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Best Approximation with Coefficient Constraints

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For given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, with $-\infty \leqslant \alpha_i < \beta_i \leqslant \infty$ $(i = 1, \ldots, n)$, and continuous functions $u_1, \ldots, u_n$, set

$$U(\boldsymbol{\alpha}; \boldsymbol{\beta}) = \left\{ \sum_1^n a_i u_i : \alpha_i \leqslant a_i \leqslant \beta_i \ \ (i = 1, \ldots, n) \right\}.$$

This paper is concerned with best approximating continuous functions, in the uniform norm, from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. We exactly characterize the $u_1, \ldots, u_n$ for which the best approximant to every continuous function is unique. We also present a general theorem characterizing all best approximants. When $(u_1, \ldots, u_n)$ is a Descartes, or a weak Descartes, system on $[0, 1]$, explicit characterizations of the best approximants in terms of equioscillations are given. These results are applied to spline spaces. They are also used to complete the characterizations in certain specific examples previously considered in the literature.

1. Introduction

LET B denote a compact Hausdorff space containing at least $n + 1$ points, and let $C(B)$ be the normed linear space of real-valued continuous functions on B endowed with the uniform norm

$$\|f\| = \max_{x \in B} |f(x)|.$$

U_n will denote an n-dimensional subspace of $C(B)$. For given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ satisfying $-\infty \leqslant \alpha_i < \beta_i \leqslant \infty$ $(i = 1, \ldots, n)$, and a given basis $(u_1, \ldots, u_n)$ for U_n, we set

$$U(\boldsymbol{\alpha}; \boldsymbol{\beta}) = \left\{ u = \sum_{i=1}^n a_i u_i : \alpha_i \leqslant a_i \leqslant \beta_i \ \ (i = 1, \ldots, n) \right\}.$$

We are interested in the problem of best approximating functions in $C(B)$ from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. Since $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ is convex, closed, and non-empty, a best approximant to each $f \in C(B)$ exists. Thus our concern will be with both characterizing best approximants, and determining when uniqueness holds.

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In Section 2, we first present a general characterization result for convex, closed, non-empty subsets of U_n, and then apply it to $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. In Section 3, we determine the $u_1, \ldots, u_n$ such that, to each $f \in C(B)$, there exists a unique best approximant from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ for every $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as above. We show that it is both necessary and sufficient that $\{u_{i_1}, \ldots, u_{i_k}\}$ be a Haar system for every choice of distinct $i_1, \ldots, i_k \in \{1, \ldots, n\}$ and every $k = 1, \ldots, n$. We also prove strong uniqueness in this case.

From Section 4 onward, we assume that $B = [0, 1]$. In Section 4, we assume that $(u_1, \ldots, u_n)$ is a 'Descartes' system and characterize via equioscillations the unique best approximant from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ to each $f \in C[0, 1]$. In Section 5, we consider the case where $(u_1, \ldots, u_n)$ is a 'weak Descartes' system, and characterize all best approximants in this case.

In doing so, we complete some characterizations which were considered in the literature, and expand upon some incomplete ones. Finally, in Section 6, we consider spline spaces and obtain a full characterization of all best approximants from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, where $(u_1, \ldots, u_n)$ is a B-spline basis for the space.

For a general review of problems of best approximation with constraints, the interested reader should consult Chalmers & Taylor [3], and the references therein.

2. Characterization: general case

Let $C(B)$ and U_n be as defined in Section 1. Let K be any closed, convex, non-empty subset of U_n. We say that u^* is a best approximant to f from K if $u^* \in K$, and

$$\|f - u^*\| = \min \{\|f - u\| : u \in K\}.$$

It is well known that a best approximant always exists. The following general result is essentially to be found in Deutsch & Maserick [5].

THEOREM 2.1 *Assume $f \in C(B) \backslash K$. Then u^* is a best approximant to f from K if and only if there exist distinct points $\{x_j\}_{j=1}^r$, with $1 \leq r \leq n + 1$, and non-zero numbers $\{c_j\}_{j=1}^r$ satisfying*

(a) $$(f - u^*)(x_j) = (\operatorname{sgn} c_j) \|f - u^*\| \quad (j = 1, \ldots, r),$$

(b) $$\sum_{j=1}^{r} c_j u^*(x_j) \geq \sum_{j=1}^{r} c_j u(x_j) \quad \textit{for all } u \in K.$$

Remark. Deutsch & Maserick prove this result using functional-analytic methods. It may also be directly proven by more standard techniques paralleling those used, for example, in Rivlin [13: p. 63], where the above theorem is proved for $K = U_n$. However, it is then necessary to replace a Caratheodory-theorem argument with a Helly-theorem argument.

The following simple consequence is well worth noting, since it will be used in the subsequent analysis.

