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CONSTRUCTIVE APPROXIMATION

Continuous Selections for the Metric Projection on C_1

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Abstract. $C_1(K)$ is the space of real continuous functions on K endowed with the usual L_1-norm where $K = \overline{\text{int } K}$ is compact in R^m. U is a finite-dimensional subspace of $C_1(K)$. The metric projection of $C_1(K)$ onto U contains a continuous selection with respect to L_1-convergence if and only if U is a unicity (Chebyshev) space for $C_1(K)$. Furthermore, if K is connected and U is not a unicity space for $C_1(K)$, then there is no continuous selection with respect to L_∞-convergence. An example is given of a U and a disconnected K with no continuous selection with respect to L_1-convergence, but many continuous selections with respect to L_∞-convergence.

1. Introduction

In what follows, K will denote a compact subset of R^m of positive finite Lebesgue measure, with $K = \overline{\text{int } K}$. $C_1 = C_1(K)$ will denote the space of real continuous functions on K endowed with the norm

$$\|f\|_1 = \int_K |f(x)|\, dx.$$

C_1 is a normed linear space, but is not complete. U will always denote a fixed n-dimensional, n finite, subspace of C_1.

For each $f \in C_1$, set

$$P(f) = \{u\colon u \in U, \|f-u\|_1 \le \|f-v\|_1, \text{all } v \in U\}.$$

$P(f)$ is the set of best approximants to f from U. It is well known that for each $f \in C_1$, $P(f)$ is convex, compact, and nonempty. The set-valued map P is called the *metric projection* onto U. Any single-valued map s from C_1 onto U for which $s(f) \in P(f)$ for each $f \in C_1$ is called a *metric selection* onto U. In this paper we are concerned with characterizing those subspaces U as above for which there exist "continuous" metric selections onto U, or for brevity, "continuous" selections.

To explain our result we introduce the following definitions.

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Definition 1. We say that U is a *unicity space* if $P(f)$ is a singleton for each $f \in C_1$.

Unicity spaces are sometimes referred to as Chebyshev spaces. U is a unicity space if to each $f \in C_1$ there exists a unique best approximant from U. Many such U exist and various characterizations of unicity spaces are known. One such, Proposition 4, will be used in this work.

We now make precise what we mean by a "continuous" selection.

Definition 2. Let $s(\cdot)$ be a metric selection onto U. We say that s is an *L_1-continuous selection* (or *L_1-continuous*) if s is continuous on C_1 with respect to L_1-convergence. That is, for $f, f_n \in C_1$ satisfying $\lim_{n\to\infty} \|f-f_n\|_1 = 0$, it follows that $\lim_{n\to\infty} s(f_n) = s(f)$. We say that s is an *L_∞-continuous selection* (or *L_∞-continuous*) if s is continuous on C_1 with respect to L_∞-convergence. That is, for $f, f_n \in C_1$ satisfying $\lim_{n\to\infty} \|f-f_n\|_\infty = 0$, it follows that $\lim_{n\to\infty} s(f_n) = s(f)$, where $\|\cdot\|_\infty$ represents the usual uniform norm on K.

Remark 1. We write $\lim_{n\to\infty} s(f_n) = s(f)$ since $s(f) \in U$ for each $f \in C_1$, U is finite-dimensional, and all norms are equivalent on U.

Remark 2. Since K has finite measure, L_1-continuity of s implies L_∞-continuity.

We can now state our result.

Theorem. *Under the above assumptions, there exists an L_1-continuous selection onto U if and only if U is a unicity space. Furthermore, if K is connected, and U is not a unicity space, then there exists no L_∞-continuous selection.*

Metric projections and continuous selections are much-studied objects. The interested reader is referred to Deutsch [3], and references therein. Relevant to this work is the following result of Lazar, Wulbert, and Morris [4]: on $L_1(K)$ there exist no (L_1-) continuous selections.

This paper is organized as follows. In Section 2 we recall some known results, including the simple fact that if U is a unicity space, then there is an L_1-continuous selection, namely, the metric projection. We also present an example of a disconnected K and a U for which there is no L_1-continuous selection, but where there are many L_∞-continuous selections.

Section 3 contains the main content of this paper. Therein is proved the remaining part of the theorem.

2. Preliminaries

We first state the easy half of the theorem. This result should be well known to the reader and is valid in a more general framework, see e.g., Cheney [1, p. 23]. Recall that if U is a unicity space, then the metric projection is single-valued.

