{*\prime}$ has at least $n-1+2r$ sign changes in (x_1, x_{n+2r}). Continuing we see that $f^{(n-2)} - s^{*(n-2)}$

has at least $2+2r$ sign changes in (x_1, x_{n+2r}). Recall that $f^{(n-2)}$ and $s^{*(n-2)}$ are continuous and convex.

Now

$$s^{*(n-2)}(x) = c + dx + \sum_{k=1}^{r} \lambda_k (n-1)!(x-\xi_k)_+^1.$$

That is, $s^{*(n-2)}$ is a convex, piecewise linear function with the knots ξ_k, $k = 1, \ldots, r$. On each (ξ_{k-1}, ξ_k), $k = 1, \ldots, r+1$, ($\xi_0 = a$, $\xi_{r+1} = b$) the function $f^{(n-2)} - s^{*(n-2)}$ has at most two sign changes, since $s^{*(n-2)}$ is linear and $f^{(n-2)}$ is convex there. Furthermore if $f^{(n-2)} - s^{*(n-2)}$ does not have two sign changes on each (ξ_{k-1}, ξ_k), $k = 1, \ldots, r+1$, then $f^{(n-2)} - s^{*(n-2)}$ does not have $2 + 2r$ sign changes in (a, b). This implies that $f^{(n-2)} - s^{*(n-2)}$ has exactly two sign changes in each (ξ_{k-1}, ξ_k), $k = 1, \ldots, r+1$, and $f - s^*$ has exactly $n + 2r$ sign changes, and in fact changes sign weakly at $\{x_i\}_{i=1}^{n+2r}$. Furthermore, $f^{(n-2)} - s^{*(n-2)}$ is necessarily positive to the right of its rightmost sign change. This implies that $f - s^*$ is positive to the right of its rightmost zero, and thus

$$(-1)^{n+i}(f - s^*)(x) \geq 0 \quad \text{for } x \in (x_i, x_{i+1}),\ i = 0, 1, \ldots, n+2r.$$

We proved this under the assumption that $f - s^*$ does not vanish identically in any (x_i, x_{i+1}). If this assumption does not hold, then we first apply a perturbation, obtain the result and then perturb back. ∎

6. Extensions to all of $\mathbb{R}$. In this section we present a method, albeit nonconstructive, for determining if an n-convex f on $E = \{x_1, \ldots, x_m\}$ has an extension to an n-convex function on all of $\mathbb{R}$. We first explain the method using the moment theory approach, and then redefine it in terms of splines.

Assume we are given the ordered points $a < x_1 < \cdots < x_m < b$, and data $\{\alpha_i\}_{i=1}^m$. We associate with them the moment $\mathbf{c} = (c_1, \ldots, c_{m-n})$ where

$$[x_j, \ldots, x_{j+n}; f] = c_j, \quad j = 1, \ldots, m-n,$$

where f is any function satisfying $f(x_i) = \alpha_i$, $i = 1, \ldots, m$. We wish to determine if $\mathbf{c} \in \mathcal{P}$, i.e., if there is an associated n-convex function f on all of $\mathbb{R}$ satisfying

$$[x_j, \ldots, x_{j+n}; f] = c_j, \quad j = 1, \ldots, m-n.$$

We shall determine whether $\mathbf{c} \in \operatorname{int} \mathcal{P}$, i.e., whether the associated measure μ is positive. (The boundary makes for technical difficulties we prefer to avoid.)

Let $[(m-n)/2] = r$ and consider $\{c_r, c_{r+1}\}$. A necessary and sufficient condition for the existence of a nonnegative Borel measure μ on $[x_r, x_{r+n+1}]$

satisfying

$$\int_{x_r}^{x_{r+n+1}} B_j(t)\,d\mu(t) = c_j, \qquad j = r, r+1,$$

is that $c_r, c_{r+1} \geq 0$ (this is equivalent to the case $m = n+2$ discussed in Section 5), and similarly with "nonnegative" replaced by "positive", and "$c_r, c_{r+1} \geq 0$" by "$c_r, c_{r+1} > 0$". Assume that such a positive measure exists. From Theorem 5.4 and Proposition 5.5 we have the existence of a unique lower principal representation for μ with respect to $\{B_r, B_{r+1}\}$, namely