COROLLARY 2.2 *Let u^* be a best approximant to f from K satisfying* (a) *and* (b). *Assume $\tilde{u}$ is any other best approximant to f from K. Then $u^*(x_j) = \tilde{u}(x_j)$ $(j = 1, \ldots, r)$.*

Proof. Let $u \in K$. From (a) and (b), we have

$$\sum_{j=1}^{r} |c_j| \, \|f - u^*\| = \sum_{j=1}^{r} c_j(f - u^*)(x_j) \leq \sum_{j=1}^{r} c_j(f - u)(x_j)$$
$$\leq \sum_{j=1}^{r} |c_j| \, \|f - u\|.$$

Setting $u = \tilde{u}$, and since $\|f - u^*\| = \|f - \tilde{u}\|$, we obtain

$$c_j(f - u^*)(x_j) = c_j(f - \tilde{u})(x_j) \quad (j = 1, \ldots, r).$$

Moreover, $c_j \neq 0$ for all j, and thus $u^*(x_j) = \tilde{u}(x_j)$ $(j = 1, \ldots, r)$. □

We now specialize the above results to the case of coefficient constraints. To this end, we first introduce the following notation. For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ satisfying $-\infty \leq \alpha_i < \beta_i \leq \infty$ $(i = 1, \ldots, n)$, set

$$\mathrm{A}(\boldsymbol{\alpha}; \boldsymbol{\beta}) = \{\boldsymbol{a} : \boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n, \ \alpha_i \leq a_i \leq \beta_i \ (i = 1, \ldots, n)\}.$$

For a given basis $(u_1, \ldots, u_n)$ for U_n, set

$$U(\boldsymbol{\alpha}; \boldsymbol{\beta}) = \left\{ \sum_{i=1}^{n} a_i u_i : \boldsymbol{a} \in \mathrm{A}(\boldsymbol{\alpha}; \boldsymbol{\beta}) \right\}.$$

To $u^* = \sum_{i=1}^{n} a_i^* u_i \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, we associate the vector $\boldsymbol{b}^* = (b_1^*, \ldots, b_n^*)$, where

$$b_i^* = \begin{cases} 1 & \text{if } a_i^* = \beta_i, \\ 0 & \text{if } \alpha_i < a_i^* < \beta_i, \\ -1 & \text{if } a_i^* = \alpha_i. \end{cases}$$

As a direct application of Theorem 2.1, we obtain the following result.

THEOREM 2.3 *Let $f \in C(B) \backslash U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. Then $u^* = \sum_{i=1}^{n} a_i^* u_i$ is a best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ if and only if there exist distinct points $\{x_j\}_{j=1}^{r}$, with $1 \leq r \leq n + 1$, and non-zero numbers $\{c_j\}_{j=1}^{r}$ satisfying*

(a) $$(f - u^*)(x_j) = (\operatorname{sgn} c_j) \, \|f - u^*\| \quad (j = 1, \ldots, r),$$

(b) $$\sum_{j=1}^{r} c_j u_i(x_j) \begin{Bmatrix} \geq 0 & \text{if } b_i^* = 1 \\ = 0 & \text{if } b_i^* = 0 \\ \leq 0 & \text{if } b_i^* = -1 \end{Bmatrix} \quad (i = 1, \ldots, n).$$

Part (a) of the next result is simply a restatement of Corollary 2.2. Part (b) is also of importance, as shall be seen in the next sections.

PROPOSITION 2.4 *Let $u^* = \sum_{i=1}^{n} a_i^* u_i$ be a best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ satisfying (a) and (b) of Theorem 2.3. If $\tilde{u} = \sum_{i=1}^{n} \tilde{a}_i u_i$ is any other best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, then*

(a) $u^*(x_j) = \tilde{u}(x_j)$ $(j = 1, \ldots, r)$

(b) *If* $\sum_{j=1}^{r} c_j u_i(x_j) \neq 0$ *then* $a_i^* = \tilde{a}_i = \begin{cases} \beta_i & \text{if } b_i^* = 1, \\ \alpha_i & \text{if } b_i^* = -1. \end{cases}$

Proof. On the basis of Corollary 2.2, it is only necessary to prove (b). As a consequence of (a), we have

$$\sum_{j=1}^{r} c_j u^*(x_j) = \sum_{j=1}^{r} c_j \tilde{u}(x_j).$$

Thus

$$\sum_{i=1}^{n} \left((a_i^* - \tilde{a}_i) \sum_{j=1}^{r} c_j u_i(x_j) \right) = 0.$$

From (b) of Theorem 2.3, we have $a_i^* - \tilde{a}_i = \beta_i - \tilde{a}_i \geq 0$ if $\sum_{j=1}^{r} c_j u_i(x_j) > 0$, while $a_i^* - \tilde{a}_i = \alpha_i - \tilde{a}_i \leq 0$ if $\sum_{j=1}^{r} c_j u_i(x_j) < 0$. Thus

$$(a_i^* - \tilde{a}_i) \sum_{j=1}^{r} c_j u_i(x_j) \geq 0 \quad (i = 1, \dots, n).$$

Therefore, $(a_i^* - \tilde{a}_i) \sum_{j=1}^{r} c_j u_i(x_j) = 0$ for $i = 1, \dots, n$, and (b) follows. □

Remarks. (i) We do not consider, in the definition of $A(\boldsymbol{\alpha}; \boldsymbol{\beta})$, the possibility of $\alpha_i = \beta_i$ for any i, since this would simply imply a reduction of the basis $u_1, \dots, u_n$ to a subset thereof, and a translation of the solution.