Proposition 1. *If U is a unicity space for C_1, then the metric projection is L_1-continuous.*

We also recall the following characterization of best approximants from U. To this end, for each $f \in C_1$, set

$$Z(f) = \{x : f(x) = 0\},$$
$$N(f) = K \backslash Z(f).$$

We will also use the notation $|A| = \text{meas}\{A\}$ for any measurable subset A of K, and write $\int_A f$ for $\int_A f(x)\,dx$.

Proposition 2. *Let $f \in C_1$. Then $u^* \in P(f)$ if and only if*

$$\left|\int_K [\text{sgn}(f-u^*)]u\right| \le \int_{Z(f-u^*)} |u|$$

for all $u \in U$.

As a consequence of Proposition 2, we easily prove this next result. We will repeatedly use Proposition 3.

Proposition 3. *Let $f \in C_1$ and $u^* \in P(f)$. Then $v \in P(f)$ if and only if:*

(a) $(f-u^*)(f-v) \ge 0$ *on* K;
(b) $\int_K [\text{sgn}(f-u^*)](v-u^*) = \int_{Z(f-u^*)} |v-u^*|$.

Proof. Assume u^*, $v \in P(f)$. Then

$$\begin{aligned} \|f-u^*\|_1 &= \int_K [\text{sgn}(f-u^*)](f-u^*) \\ &= \int_K [\text{sgn}(f-u^*)](f-v) + \int_K [\text{sgn}(f-u^*)](v-u^*) \\ &\le \int_{N(f-u^*)} |f-v| + \int_{Z(f-u^*)} |v-u^*| \\ &= \int_{N(f-u^*)} |f-v| + \int_{Z(f-u^*)} |v-f| \\ &= \|f-v\|_1 \\ &= \|f-u^*\|_1. \end{aligned}$$

Since equality holds, it is necessary that:

(a′) $\int_K [\text{sgn}(f-u^*)](f-v) = \int_{N(f-u^*)} |f-v|$;
(b′) $\int_K [\text{sgn}(f-u^*)](v-u^*) = \int_{Z(f-u^*)} |v-u^*|$.

Thus (a) follows from (a′) and continuity, while (b) is (b′). If (a) and (b) hold, $u^* \in P(f)$ and $v \in U$, then by the above reasoning $v \in P(f)$. ∎

The next result is a characterization of unicity spaces for C_1. It is due to Cheney and Wulbert [2], with very minor modifications. Because of its importance, we present a proof thereof.

Proposition 4. *U is not a unicity space if and only if there exists a measurable function h on K for which* $h^2 \equiv 1$, *and a* $u^* \in U \backslash \{0\}$ *such that*:

(a) $\int_K hu = 0$, *all* $u \in U$;

(b) $h|u^*| \in C_1$ (*i.e.*, $h|u^*|$ *is continuous*).

Proof. Assume there exists an h and u^* as above. Set $f = h|u^*| \in C_1$. Since $\operatorname{sgn} f = h$ on $N(f)$, we have, from (a),

$$\int_K [\operatorname{sgn} f] u = -\int_{Z(f)} hu, \qquad \text{all} \quad u \in U.$$

Thus

$$\left| \int_K [\operatorname{sgn} f] u \right| \le \int_{Z(f)} |u|, \qquad \text{all} \quad u \in U,$$

and from Proposition 2, $0 \in P(f)$. For each $a \in \mathbf{R}$, $|a| < 1$, it is easily checked that $\operatorname{sgn} f = \operatorname{sgn}(f - au^*)$, and $Z(f) = Z(f - au^*)$. Thus

$$\left| \int_K [\operatorname{sgn}(f - au^*)] u \right| \le \int_{Z(f - au^*)} |u|, \qquad \text{all} \quad u \in U,$$

and from Proposition 2, $au^* \in P(f)$ for all $|a| < 1$. U is not a unicity space.

We now assume that U is not a unicity space. Let $f \in C_1$ be such that u_1, $u_2 \in P(f)$, $u_1 \ne u_2$. Set $f^* = f - (u_1 + u_2)/2$ and $u^* = (u_1 - u_2)/2$. It follows that 0, $\pm u \in P(f^*)$, and also that

$$2|f^*(x)| = |f^*(x) - u^*(x)| + |f^*(x) + u^*(x)|$$

for all $x \in K$. If $x \in Z(f^*)$, then $(f^* \pm u^*)(x) = 0$, implying that $x \in Z(u^*)$. Thus $Z(f^*) \subseteq Z(u^*)$.

Since $0 \in P(f^*)$, from the Hahn-Banach theorem there exists an $h \in L_\infty(K)$ for which:

(1) $\|h\|_\infty = 1$;
(2) $\int_K hu = 0$, all $u \in U$;
(3) $\int_K hf^* = \|f^*\|_1$.