$$c_j = \lambda_{11} B_j(\xi_{11}), \qquad j = r, r+1,$$

where $\lambda_{11} > 0$ and $x_{r+1} < \xi_{11} < x_{r+n}$. It is easily verified that both B_{r-1} and B_{r+2} are in the convexity cone of the WT-system span$\{B_r, B_{r+1}\}$. Applying Proposition 5.5(c), we see that for any nonnegative Borel measure ν satisfying

$$\int_{x_r}^{x_{r+n+1}} B_j(t)\,d\nu(t) = c_j, \qquad j = r, r+1,$$

we have

$$\int_{x_r}^{x_{r+n+1}} B_j(t)\,d\nu(t) \geq \lambda_{11} B_j(\xi_{11}), \qquad j = r-1, r+2.$$

Thus if there exists a nonnegative Borel measure ν on $[x_{r-1}, x_{r+n+2}]$ satisfying

$$\int_{x_{r-1}}^{x_{r+n+2}} B_j(t)\,d\nu(t) = c_j, \qquad j = r-1, r, r+1, r+2,$$

then necessarily

$$c_j = \int_{x_{r-1}}^{x_{r+n+2}} B_j(t)\,d\nu(t) \geq \lambda_{11} B_j(\xi_{11}), \qquad j = r-1, r+2.$$

On the other hand, if

$$c_j \geq \lambda_{11} B_j(\xi_{11}), \qquad j = r-1, r+2,$$

then there exists a nonnegative Borel measure ν on $[x_{r-1}, x_{r+n+2}]$ satisfying

$$\int_{x_{r-1}}^{x_{r+n+2}} B_j(t)\,d\nu(t) = c_j, \qquad j = r-1, r, r+1, r+2.$$

One such nonnegative measure is given by

$$\mu_1 = \lambda_{11}\delta_{\xi_{11}} + a\delta_{\eta_1} + b\delta_{\eta_2}$$

where we can arbitrarily choose $\eta_1 \in (x_{r-1}, x_r)$, $\eta_2 \in (x_{r+n+1}, x_{r+n+2})$, and define $a, b \geq 0$ accordingly. (Note that η_1 lies in the support of B_{r-1} but not in those of B_r, B_{r+1} and B_{r+2}, while η_2 lies in the support of B_{r+2} but not in those of B_{r-1}, B_r and B_{r+1}.)

Now, if

$$c_{r-1} = \lambda_{11} B_{r-1}(\xi_{11}),$$

i.e., $a = 0$, then there exists a nontrivial linear combination of $\{B_j\}_{j=r-1}^{r+2}$ that vanishes at ξ_{11} and η_2, implying that μ_1 is not a positive measure. Thus $(c_{r-1}, c_r, c_{r+1}, c_{r+2})$ is a boundary point of the associated moment cone in $\mathbb{R}^4$, and $\mathbf{c} \notin \operatorname{int} \mathcal{P}$. The similar result holds if $c_{r+2} = \lambda_{11} B_{r+2}(\xi_{11})$, i.e., $b = 0$.

If $a, b > 0$, then there exists no nontrivial nonnegative linear combination of $\{B_j\}_{j=r-1}^{r+2}$ that vanishes at ξ_{11}, η_1 and η_2. Thus $(c_{r-1}, c_r, c_{r+1}, c_{r+2})$ is an interior point of the associated moment cone in $\mathbb{R}^4$, and μ_1 is a positive measure relative to $\operatorname{span}\{B_{r-1}, B_r, B_{r+1}, B_{r+2}\}$. From Theorem 5.4 and Proposition 5.5 we now obtain the existence of a unique lower principal representation for μ_1 with respect to the WT-system $\operatorname{span}\{B_{r-1}, B_r, B_{r+1}, B_{r+2}\}$, namely

$$c_j = \lambda_{12} B_j(\xi_{12}) + \lambda_{22} B_j(\xi_{22}), \qquad j = r-1, r, r+1, r+2,$$