(ii) The choice of $A(\boldsymbol{\alpha}; \boldsymbol{\beta})$ and $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ is a special case of the following general situation. Let $D = [d_{ij}]_{i=1}^{m}{}_{j=1}^{n}$. Set $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$, with $-\infty \leq \alpha_i < \beta_i \leq \infty$ $(i = 1, \dots, m)$, and

$$A(D; \boldsymbol{\alpha}; \boldsymbol{\beta}) = \left\{ \boldsymbol{a} : \boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{R}^n,\ \alpha_i \leq \sum_{j=1}^{n} d_{ij} a_j \leq \beta_i \ (i = 1, \dots, m) \right\},$$

$$U(D; \boldsymbol{\alpha}; \boldsymbol{\beta}) = \left\{ \sum_{i=1}^{n} a_i u_i : \boldsymbol{a} \in A(D; \boldsymbol{\alpha}; \boldsymbol{\beta}) \right\}.$$

We do not deal with this general case. However, it should be noted that, if rank $D = m$, there then exists a basis $(v_1, \dots, v_n)$ for U_n for which

$$U(D; \boldsymbol{\alpha}; \boldsymbol{\beta}) = \left\{ \sum_{i=1}^{n} a_i' v_i : \alpha_i \leq a_i' \leq \beta_i \ (i = 1, \dots, m) \right\}.$$

3. Uniqueness and strong uniqueness

We are interested in determining when to each $f \in C(B)$ there exists a *unique* best approximant from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. For ease of exposition, we present the following definition.

Definition 3.1 If to each $f \in C(B)$ there exists a unique best approximant from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, then we say that $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ is a *unicity set* for $C(B)$.

Unicity sets are often referred to as *Haar sets* in the literature. The reason for this latter term is due to the following classic theorem of Haar [6].

Haar Theorem *An m-dimensional subspace* $U_m = \operatorname{span}\{u_1, \dots, u_m\}$ *of* $C(B)$ *is a unicity set for* $C(B)$ *if and only if* $\det[u_i(x_j)]_{i,j=1}^{m} \neq 0$ *for every choice of m distinct points* $\{x_j\}_{j=1}^{m}$ *in B.*

Such a subspace U_m will be referred to as a *Haar space*, and $\{u_1, \dots, u_m\}$ as a *Haar system*.

To determine whether $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ is a unicity set, we make use of the seemingly innocuous part (b) of Proposition 2.4. Let us see how.

Let $u^* \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ be a best approximant to $f \in C(B) \backslash U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. Assume that the points $\{x_j\}_1^r$ and non-zero numbers $\{c_j\}_1^r$ satisfy (a) and (b) of Theorem 2.3 with respect to u^*. Set

$$I = \left\{ i : \sum_{j=1}^{r} c_j u_i(x_j) = 0 \right\},$$

and $J = \{1, \dots, n\} \backslash I$.

From (b) of Proposition 2.4, for every best approximant $u = \sum_{i=1}^n a_i u_i \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ to f, we have that $a_i^* = a_i$ for all $i \in J$. Thus our original problem is equivalent to that of approximating

$$f^* = f - \sum_{i \in J} a_i^* u_i$$

from $U^*(\boldsymbol{\alpha}^*; \boldsymbol{\beta}^*) = \{\sum_{i \in I} a_i u_i : \alpha_i \leq a_i \leq \beta_i \ (i \in I)\}$. Let $U_I = \operatorname{span}\{u_i : i \in I\}$. For each $i \in I$, we have, from (b) of Theorem 2.3, that

$$\text{(b}'\text{)} \qquad \sum_{j=1}^{r} c_j u_i(x_j) = 0,$$

while (a) of Theorem 2.3 holds for f^* and $\sum_{i \in I} a_i^* u_i$. (We also have $1 \leq r \leq n+1$ rather than $1 \leq r \leq |I| + 1$, where $|I|$ is the number of indices in I, but this is unimportant). Conditions (a) and (b′) imply (by Theorem 2.3 with $\alpha_i = -\infty$ and $\beta_i = \infty$ for all $i \in I$) that $\sum_{i \in I} a_i^* u_i$ is a best approximant to f^* from U_I with *no* constraints whatsoever on the coefficients. If $u = \sum_1^n a_i u_i$ is any best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, then $\sum_{i \in I} a_i u_i$ is necessarily a best approximant to f^* from U_I. (The converse need not hold.) This simple observation together with Haar's theorem immediately implies the following result (see also Rozema & Smith [15]).

PROPOSITION 3.1 *Let $N = \{i : \alpha_i = -\infty, \beta_i = \infty\}$. If $\{u_{i_1}, \dots, u_{i_k}\}$ is a Haar system for every choice of distinct $i_1, \dots, i_k \in \{1, \dots, n\}$ satisfying $N \subseteq \{i_1, \dots, i_k\}$, then $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ is a unicity set for $C(B)$.*

We do not know if the converse result is valid in this generalization. It is however true with a slight additional assumption.

PROPOSITION 3.2 *Let $N = \{i : \alpha_i = -\infty, \beta_i = \infty\}$, and assume that, for all $i \notin N$, $-\infty < \alpha_i < \beta_i < \infty$. If $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ is a unicity set for $C(B)$, then $\{u_{i_1}, \dots, u_{i_k}\}$ is a Haar system for every choice of distinct $i_1, \dots, i_k \in \{1, \dots, n\}$ satisfying $N \subseteq \{i_1, \dots, i_k\}$.*

Proof. Let $N \subseteq \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and assume $U_k = \operatorname{span}\{u_{i_1}, \dots, u_{i_k}\}$ is *not* a Haar space. We will prove that $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ is not a unicity space for $C(B)$.