From Lyapunov's theorem, it follows that we may assume that the above h satisfies $h^2 \equiv 1$ (see Phelps [5]). From (3), $h = \operatorname{sgn} f^*$ a.e. on $N(f^*)$. Thus we may also assume that $h = \operatorname{sgn} f^*$ on $N(f^*)$. Since $Z(f^*) \subseteq Z(u^*)$, it is now seen that $h|u^*|$ is continuous. ∎

We close this section by providing an example of a K and U with no L_1-continuous selection, but with many L_∞-continuous selections.

Example. $K = [-2, -1] \cup [1, 2]$ and U is the one-dimensional space spanned by the constant functions. Set $h = 1$ on $[-2, -1]$, and $h = -1$ on $[1, 2]$. Then it follows from Proposition 4 that U is not a unicity space.

Let

$$f_n(x)=\begin{cases}1, & x\in[-2,-1-1/n],\\ -2nx-(1+2n), & x\in[-1-1/n,-1],\\ -1, & x\in[1,2],\end{cases}$$

$$g_n(x)=\begin{cases}1, & x\in[-2,-1],\\ -2nx+(1+2n), & x\in[1,1+1/n],\\ -1, & x\in[1+1/n,2],\end{cases}$$

for all $n\in\mathbf{N}$. Obviously f_n, g_n, $h\in C_1$, and

$$\lim_{n\to\infty}\|h-f_n\|_1=\lim_{n\to\infty}\|h-g_n\|_1=0.$$

From Propositions 2 and 3, we obtain $P(f_n)=\{-1\}$ and $P(g_n)=\{1\}$ for all n. Thus there exists no L_1-continuous selection for U on K.

We now assume that $f_n, f\in C_1$, and $\lim_{n\to\infty}\|f-f_n\|_\infty=0$.

Claim 1. *If $P(f)$ is a singleton, then $\lim_{n\to\infty}s(f_n)=s(f)$ $(=P(f))$ for any choice of $s(f_n)\in P(f_n)$.*

The proof of this claim follows the method of the proof of Proposition 1, see Cheney [1, p. 23].

Claim 2. *Let $f\in C_1$ and assume that $P(f)$ is not a singleton. Then f satisfies either*

(a) $$\min_{-2\le x\le -1}f(x)>\max_{1\le x\le 2}f(x)$$

or

(b) $$\max_{-2\le x\le -1}f(x)<\min_{1\le x\le 2}f(x).$$

Proof. Assume the constant functions α and β $(\alpha<\beta)$ are in $P(f)$. Then $\lambda\alpha+(1-\lambda)\beta\in P(f)$ for all $\lambda\in[0,1]$. There exist α^1, $\beta^1\in P(f)$, $\alpha\le\alpha^1<\beta^1\le\beta$ such that $|Z(f-\alpha^1)|=|Z(f-\beta^1)|=0$. Thus, from Proposition 2,

$$\int_K \operatorname{sgn}(f-\alpha^1)=\int_K \operatorname{sgn}(f-\beta^1)=0,$$

and hence

$$|\{x\colon (f-\alpha^1)(x)>0\}|=|\{x\colon (f-\alpha^1)(x)<0\}|=1$$

and

$$|\{x\colon (f-\beta^1)(x)>0\}|=|\{x\colon (f-\beta^1)(x)<0\}|=1.$$

Furthermore, since $\alpha^1<\beta^1$,

$$\{x\colon (f-\beta^1)(x)>0\}\subseteq\{x\colon (f-\alpha^1)(x)>0\}$$

and

$$\{x: (f-\beta^1)(x)<0\} \supseteq \{x: (f-\alpha^1)(x)<0\}.$$

If f does not satisfy

$$\min_{-2\le x\le -1} f(x) \ge \beta^1 > \alpha^1 \ge \max_{1\le x\le 2} f(x)$$

or

$$\max_{-2\le x\le -1} f(x) \le \alpha^1 < \beta^1 \le \min_{1\le x\le 2} f(x),$$

then a contradiction ensues from the continuity of f on $[-2,-1]$ and on $[1,2]$. ■

Claim 3. *Assume that $P(f)$ is not a singleton and that*

$$C = \min_{-2\le x\le -1} f(x) > \max_{1\le x\le 2} f(x) = c.$$

Then the constant function α is in $P(f)$ if and only if $\alpha \in [c, C]$.

Proof. Let $\alpha \in (c, C)$. Then

$$\operatorname{sgn}(f-\alpha)(x) = \begin{cases} 1, & x\in[-2,-1], \\ -1, & x\in[1,2]. \end{cases}$$

From Proposition 2 it follows that $\alpha \in P(f)$. Since $P(f)$ is closed, $[c, C] \subseteq P(f)$.