where $\lambda_{12}, \lambda_{22} > 0$ and $x_r < \xi_{12} < x_{r+n-1}, x_{r+2} < \xi_{22} < x_{r+n+1}$. The B-splines B_{r-2} and B_{r+3} are in the convexity cone of the WT-system $\operatorname{span}\{B_{r-1}, B_r, B_{r+1}, B_{r+2}\}$. This implies that for any nonnegative Borel measure ν satisfying

$$\int_{x_{r-2}}^{x_{r+n+3}} B_j(t)\, d\nu(t) = c_j, \qquad j = r-1, r, r+1, r+2,$$

we have

$$c_j \geq \lambda_{12} B_j(\xi_{12}) + \lambda_{22} B_j(\xi_{22}), \qquad j = r-2, r+3. \tag{6.1}$$

Furthermore, if (6.1) holds, then there exists a nonnegative Borel measure ν on $[x_{r-2}, x_{r+n+3}]$ satisfying

$$\int_{x_{r-2}}^{x_{r+n+3}} B_j(t)\, d\nu(t) = c_j, \qquad j = r-2, \ldots, r+3.$$

We now continue exactly as above. The vector $\mathbf{c}$ is in $\operatorname{int} \mathcal{P}$ if and only if there is strict inequality in (6.1) for $j = r-2$ and $j = r+3$. If this holds then we can easily construct a positive measure with respect to the WT-system $\operatorname{span}\{B_{r-2}, \ldots, B_{r+3}\}$ by adding to the right-hand side point functionals, with positive weights, at a point in (x_{r-2}, x_{r-1}) and a point in (x_{r+n+2}, x_{r+n+3}).

We continue this process in order to eventually verify if $\mathbf{c} \in \operatorname{int}\mathcal{P}$. Depending on the parity of $m-n$ we will either have two conditions or one condition to check at the very last step.

In terms of splines, our method of determining if there exists an n-convex f on $\mathbb{R}$ satisfying $f(x_i)=\alpha_i$, $i=1,\ldots,m$, is the following. This equivalence is a consequence of Theorem 5.6 and the explanation prior to Theorem 5.6. It is based on the fact that if $(c_{r-l+1},\ldots,c_{r+l})$ is in the interior of the associated moment cone in $\mathbb{R}^{2l}$, then there exists a spline of the form

$$s_l^*(x) = p_l(x) + \sum_{k=1}^{l} \lambda_{kl}(x-\xi_{kl})_+^{n-1},$$

where p_l is a polynomial of degree at most $n-1$, satisfying

$$s_l^*(x_j) = f(x_j) = \alpha_j, \qquad j = r-l+1,\ldots,r+l+n,$$

with $\lambda_{kl} > 0$, $k=1,\ldots,l$, and

$$x_{r-l+2k} < \xi_{kl} < x_{r-l+2k+n-1}, \qquad k=1,\ldots,l.$$

The moment $(c_{r-l},c_{r-l+1},\ldots,c_{r+l},c_{r+l+1})$ is then in the interior of the associated moment cone in $\mathbb{R}^{2l+2}$ if and only if

$$(-1)^n f(x_{r-l}) > (-1)^n s_l^*(x_{r-l}) \quad \text{and} \quad f(x_{r+l+n+1}) > s_l^*(x_{r+l+n+1}).$$

The drawback in these two equivalent methods of determining whether an n-convex function on E is extendable to all of $\mathbb{R}$ is that we do not know how to construct these lower principal representations, or the equivalent splines s_l^*. It is doubtful if there are more elementary criteria.

7. Envelopes. In this section we determine upper and lower pointwise bounds on the set of all n-convex functions f on (a,b) satisfying $f(x_i)=\alpha_i$, $i=1,\ldots,m$.