From Haar's theorem, there exists an $f \in C(B)$ and $u^1, u^2 \in U_k$, with $u^1 \not\equiv u^2$, such that u^1 and u^2 are best approximants to f from U_k. Since $\lambda u^1 + (1-\lambda) u^2$ is also a best approximant to f for every $\lambda \in (0, 1)$, and since $u^i + u$ $(i = 1, 2)$ are best approximants to $f + u$ from U_k for any fixed $u \in U_k$, we may assume

$$u^1 = \sum_{j=1}^{k} a^1_{i_j} u_{i_j}, \qquad u^2 = \sum_{j=1}^{k} a^2_{i_j} u_{i_j},$$

where

$$\alpha_{i_j} < a^1_{i_j}, a^2_{i_j} < \beta_{i_j} \quad (j = 1, \ldots, k).$$

From Theorem 2.3 and Proposition 2.4, it follows that there exist points $\{x_j\}_1^r$, with $1 \leq r \leq k+1$, and non-zero numbers $\{c_j\}_1^r$, satisfying

(a) $$(f - u^i)(x_j) = (\operatorname{sgn} c_j)\, \|f - u^i\| \quad (i = 1, 2; j = 1, \ldots, r),$$

(b) $$\sum_{j=1}^{r} c_j u_i(x_j) = 0 \quad \text{for } i \in \{i_1, \ldots, i_k\}.$$

Set $Q = \{1, \ldots, n\} \backslash \{i_1, \ldots, i_k\}$. If $Q = \varnothing$, then there is nothing to prove, since $u^1, u^2 \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. Assume $Q \neq \varnothing$. Since $N \subseteq \{i_1, \ldots, i_k\}$, then, by our assumption, $-\infty < \alpha_i < \beta_i < \infty$ for each $i \in Q$. Divide Q as follows. Set $Q = Q_{-1} \cup Q_0 \cup Q_1$, where

$$i \in \begin{cases} Q_{-1} & \text{if } \sum_{j=1}^{r} c_j u_i(x_j) < 0, \\ Q_0 & \text{if } \sum_{j=1}^{r} c_j u_i(x_j) = 0, \\ Q_1 & \text{if } \sum_{j=1}^{r} c_j u_i(x_j) > 0. \end{cases}$$

For $i \in Q_0$, choose $\gamma_i \in [\alpha_i, \beta_i]$. Set

$$\tilde{f} = f + \sum_{i \in Q_0} \gamma_i u_i + \sum_{i \in Q_1} \beta_i u_i + \sum_{i \in Q_{-1}} \alpha_i u_i,$$

$$\tilde{u}^j = u^j + \sum_{i \in Q_0} \gamma_i u_i + \sum_{i \in Q_1} \beta_i u_i + \sum_{i \in Q_{-1}} \alpha_i u_i \quad (j = 1, 2).$$

It is now easily checked, using (a), (b), and Theorem 2.3, that $\tilde{u}^1$ and $\tilde{u}^2$ are two best approximants to $\tilde{f}$ from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. □

As an immediate consequence of Propositions 3.1 and 3.2, we have the following.

COROLLARY 3.3 *$U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ is a unicity set for $C(B)$ for every choice of $(\boldsymbol{\alpha}; \boldsymbol{\beta})$ satisfying $-\infty \leq \alpha_i < \beta_i \leq \infty$ $(i = 1, \ldots, n)$ if and only if $\{u_{i_1}, \ldots, u_{i_k}\}$ is a Haar system for every choice of distinct $i_1, \ldots, i_k \in \{1, \ldots, n\}$.*

Under the assumptions of Proposition 3.2, not only is $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ a unicity set for $C(B)$, but it is also true that strong unicity holds. The proof of this fact is somewhat standard, but we include it for completeness (see also Chambers & Taylor [4]).

PROPOSITION 3.4 *Let $N = \{i : \alpha_i = -\infty, \beta_i = \infty\}$, and assume that, for all $i \notin N$, $-\infty < \alpha_i < \beta_i < \infty$. Assume that $\{u_{i_1}, \ldots, u_{i_k}\}$ is a Haar system for every choice* *of distinct $i_1, \ldots, i_k$ satisfying $N \subseteq \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$. Let $f \in C(B)$, and let u^* denote the unique best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. Then there exists $\gamma > 0$ (dependent on f) such that, for all $u \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$,*

$$\|f-u\| \geq \|f-u^*\| + \gamma \|u^*-u\|.$$

Proof. If $f = u^*$, set $\gamma = 1$. Assume $f \neq u^*$. Let $\{x_j\}_1^r$ and $\{c_j\}_1^r$ be as in Theorem 2.3. Let $u^* = \sum_{i=1}^n a_i^* u_i$, and define $I = \{i : \sum_{j=1}^r c_j u_i(x_j) = 0\}$ and $J = \{1, \ldots, n\} \setminus I$. Let $u = \sum_{i=1}^n a_i u_i \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, with $u \neq u^*$.