Assume $\alpha \notin [c, C]$. Let $\beta \in (c, C) \subseteq P(f)$. From Proposition 3 applied to β and α it follows that $\alpha \notin P(f)$. ■

Let $\lambda, \mu \in [0,1]$. Define $s(f)$ as follows:

(1) If $P(f)$ is a singleton, $s(f) = P(f)$.

(2) If $C(f) = \min_{-2\le x\le -1} f(x) > \max_{1\le x\le 2} f(x) = c(f)$,

$$s(f) = \lambda C(f) + (1-\lambda)c(f),$$

(3) If $d(f) = \max_{-2\le x\le -1} f(x) < \min_{1\le x\le 2} f(x) = D(f)$,

$$s(f) = \mu D(f) + (1-\mu) d(f),$$

Claim 4. *s, as above, is an L_∞-continuous selection.*

Proof. Assume $f, f_n \in C_1$, f satisfies (2) and $\lim_{n\to\infty} \|f-f_n\|_\infty = 0$. Then for n sufficiently large f_n satisfies (2). Furthermore, if $C(f_n) = \min_{-2\le x\le -1} f_n(x)$ and $c(f_n) = \max_{1\le x\le 2} f_n(x)$, then $\lim_{n\to\infty} C(f_n) = C(f)$ and $\lim_{n\to\infty} c(f_n) = c(f)$. Thus $\lim_{n\to\infty} s(f_n) = s(f)$. The rest of the claim follows easily. ■

3. Main Result

In this section we assume that U is *not* a unicity space. In addition we assume that K is connected and we will prove that there is no L_∞-continuous selection. At the end of this section, we will indicate how a simple modification of the proof shows that there is no L_1-continuous selection for disconnected K.

The proof is somewhat lengthy and as such we divide it into a series of lemmas and propositions.

Recall that since U is not a unicity space, there exists a measurable h satisfying $h^2 \equiv 1$ on K and a $u^* \in U\backslash\{0\}$ for which

$$\int_K hu = 0, \qquad \text{all} \quad u \in U,$$

and $h|u^*|$ is continuous. We fix h throughout.

Lemma 5. *Let* $W = \{u\colon u \in U, h|u| \text{ continuous}\}$. *Then* $W = \{u\colon u \in U, hu \text{ continuous}\}$, *and* W *is a subspace of* U.

Proof. $h|u|$ is continuous if and only if $u(x) = 0$ at each pointof discontinuity x of h. Similarly, hu is continuous if and only if $u(x) = 0$ at each point of discontinuity x of h. The lemma follows. ■

Lemma 6. *Let* $v \in W$ *and* $u \in U$. *If* $|v| \geq hu$ *on* K, *then* $u \in W$,

Proof. Let x be a point of discontinuity of h. Then $0 = |v(x)| \geq (hu)(x)$. We must show that $u(x) = 0$. Assume $u(x) = c \neq 0$. Since h is discontinuous at x, in any neighborhood of x there exists a point y for which $(hu)(y) \geq |c|/2$. But $v(x) = 0$, v is continuous at x, and $|v| \geq hu$ on K. A contradiction ensues. Thus $u(x) = 0$, and $u \in W$. ■

Set $\bar{W} = \{u\colon u \in W, \|u\|_1 = 1\}$ and $J(u) = \{x\colon (hu)(x) \leq 0\}$. Note that $J(cu) = J(u)$ for all $c > 0$. The mapping $u \to |J(u)|$ is upper semicontinuous, and $\bar{W}$ is compact. Thus there exists a $w \in \bar{W}$ such that $|J(w)| \geq |J(u)|$ for all $u \in \bar{W}$. Therefore $|J(w)| \geq |J(u)|$ for all $u \in W\backslash\{0\}$. Since $\int_K hu = 0$, all $u \in U$, we have $0 < |J(w)| < |K|$.

Lemma 7. *Let* w *be as above. If* $u \in W$ *and* $|w| \geq hu$, *then* $J(w) \subseteq J(w+u)$. *If* $w \neq -u$, *then* $|J(w)| = |J(w+u)|$ *and* $h(w+u) > 0$ *a.e. on* $K\backslash J(w)$.

Proof. Let $x \in J(w)$. Then $(hw)(x) \leq 0$, and $|w(x)| = -(hw)(x) \geq (hu)(x)$. Thus $(h(w+u))(x) \leq 0$ implying that $J(w) \subseteq J(w+u)$. If $w + u \neq 0$, then, from the definition of w and Lemma 5, the remaining claims of the lemma follow. ■

Set

$$V = \{v\colon v \in \bar{W}, J(w) \subseteq J(v)\}.$$

V is a compact subset of a finite-dimensional subspace of $C(K)$. Hence equicontinuous and equibounded. For every $v \in V$, let

$$B_v = \{x\colon (hv)(x) < 0\}.$$

Then $B_v \subseteq J(w)$ for all $v \in V$, and $J(w)\backslash B_v \subseteq Z(v)$. Furthermore, $B_v \neq \varnothing$.