Assume f is n-convex on (a,b). Again, for convenience, we assume that the associated moment $\mathbf{c}$ is in $\operatorname{int}\mathcal{P}$. First assume that $m=n+2r$. Then from Theorem 5.6 we obtain a spline s^* with exactly r knots satisfying

$$(-1)^{n+i}(f-s^*)(x) \ge 0 \quad \text{for } x \in (x_i,x_{i+1}),\ i=0,1,\ldots,n+2r.$$

This gives half the pointwise bounds on any n-convex f on (a,b) satisfying $f(x_i)=\alpha_i$, $i=1,\ldots,n+2r$. How may we obtain the other half? Consider this same Theorem 5.6, but only with respect to the points $\{x_2,\ldots,x_{n+2r-1}\}$, i.e., the B-splines $\{B_2,\ldots,B_{2r-1}\}$. From the previous analysis we obtain a function

$$S^*(x) = p(x) + \sum_{k=1}^{r-1} \mu_k(x-\eta_k)_+^{n-1},$$

where $x_{2k+1} < \eta_k < x_{n+2k}$ and $\mu_k > 0$, $k = 1, \ldots, r-1$, satisfying

$$(-1)^{n+i-1}(f - S^*)(x) \geq 0 \quad \text{for } x \in (x_i, x_{i+1}),\ i = 1, \ldots, n+2r-1.$$

These new bounds are exact even though the n-convex S^* does not interpolate f at x_1 and x_{n+2r}.

THEOREM 7.1. *Let f, s^* and S^* be as above. Then*

$$\begin{aligned}
(-1)^{n+i}s^*(x) \leq (-1)^{n+i}f(x) &\leq (-1)^{n+i}S^*(x) \\
&\quad \textit{on } (x_i, x_{i+1}),\ i = 1, \ldots, n+2r-1, \\
s^*(x) \leq f(x) &\quad \textit{on } (x_{n+2r}, b) \\
(-1)^n s^*(x) \leq (-1)^n f(x) &\quad \textit{on } (a, x_1).
\end{aligned}$$

Furthermore these bounds are tight.

Proof. The inequalities of Theorem 7.1 always hold. We will prove that these bounds are tight. Those given by s^* are exact since s^* is an n-convex function that interpolates f at $\{x_i\}_{i=1}^{n+2r}$. We must prove the exactness of the bounds given by S^*.

By construction, we have $(f - S^*)(x_{n+2r}) \geq 0$ and $(-1)^n(f - S^*)(x_1) \geq 0$. Given $\varepsilon > 0$ small, let $y_1 \in (x_1, x_1 + \varepsilon)$ and $y_2 \in (x_{n+2r} - \varepsilon, x_{n+2r})$. Set

$$S_\varepsilon(x) = S^*(x) + A(x - y_2)_+^{n-1} + B(y_1 - x)_+^{n-1}$$

where A and B are chosen so that

$$S_\varepsilon(x_1) = f(x_1), \qquad S_\varepsilon(x_{n+2r}) = f(x_{n+2r}).$$

Thus $A \geq 0$ and $(-1)^n B \geq 0$. A simple analysis (see the last part of Section 3) shows that S_ε is n-convex. Thus we have constructed an n-convex function that, for all $\varepsilon > 0$ small, interpolates f at all the $\{x_i\}_{i=1}^{n+2r}$ and satisfies $S_\varepsilon(x) = S^*(x)$ for $x \in [x_1 + \varepsilon, x_{n+2r} - \varepsilon]$. Thus the bounds given by S^* in the statement of Theorem 7.1 are exact.

On (x_{n+2r}, b) and (a, x_1) there are in fact only one-sided bounds on f. For example, the function $s^*(x) + A(x - x_{n+2r})_+^{n-1}$ for $A \geq 0$ arbitrary is an n-convex function interpolating f at $\{x_i\}_{i=1}^{n+2r}$, and the set of such functions is not bounded above at any $x > x_{n+2r}$. A similar result holds on (a, x_1). ∎

In Theorem 7.1 we considered the case of $n + 2r$ interpolation points. Now assume that we are given $n + 2r + 1$ points

$$x_0 = a < x_1 < \cdots < x_{n+2r+1} < b = x_{n+2r+2}.$$

Again we assume that the corresponding moments are in the interior of $\mathcal{P}$, and thus we can apply Theorem 5.4 and Proposition 5.5, as above, both to the points $\{x_1, \ldots, x_{n+2r}\}$, i.e., the B-splines $\{B_1, \ldots, B_{2r}\}$, and to the points $\{x_2, \ldots, x_{n+2r+1}\}$, i.e., the B-splines $\{B_2, \ldots, B_{2r+1}\}$. From this analysis we obtain splines σ^* and Σ^*, respectively. Each of them has r knots satisfying certain inequalities. From Theorem 5.6 we have