For $i \in J$, we have

$$(a_i^* - a_i) \sum_{j=1}^r c_j u_i(x_j) \geq 0,$$

since $a_i^* = \beta_i$ if $b_i^* = 1$, and $a_i^* = \alpha_i$ if $b_i^* = -1$. Thus

$$\sum_{j=1}^r c_j (u^*-u)(x_j) = \sum_{i=1}^n \left((a_i^* - a_i) \sum_{j=1}^r c_j u_i(x_j) \right) \geq 0.$$

If $u^*(x_j) \neq u(x_j)$ for some $j \in \{1, \ldots, r\}$, then

$$\max_{j=1,\ldots,r} (\operatorname{sgn} c_j)(u^*-u)(x_j) > 0.$$

If $u^*(x_j) = u(x_j)$ $(j = 1, \ldots, r)$, then it also follows from the above that $a_i^* = a_i$ for all $i \in J$. Thus

$$\sum_{j=1}^r c_j u_i(x_j) = 0 \quad (i \in I),$$

$$\sum_{i \in I} (a_i^* - a_i) u_i(x_j) = u^*(x_j) - u(x_j) = 0 \quad (j = 1, \ldots, r).$$

Since $N \subseteq I$, we have that $U_I = \operatorname{span}\{u_i : i \in I\}$ is a $|I|$-dimensional Haar space and therefore $\operatorname{rank}[u_i(x_j)]_{i \in I, j=1}^{r} = \min\{|I|, r\}$. Since $c_j \neq 0$, it follows that $a_i^* = a_i$, for all $i \in I$. Thus $u = u^*$, a contradiction.

We have proved that, for all $u \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, with $u \neq u^*$,

$$\max_{j=1,\ldots,r} (\operatorname{sgn} c_j)(u^*-u)(x_j) > 0.$$

Since U_n is a finite-dimensional subspace, a compactness argument implies that

$$\max_{j=1,\ldots,r} (\operatorname{sgn} c_j)(u^*-u)(x_j) \geq \gamma \|u^*-u\|$$

for all $u \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ and some $\gamma > 0$.

Assume that $u \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, and that $k \in \{1, \ldots, r\}$ satisfies

$$(\operatorname{sgn} c_k)(u^*-u)(x_k) \geq \gamma \|u^*-u\|.$$

Then, using also (a) of Theorem 2.3,

$$\begin{aligned} \|f-u\| &\geq (\operatorname{sgn} c_k)(f-u)(x_k) \\ &= (\operatorname{sgn} c_k)(f-u^*)(x_k) + (\operatorname{sgn} c_k)(u^*-u)(x_k) \\ &\geq \|f-u^*\| + \gamma \|u^*-u\|. \quad \square \end{aligned}$$

4. Descartes systems

We will now restrict ourselves to the case where B is a real interval. For convenience we assume $B = [0, 1]$. We will also assume that the conditions of Corollary 3.3 hold. Moreover, for ease of exposition, we further assume that $(u_1, \ldots, u_n)$ is a Descartes system.

DEFINITION 4.1 $(u_1, \ldots, u_n)$ is a *Descartes system* on $[0, 1]$ if

$$\det\left[u_{i_r}(x_s)\right]_{r,s=1}^{k} > 0$$

for every choice of $1 \leq i_1 < \cdots < i_k \leq n$ and $0 \leq x_1 < \cdots < x_k \leq 1$, for $k = 1, \ldots, n$.

We will use a well-known consequence of the Descartes property which is actually equivalent to it. But first we introduce some ancillary notation.

(a) For $f \in C[0, 1]$, $z(f)$ will denote the number of distinct zeros of f on $[0, 1]$.

(b) For $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $s^-(\boldsymbol{a})$ denotes the number of *weak sign changes* of $\boldsymbol{a}$, and is defined by

$$s^-(\boldsymbol{a}) = \max\{k : a_{i_j}a_{i_{j+1}} < 0 \ (j = 1, \ldots, k),\ 1 \leq i_1 < \cdots < i_{k+1} \leq n\}.$$

The (weak) *orientation* of $\boldsymbol{a}$ is the sign of its last non-zero coefficient.

(c) If $z(f) = s^-(\boldsymbol{a})$, with $\boldsymbol{a} \in \mathbb{R}^n \backslash \{\boldsymbol{0}\}$, we say that the *sign orientations* agree if the following hold:

(i) If $f(1) \neq 0$, then $\operatorname{sgn} f(1)$ is equal to the orientation of $\boldsymbol{a}$.

(ii) If $f(1) = 0$, then the orientation of $\boldsymbol{a}$ is equal to $-\operatorname{sgn} f(1-\varepsilon)$ for all $\varepsilon > 0$ sufficiently small.

With this notation, we can now state (see e.g. Karlin & Studden [8: p. 25] and Karlin [7: p. 233]) the following result.

PROPOSITION 4.1 *If $(u_1, \ldots, u_n)$ is a Descartes system on $[0, 1]$ and $u = \sum_{i=1}^{n} a_i u_i$, with $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{a} \neq \boldsymbol{0}$, then $z(u) \leq s^-(\boldsymbol{a})$. Further if $z(u) = s^-(\boldsymbol{a})$, then u changes sign at each of its interior zeros and the sign orientations agree.*

Before stating the main result of this section, we must introduce additional notation.