Lemma 8. *There exists a $w^* \in V$ such that, for all $v \in V$, $B_v \subseteq B_{w^*}$.*

Proof. Assume $v_1, \ldots, v_k \in V$ and $B_{v_i} \subsetneqq B_{v_{i+1}}$, $i = 1, \ldots, k-1$. We claim that $v_1, \ldots, v_k$ are linearly independent. To see this, choose $x_j \in B_{v_j} \backslash B_{v_{j-1}}$, $j = 1, \ldots, k$ (where $B_{v_0} = \varnothing$). Now $v_j(x_j) \neq 0$ by definition, and $v_i(x_j) = 0$ for all $i < j$ since $x_j \in B_{v_j} \backslash B_{v_{j-1}} \subseteq B_{v_j} \backslash B_{v_i} \subseteq J(w) \backslash B_{v_i} \subseteq Z(v_i)$. Thus the matrix $(v_i(x_j))_{i,j=1}^k$ is of rank k, and $v_1, \ldots, v_k$ are linearly independent.

Now, choose $v_1 \in V$. If there exists a $v_2 \in V$ for which $B_{v_1} \subsetneqq B_{v_2}$, then replace v_1 by v_2. Continue this process. Since $V \subseteq U$ and $\dim U < \infty$, it follows that this process stops after a finite number of steps.

Thus there exists a $w^* \in V$ such that if $v \in V$ and $B_{w^*} \subseteq B_v$, then $B_{w^*} = B_v$. We claim that w^* satisfies the conditions of the lemma. Assume $v \in V$ and $B_v \not\subseteq B_{w^*}$. Thus there exists a $y \in K$ such that $(hv)(y) < 0 = (hw^*)(y)$. Set $z = (v + w^*)/\|v + w^*\|_1$. It follows that $z \in V$ and $B_{w^*} \subsetneqq B_z$. This is a contradiction. ∎

Replace w by w^* in the definition of V. However, for ease of notation we continue to denote it by w. Thus we now assume that

$$V = \{v \colon v \in \bar{W}, J(w) \subseteq J(v)\}$$

satisfies:

(1) $|J(w)| \geq |J(u)|$, all $u \in W \backslash \{0\}$;
(2) $B_v \subseteq B_w$ for all $v \in V$.

For w as above, set

$$A = \{x \colon (hw)(x) > 0\}$$

and

$$B = \{x \colon (hw)(x) < 0\},$$

i.e., $B = B_w$. Note that for all $v \in V$, $hv > 0$ a.e. on A.

Since K is connected, there exists an $\alpha \in \partial A \cap Z(w)$ such that for each $\varepsilon > 0$, sufficiently small,

$$A_\varepsilon = A \cap \{x \colon |x - \alpha| < \varepsilon\}$$

satisfies $|A_\varepsilon| > 0$.

Similarly, there exists a $\beta \in \partial B \cap Z(w)$ such that for each $\varepsilon > 0$, sufficiently small,

$$B_\varepsilon = B \cap \{x \colon |x - \beta| < \varepsilon\}$$

satisfies $|B_\varepsilon| > 0$.

For all $v \in V$, $hv > 0$ a.e. on A_ε and $v(\alpha) = 0$ $(Z(w) \subseteq Z(v))$. For all $v \in V$, $hv \leq 0$ on B_ε and $v(\beta) = 0$ $(Z(w) \subseteq Z(v))$.

For each $\varepsilon > 0$ choose $v_\varepsilon V$ to satisfy

$$\int_{B_\varepsilon} |v_\varepsilon| \geq \int_{B_\varepsilon} |v|, \qquad \text{all} \quad v \in V.$$

V is compact so that such a v_ε exists.