$$(-1)^{n+i}(f-\sigma^*)(x) \geq 0 \quad \text{for } x \in (x_i, x_{i+1}),\ i = 0,1,\ldots,n+2r,$$

while

$$(-1)^{n+i-1}(f-\Sigma^*)(x) \geq 0 \quad \text{for } x \in (x_i, x_{i+1}),\ i = 1,\ldots,n+2r+1.$$

Neither σ^* nor Σ^* interpolates f at all $\{x_i\}_{i=1}^{n+2r+1}$. However, the analysis of Theorem 7.1 is easily applied.

Summarizing we have

THEOREM 7.2. *Let f, σ^* and Σ^* be as above. Then*

$$\begin{aligned} &(-1)^{n+i}\sigma^*(x) \leq (-1)^{n+i} f(x) \leq (-1)^{n+i}\Sigma^*(x) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } (x_i, x_{i+1}),\ i = 1,\ldots,n+2r, \\ &(-1)^n\sigma^*(x) \leq (-1)^n f(x) \qquad\quad \textit{for } x \in (a, x_1), \\ &\qquad \Sigma^*(x) \leq f(x) \qquad\qquad\qquad \textit{for } x \in (x_{n+2r+1}, b). \end{aligned}$$

Furthermore these bounds are tight.

The above results were proven under the assumption that the corresponding moments are in the interior of $\mathcal{P}$. What happens on the boundary? The boundary of the moment cone associated with a WT-system can be somewhat complicated. Therefore it seems easier in our case to simply perturb to the interior of $\mathcal{P}$, obtain the result of Theorem 7.1 or Theorem 7.2, as appropriate, and then perturb back. We will not delve into the details of this perturbation. Suffice it to say that if $\mathbf{c}$ is on the boundary of $\mathcal{P}$, then $\mathbf{c} + \boldsymbol{\varepsilon} \in \operatorname{int}\mathcal{P}$, where $\boldsymbol{\varepsilon} = (\varepsilon,\ldots,\varepsilon)$, $\varepsilon > 0$. If f is n-convex and satisfies $f(x_i) = \alpha_i$, $i = 1,\ldots,m$, and

$$[x_j,\ldots,x_{j+n}; f] = c_j, \qquad j = 1,\ldots,m-n,$$

then $g(x) = f(x) + \varepsilon x^n$ is n-convex, and satisfies $g(x_i) = \alpha_i + \varepsilon x_i^n$, $i = 1,\ldots,m$, and

$$[x_j,\ldots,x_{j+n}; g] = c_j + \varepsilon, \qquad j = 1,\ldots,m-n.$$

We can take a subsequence, as $\varepsilon \to 0$, of the associated perturbed splines, say s_ε^* and S_ε^*, that converges uniformly to some s^* and S^*, respectively. Thus s^*, corresponding to the s^* of Theorem 5.6, is necessarily a spline of the form

$$s^*(x) = p(x) + \sum_{k=1}^{r} \lambda_k (x-\xi_k)_+^{n-1}$$

where p is a polynomial of degree at most $n-1$, $x_{2k} \leq \xi_k \leq x_{n+2k-1}$ and $\lambda_k \geq 0$, $k = 1,\ldots,r$. (At least one of these inequalities must be an equality, or otherwise $\mathbf{c} \in \operatorname{int}\mathcal{P}$.) A similar result holds for S^*, σ^* and Σ^*.

Hermann Burchard in his thesis [6] considered, among other things, much of the subject matter of this paper and especially this section. Brief announcements appeared in Burchard [7] and [8]. Burchard considered this problem in the more general context of "generalized convex functions" given via the kernel of disconjugate differential equations (see Karlin and Studden [13, Chap. XI]). We have restricted ourselves in this paper to the differential equation $y^{(n)} = 0$. Burchard obtained results concerning the "envelopes" of n-convex interpolating functions as outlined in this section, although his results are less exact and hold under certain restrictions.

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