(d) Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n \backslash \{\boldsymbol{0}\}$. Then $s^+(\boldsymbol{a})$ denotes the number of *strong sign changes* of $\boldsymbol{a}$, and is defined by

$$s^+(\boldsymbol{a}) = \max_{\boldsymbol{a}' \in A} s^-(\boldsymbol{a}'),$$

where A is the set of all vectors $\boldsymbol{a}' = (a_1', \ldots, a_n')$ satisfying $a_i' = a_i$ for all $a_i \neq 0$. The vector $\boldsymbol{a}$ is said to have (strong) *positive orientation* if $a_n > 0$, or if $a_n = a_{n-1} = \cdots = a_{m+1} = 0$ and $a_m(-1)^{n-m} > 0$. Otherwise it is said to have (strong) *negative orientation.*

(e) Let $g \in C[0, 1]$, with $g \not\equiv 0$. We say that g *equioscillates* on exactly k points if there exist points $0 \leq x_1 < \cdots < x_k \leq 1$ and a sign $\varepsilon \in \{-1, 1\}$ for which $g(x_i) = \varepsilon(-1)^{i+k}\|g\|$ $(i = 1, \ldots, k)$ and there exists no such set of $k+1$ points. We also say that g has *positive orientation* if $\varepsilon = 1$, and *negative orientation* if $\varepsilon = -1$.

We recall that, for $u^* = \sum_1^n a_i^* u_i \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, we set $\boldsymbol{b}^* = (b_1^*, \ldots, b_n^*)$, where

$$b_i^* = \begin{cases} 1 & \text{if } a_i^* = \beta_i, \\ 0 & \text{if } \alpha_i < a_i^* < \beta_i, \\ -1 & \text{if } a_i^* = \alpha_i. \end{cases}$$

Our next result is a characterization of the best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. It differs from Theorem 2.3 in that it is geometric in nature. It should also be viewed as a semi-discrete version of results in Pinkus [11].

THEOREM 4.2 *Assume $(u_1, \ldots, u_n)$ is a Descartes system on* $[0, 1]$. *To each $f \in C[0, 1]$ there exists a unique best approximant u^* from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. The function u^* is uniquely characterized as follows:*

(i) *If $\boldsymbol{b}^* = \boldsymbol{0}$, then $f - u^*$ equioscillates on at least $n + 1$ points.*

(ii) *If $\boldsymbol{b}^* \neq \boldsymbol{0}$, then $f - u^*$ equioscillates on at least $s^+(\boldsymbol{b}^*) + 1$ points. Further, if the number of points of equioscillation of $f - u^*$ is exactly $s^+(\boldsymbol{b}^*) + 1$, then $f - u^*$ and $\boldsymbol{b}^*$ have orientations of the same sign.*

In the proof of Theorem 4.2 we will use the following lemma.

LEMMA 4.3 *Let $u^* = \sum_1^n a_i^* u_i$, $\boldsymbol{a}^* = (a_1^*, \ldots, a_n^*)$, and $\boldsymbol{b}^* = (b_1^*, \ldots, b_n^*)$ be as defined above. Let $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \tilde{a}_n) \in A(\boldsymbol{\alpha}; \boldsymbol{\beta})$. Assume $\boldsymbol{b}^* \neq \boldsymbol{0}$ and $\boldsymbol{a}^* \neq \tilde{\boldsymbol{a}}$. Then*

$$s^-(\boldsymbol{a}^* - \tilde{\boldsymbol{a}}) \leq s^+(\boldsymbol{b}^*).$$

Further, if equality holds, then the (weak) orientation of $\boldsymbol{a}^* - \tilde{\boldsymbol{a}}$ agrees with the (strong) orientation of $\boldsymbol{b}^*$.

Proof. If $b_i^* = 1$, then $a_i^* - \tilde{a}_i = \beta_i - \tilde{a}_i \geq 0$. Similarly, if $b_i^* = -1$, then $a_i^* - \tilde{a}_i \leq 0$. From these facts it follows that $s^-(\boldsymbol{a}^* - \tilde{\boldsymbol{a}}) \leq s^+(\boldsymbol{b}^*)$. The case of equality is also easily checked. □

Proof of Theorem 4.2. Assume that $f \in C[0, 1] \backslash U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ and $u^* = \sum_1^n a_i^* u_i$ is the unique (from Corollary 3.3) best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$.

From Theorem 2.3, there exist points $0 \leq x_1 < \cdots \leq x_r \leq 1$, with $1 \leq r \leq n + 1$, and non-zero numbers $c_1, \ldots, c_r$ satisfying

(a) $$(f - u^*)(x_j) = (\operatorname{sgn} c_j)\, \|f - u^*\| \quad (j = 1, \ldots, r),$$

(b) $$\sum_{j=1}^r c_j u_i(x_j) \begin{Bmatrix} \geq 0 & \text{if } b_i^* = 1 \\ = 0 & \text{if } b_i^* = 0 \\ \leq 0 & \text{if } b_i^* = -1 \end{Bmatrix} \quad (i = 1, \ldots, n).$$

From (a), the number of equioscillations of $f - u^*$ is at least $s^-(\boldsymbol{c}) + 1$ $(= s^+(\boldsymbol{c}) + 1)$, where $\boldsymbol{c} = (c_1, \ldots, c_r)$.