Lemma 9. *Assume $u \in U$ satisfies $|v_\varepsilon| \geq hu$ on K, and $-|v_\varepsilon| \geq hu$ on A_ε. Then $u = -v_\varepsilon$.*

Proof. Assume $u \neq -v_\varepsilon$. From Lemmas 6 and 7 with w replaced by v_ε, $u \in W$ and $h(v_\varepsilon + u) > 0$ a.e. on $A = K \backslash J(w)$. On A, $|v_\varepsilon| = hv_\varepsilon$. Thus on A_ε, $-hv_\varepsilon = -|v_\varepsilon| \geq hu$, implying that $h(v_\varepsilon + u) \leq 0$ on A_ε. This contradicts the fact that $h(v_\varepsilon + u) > 0$ a.e. on A. ■

Lemma 10. *Assume $u \in U$ satisfies $|v_\varepsilon| \geq hu$ on K, and $-|v_\varepsilon| \geq hu$ on B_ε. Then $\|v_\varepsilon + u\|_1 = 2$.*

Proof. From Lemmas 6 and 7, $u \in W$ and $z = (v_\varepsilon + u)/\|v_\varepsilon + u\|_1 \in V$ if $u \neq -v_\varepsilon$. Since $-|v_\varepsilon| \geq hu$ on B_ε, and $\int_{B_\varepsilon} |v_\varepsilon| \geq \int_{B_\varepsilon} |w| > 0$, it easily follows that $u \neq -v_\varepsilon$. Thus $z \in V$. Now,

$$\begin{aligned}
\|v_\varepsilon + u\|_1 &= \int_K [\operatorname{sgn}(v_\varepsilon + u)](v_\varepsilon + u) \\
&= \int_K [\operatorname{sgn} h(v_\varepsilon + u)] h(v_\varepsilon + u) \\
&= \int_{h(v_\varepsilon+u)>0} h(v_\varepsilon + u) - \int_{h(v_\varepsilon+u)\leq 0} h(v_\varepsilon + u).
\end{aligned}$$

Since $h(v_\varepsilon + u) > 0$ a.e. on $hv_\varepsilon > 0$, and $h(v_\varepsilon + u) \leq 0$ a.e. on $hv_\varepsilon \leq 0$,

$$\begin{aligned}
\|v_\varepsilon + u\|_1 &= \int_{hv_\varepsilon>0} h(v_\varepsilon + u) - \int_{hv_\varepsilon\leq 0} h(v_\varepsilon + u) \\
&= \int_{hv_\varepsilon>0} hv_\varepsilon - \int_{hv_\varepsilon\leq 0} hv_\varepsilon + \int_{hv_\varepsilon>0} hu - \int_{hv_\varepsilon\leq 0} hu.
\end{aligned}$$

Now,

$$\int_{hv_\varepsilon>0} hv_\varepsilon - \int_{hv_\varepsilon\leq 0} hv_\varepsilon = \|hv_\varepsilon\|_1 = \|v_\varepsilon\|_1 = 1,$$

$$\int_{hv_\varepsilon>0} hu + \int_{hv_\varepsilon\leq 0} hu = 0, \qquad \text{all} \quad u \in U,$$

and on $hv_\varepsilon > 0$, $hv_\varepsilon = |v_\varepsilon| \geq hu$.
Thus,

$$\|v_\varepsilon + u\|_1 = 1 + 2\int_{hv_\varepsilon>0} hu \leq 1 + 2\int_{hv_\varepsilon>0} hv_\varepsilon = 1 + \|v_\varepsilon\|_1 = 2.$$

On B_ε, $hv_\varepsilon \leq 0$ and $0 \geq hv_\varepsilon = -|v_\varepsilon| \geq hu$. Thus $0 \geq 2hv_\varepsilon \geq h(v_\varepsilon + u)$ implying that $|v_\varepsilon + u| \geq 2|v_\varepsilon|$ on B_ε. If $\|v_\varepsilon + u\|_1 < 2$, then

$$\int_{B_\varepsilon} |z| > \int_{B_\varepsilon} |v_\varepsilon|,$$

contradicting our choice of v_ε. ■

We now come to the main content of the proof of the theorem.

Set $H_\varepsilon = h|v_\varepsilon|$, and construct

$$G_\varepsilon(x) = \begin{cases} H_\varepsilon(x), & x \notin A_{2\varepsilon}, \\ \sigma_\varepsilon(x), & x \in A_{2\varepsilon} \backslash \bar{A}_\varepsilon, \\ -v_\varepsilon(x), & x \in \bar{A}_\varepsilon, \end{cases}$$

where $|\sigma_\varepsilon| < |v_\varepsilon|$ on $(A_{2\varepsilon} \backslash \bar{A}_\varepsilon) \backslash Z(v_\varepsilon)$, $\sigma_\varepsilon = 0$ on $(A_{2\varepsilon} \backslash \bar{A}_\varepsilon) \cup Z(v_\varepsilon)$, and $G_\varepsilon \in C_1$. (Note that on $\bar{A}_\varepsilon$, $G_\varepsilon = -H_\varepsilon$.) Such a construction is possible.

Similarly, let

$$F_\varepsilon(x) = \begin{cases} H_\varepsilon(x), & x \notin B_{2\varepsilon}, \\ \gamma_\varepsilon(x), & x \in B_{2\varepsilon} \backslash \bar{B}_\varepsilon, \\ v_\varepsilon(x), & x \in \bar{B}_\varepsilon, \end{cases}$$

where $|\gamma_\varepsilon| < |v_\varepsilon|$ on $(B_{2\varepsilon} \backslash \bar{B}_\varepsilon) \backslash Z(v_\varepsilon)$, $\gamma_\varepsilon = 0$ on $(B_{2\varepsilon} \backslash \bar{B}_\varepsilon) \cup Z(v_\varepsilon)$, and $F_\varepsilon \in C_1$.

Proposition 11. *$P(G_\varepsilon) = \{-v_\varepsilon\}$, and if $u \in P(F_\varepsilon)$, then $\|v_\varepsilon + u\|_1 = 2$.*

Proof. The idea is to prove:

(I) $-v_\varepsilon \in P(G_\varepsilon)$ and if $u \in P(G_\varepsilon)$, then $|v_\varepsilon| \geq hu$ on K, and $-|v_\varepsilon| \geq hu$ on A_ε;
(II) $v_\varepsilon \in P(F_\varepsilon)$ and if $u \in P(F_\varepsilon)$, then $|v_\varepsilon| \geq hu$ on K, and $-|v_\varepsilon| \geq hu$ on B_ε.

We then apply Lemmas 9 and 10 to (I) and (II), respectively, to obtain the desired results. The proofs of (I) and (II) are totally analogous. As such we prove only (I).

We first consider $\operatorname{sgn}(G_\varepsilon + v_\varepsilon)$:

(i) On $\bar{A}_\varepsilon$.

$$\operatorname{sgn}(G_\varepsilon + v_\varepsilon) = \operatorname{sgn}(-v_\varepsilon + v_\varepsilon) = 0.$$

(ii) On $A_{2\varepsilon} \backslash \bar{A}_\varepsilon$,

$$\begin{aligned} \operatorname{sgn}(G_\varepsilon + v_\varepsilon) &= \operatorname{sgn}(\sigma_\varepsilon + v_\varepsilon) \\ &= \operatorname{sgn} v_\varepsilon \\ &= \operatorname{sgn} H_\varepsilon. \end{aligned}$$

(iii) Off $A_{2\varepsilon}$,

$$\begin{aligned} \operatorname{sgn}(G_\varepsilon + v_\varepsilon) &= \operatorname{sgn}(H_\varepsilon + v_\varepsilon) \\ &= \begin{cases} 0, & H_\varepsilon = -v_\varepsilon, \\ \operatorname{sgn} H_\varepsilon, & H_\varepsilon = v_\varepsilon \neq 0. \end{cases} \end{aligned}$$

Thus, for all $u \in U$,

(a) $$\int_{Z(G_\varepsilon + v_\varepsilon)} |u| = \int_{H_\varepsilon = -v_\varepsilon \neq 0} |u| + \int_{Z(H_\varepsilon)} |u| + \int_{\bar{A}_\varepsilon} |u|;$$

(b) $$\begin{aligned} \int_K \operatorname{sgn}(G_\varepsilon + v_\varepsilon) u &= \int_{\substack{H_\varepsilon = v_\varepsilon \neq 0 \\ x \notin A_{2\varepsilon}}} (\operatorname{sgn} H_\varepsilon) u + \int_{A_{2\varepsilon} \backslash \bar{A}_\varepsilon} (\operatorname{sgn} H_\varepsilon) u \\ &= \int_{H_\varepsilon = v_\varepsilon \neq 0} (\operatorname{sgn} H_\varepsilon) u - \int_{\bar{A}_\varepsilon} (\operatorname{sgn} H_\varepsilon) u. \end{aligned}$$

For every $u \in U$,

$$0 = \int_K hu = \int_{H_\varepsilon = v_\varepsilon \neq 0} (\operatorname{sgn} H_\varepsilon)u + \int_{H_\varepsilon = -v_\varepsilon \neq 0} (\operatorname{sgn} H_\varepsilon)u + \int_{Z(H_\varepsilon)} hu.$$

Therefore,

$$\text{(b}'\text{)} \int_K \operatorname{sgn}(G_\varepsilon + v_\varepsilon)u = -\int_{H_\varepsilon = -v_\varepsilon \neq 0} (\operatorname{sgn} H_\varepsilon)u - \int_{Z(H_\varepsilon)} hu - \int_{\tilde{A}_\varepsilon} (\operatorname{sgn} H_\varepsilon)u.$$

From (a) and (b′),

$$\left| \int_K \operatorname{sgn}(G_\varepsilon + v_\varepsilon)u \right| \leq \int_{Z(G_\varepsilon + v_\varepsilon)} |u|.$$

Thus, from Proposition 2, $-v_\varepsilon \in P(G_\varepsilon)$.