If $\boldsymbol{b}^* = \boldsymbol{0}$, then it is well known that, since $\{u_1, \ldots, u_n\}$ is a Haar system, we must have $s^+(\boldsymbol{c}) + 1 = n + 1$, and (i) holds.

Assume $\boldsymbol{b}^* \neq \boldsymbol{0}$. The matrix $[u_i(x_j)]_{i=1}^n {}_{j=1}^r$ is strictly totally positive (STP). Therefore, if $\sum_{j=1}^r c_j u_i(x_j) = 0$ $(i = 1, \ldots, n)$, then $s^+(\boldsymbol{c}) + 1 = n + 1$ and (ii) holds. Otherwise it follows (see Karlin [7: p. 219]) that

$$s^+\left(\left(\sum_{j=1}^r c_j u_i(x_j)\right)_{i=1}^n\right) \leq s^-(\boldsymbol{c}).$$

Further, from (b) and the definition of $\boldsymbol{b}^*$,

$$\mathrm{s}^+(\boldsymbol{b}^*) \leq \mathrm{s}^+\Big(\Big(\sum_{j=1}^{r} c_j u_i(x_j)\Big)_{i=1}^{n}\Big).$$

Thus $f - u^*$ equioscillates on at least $\mathrm{s}^+(\boldsymbol{b}^*)+1$ points. If the number of equioscillations is exactly $\mathrm{s}^+(\boldsymbol{b}^*)+1$, then equality holds in the above two inequalities. From Karlin [7: p. 223], the (strong) orientations of $\boldsymbol{b}^*$ and $\boldsymbol{c}$ must agree, implying (ii).

Assume that $f \in \mathrm{C}[0,1]\backslash U(\boldsymbol{\alpha};\boldsymbol{\beta})$, and that $u^* = \sum_1^n a_i^* u_i \in U(\boldsymbol{\alpha};\boldsymbol{\beta})$ satisfies (i) or (ii). If $f - u^*$ equioscillates on at least $n+1$ points, then, since U_n is a Haar space, it follows that u^* is the unique best approximant to f from all of U_n and thus from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$.

We therefore assume that (ii) holds, $\mathrm{s}^+(\boldsymbol{b}^*)+1 = k+1$, and $f - u^*$ equioscillates on $m+1$ points, with $k \leq m < n$. Now assume that

$$\tilde{u} \in U(\boldsymbol{\alpha};\boldsymbol{\beta}), \qquad \tilde{u} = \sum_1^n \tilde{a}_i u_i, \qquad \tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \tilde{a}_n),$$

and suppose (for contradiction) that $\|f - \tilde{u}\| < \|f - u^*\|$. We will prove that this cannot occur. Since $f - u^*$ equioscillates on $m+1$ points, from Proposition 4.1,

$$m \leq \mathrm{z}\big((f-u^*)-(f-\tilde{u})\big) = \mathrm{z}(\tilde{u}-u^*) = \mathrm{z}\Big(\sum_1^n (\tilde{a}_i - a_i^*)u_i\Big) \leq \mathrm{s}^-(\tilde{\boldsymbol{a}}-\boldsymbol{a}^*).$$

From Lemma 4.3, $\mathrm{s}^-(\tilde{\boldsymbol{a}}-\boldsymbol{a}^*) \leq \mathrm{s}^+(\boldsymbol{b}^*) = k$. By assumption, $m \geq k$. A contradiction ensues unless $m = k$. If $m = k$, then (ii) gives us that $f - u^*$ and $\boldsymbol{b}^*$ have the same orientation; also, from the above, $m = k$ implies

$$k = \mathrm{z}(\tilde{u}-u^*) = \mathrm{s}^-(\tilde{\boldsymbol{a}}-\boldsymbol{a}^*) = \mathrm{s}^+(\tilde{\boldsymbol{b}}^*).$$

The (strong) orientation of $\boldsymbol{b}^*$ agrees by assumption with the orientation of $f - u^*$ which, in turn, is the same as the 'orientation' of $\tilde{u} - u^*$ $\big(=(f-u^*)-(f-\tilde{u})\big)$ since $\|f-u^*\| > \|f-\tilde{u}\|$. From Proposition 4.1, we obtain that the (strong) orientation of $\boldsymbol{b}^*$ agrees with the (weak) orientation of $\tilde{\boldsymbol{a}}-\boldsymbol{a}^*$. However, from Lemma 4.3, the converse holds. This contradiction implies that, for all $u \in U(\boldsymbol{\alpha};\boldsymbol{\beta})$, we have $\|f-u\| \geq \|f-u^*\|$. Moreover, from Corollary 3.3, the best approximant to f from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$ is unique. Thus $\|f-u\| > \|f-u^*\|$ for all $u \in U(\boldsymbol{\alpha};\boldsymbol{\beta})$ with $u \neq u^*$. □

As examples of Descartes systems of $[0,1]$, consider

(1) $(\mathrm{e}^{a_1 x}, \ldots, \mathrm{e}^{a_n x})$ for any $a_1 < \cdots < a_n$,
(2) $((x+c)^{a_1}, \ldots, (x+c)^{a_n})$ for any $a_1 < \cdots < a_n$ and $c > 0$.

Many other examples exist, and the interested reader can consult Karlin & Studden [8].