Assume $u \in P(G_\varepsilon)$, $u \neq -v_\varepsilon$. From Proposition 3 we have:

(A) $(G_\varepsilon + v_\varepsilon)(G_\varepsilon - u) \geq 0$ on K;
(B) $\int_K \operatorname{sgn}(G_\varepsilon + v_\varepsilon)(u + v_\varepsilon) = \int_{Z(G_\varepsilon + v_\varepsilon)} |u + v_\varepsilon|$.

From (i)-(iii), (a), and (b′) we obtain:

(1) On $H_\varepsilon = v_\varepsilon \neq 0$, $x \notin A_{2\varepsilon}$, $(\operatorname{sgn} H_\varepsilon)(H_\varepsilon - u) \geq 0$.
(2) On $A_{2\varepsilon} \backslash \tilde{A}_\varepsilon$, $(\operatorname{sgn} H_\varepsilon)(\sigma_\varepsilon - u) \geq 0$.
(3) On $H_\varepsilon = -v_\varepsilon \neq 0$, $-(\operatorname{sgn} H_\varepsilon)(u + v_\varepsilon) \geq 0$.
(4) On $Z(H_\varepsilon)$, $-h(u + v_\varepsilon) \geq 0$.
(5) On $\tilde{A}_\varepsilon$, $-(\operatorname{sgn} H_\varepsilon)(u + v_\varepsilon) \geq 0$.

Since $\operatorname{sgn} H_\varepsilon = h$ off $Z(H_\varepsilon)$, it follows from the construction, and from (1)-(5), that $|v_\varepsilon| \geq hu$ on all of K, and $-|v_\varepsilon| \geq hu$ on A_ε. ∎

Proof of the Theorem. V is compact and equicontinuous. There therefore exists a subsequence $\varepsilon_n \downarrow 0$ and a $v^* \in V$ such that $\lim_{n\to\infty} \|v^* - v_{\varepsilon_n}\|_\infty = 0$. Set $H^* = h|v^*|$ and recall that $H_\varepsilon = h|v_\varepsilon|$. Then $\lim_{n\to\infty} \|H^* - H_{\varepsilon_n}\|_\infty = 0$. By definition,

$$\|H_\varepsilon - G_\varepsilon\|_\infty \leq 2 \max_{x \in \tilde{A}_{2\varepsilon}} |v_\varepsilon(x)|,$$

$$\|H_\varepsilon - F_\varepsilon\|_\infty \leq 2 \max_{x \in \tilde{B}_{2\varepsilon}} |v_\varepsilon(x)|.$$

By construction, $v(\alpha) = v(\beta) = 0$ for all $v \in V$. From the equicontinuity of V it therefore follows that

$$\lim_{n\to\infty} \|H_{\varepsilon_n} - G_{\varepsilon_n}\|_\infty = \lim_{n\to\infty} \|H_{\varepsilon_n} - F_{\varepsilon_n}\|_\infty = 0.$$

Thus

$$\lim_{n\to\infty} \|H^* - G_{\varepsilon_n}\|_\infty = \lim_{n\to\infty} \|H^* - F_{\varepsilon_n}\|_\infty = 0.$$

From Proposition 11, $P(G_{\varepsilon_n}) = \{-v_{\varepsilon_n}\}$ and, for every $u \in P(F_{\varepsilon_n})$, $\operatorname{dist}(u, -v_{\varepsilon_n}) = \|u + v_{\varepsilon_n}\|_1 = 2$. Thus there cannot exist an L_∞-continuous selection onto U.

Assume now that K is *not* connected. The above proof fails since we cannot necessarily find the requisite α and β. That this may indeed be the case is illustrated by the example in Section 2. What we do in this case is simply choose α to be any point in A, and β to be any point in B. A reading of the proof shows that all of the lemmas and propositions still hold. (Lemma 8 is unnecessary.) We do lose the uniform convergence of $H_{\varepsilon_n} - G_{\varepsilon_n}$ and $H_{\varepsilon_n} - F_{\varepsilon_n}$ to zero. However, it is easily seen that

$$\lim_{n\to\infty} \|H_{\varepsilon_n} - G_{\varepsilon_n}\|_1 = \lim_{n\to\infty} \|H_{\varepsilon_n} - F_{\varepsilon_n}\|_1 = 0.$$

It therefore follows that there cannot exist an L_1-continuous selection onto U. ■

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