An important example is the following:

Let π_{n-1} denote the space of algebraic polynomials of degree $\leq n-1$, and let $\{x_i\}_1^n$ be distinct points which lie *outside* $[0,1]$. Set

$$K = \{p \in \pi_{n-1} : \gamma_i \leq p(x_i) \leq \delta_i \ (i = 1, \ldots, n)\},$$

where $-\infty \leq \gamma_i < \delta_i \leq \infty$ $(i = 1, \ldots, n)$. There exist $u_1, \ldots, u_n \in \pi_{n-1}$, and a pair $(\boldsymbol{\alpha}; \boldsymbol{\beta})$ for which $K = U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, and $(u_1, \ldots, u_n)$ is a Descartes system on $[0, 1]$. To see this, assume

$$x_r < \cdots < x_1 < 0 < 1 < x_n < \cdots < x_{r+1} \quad (0 \leq r \leq n).$$

For $i = 1, \ldots, r$, let $u_i \in \pi_{n-1}$ satisfy $u_i(x_j) = (-1)^{i+1}\delta_{i,j}$ $(j = 1, \ldots, n)$, and let

$$\alpha_i = \begin{cases} \gamma_i & \text{for } i \text{ odd}, \\ -\delta_i & \text{for } i \text{ even}, \end{cases} \qquad \beta_i \begin{cases} \delta_i & \text{for } i \text{ odd}, \\ -\gamma_i & \text{for } i \text{ even}. \end{cases}$$

For $i = r + 1, \ldots, n$, let $u_i \in \pi_{n-1}$ satisfy $u_i(x_j) = (-1)^{i+n}\delta_{i,j}$ $(j = 1, \ldots, n)$, and let

$$\alpha_i = \begin{cases} \gamma_i & \text{for } n + i \text{ even}, \\ -\delta_i & \text{for } n + i \text{ odd}, \end{cases} \qquad \beta_i = \begin{cases} \delta_i & \text{for } n + i \text{ even}, \\ -\gamma_i & \text{for } n + i \text{ odd}. \end{cases}$$

It is not difficult to check that $K = U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. With a bit more work, using the common zeros of any subset of the polynomials $\{u_i : i = 1, \ldots, n\}$, it also follows that $(u_1, \ldots, u_n)$ is a Descartes system on $[0, 1]$ and therefore Theorem 4.2 can be applied.

5. Weak Descartes systems

DEFINITION 5.1 $(u_1, \ldots, u_n)$ is a *weak Descartes system* on $[0, 1]$ if the u_i are linearly independent thereon, and

$$\det\left[u_{i_r}(x_s)\right]_{r,s=1}^{k} \geq 0$$

for every choice of $1 \leq i_1 < \cdots < i_k \leq n$ and $0 \leq x_1 < \cdots < x_k \leq 1$, for $k = 1, \ldots, n$.

When dealing with weak Descartes systems rather than Descartes systems, we lose the uniqueness of the best approximant to each $f \in C[0, 1]$. However, there is always at least one best approximant that is 'well behaved' (as the u^* of Theorem 4.2). This is an immediate consequence of 'smoothing' (see Karlin [7, p. 103]) and a known weaker form of Proposition 4.1. As such, we only state and do not prove this next result.

THEOREM 5.1 *Assume $(u_1, \ldots, u_n)$ is a weak Descartes system on $[0, 1]$. To each $f \in C[0, 1]$ there exists a best approximant u^* from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ that satisfies* (i) *and* (ii) *of Theorem* 4.2. *Conversely, if $u^* \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ satisfies* (i) *and* (ii) *of Theorem* 4.2, *then u^* is a best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$.*

Thus, if to a particular $f \in C[0, 1]$ there exists a unique best approximant from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, then, from Theorem 5.1, we can characterize it explicitly.

In some cases, $(u_1, \ldots, u_n)$ is a weak Descartes system that comes very close to being a Descartes system. For example

(1) $(1, x, \ldots, x^{n-1})$ is a Descartes system on $(0, 1]$, but not on $[0, 1]$.

(2) $((1-x)^{n-1}, x(1-x)^{n-2}, \ldots, x^{n-1})$ is a Descartes system on $(0, 1)$.

(3) If $\{u_i\}_1^n$ is an (extended complete Tchebycheff) ECT-system on $[0, 1]$ satisfying $u_i^{(j)}(0) = 0$ $(j = 0, \ldots, i-2; i = 2, \ldots, n)$ (see Karlin & Studden

[8: p. 378]), then it is a Descartes system on $(0,1]$, but not on $[0,1]$. (Example 1 is a special case.)

There is a 'natural' set of functions for which uniqueness always holds when approximating from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$, with respect to each of the above examples. To explain this, we consider two classes of functions.

DEFINITION 5.2 The system $(u_1,\ldots,u_n)$ is said to satisfy *Condition* A if $(u_1,\ldots,u_n)$ is a Descartes system on $(0,1]$, with $u_i(0)=\delta_{i,1}$ $(i=1,\ldots,n)$. The system $(u_1,\ldots,u_n)$ is said to satisfy *Condition* B if $(u_1,\ldots,u_n)$ is a Descartes system on $(0,1)$, with $u_i(0)=\delta_{i,1}$ and $u_i(1)=\delta_{i,n}$ $(i=1,\ldots,n)$.

We first state these next two results.

PROPOSITION 5.2 *Assume $(u_1,\ldots,u_n)$ satisfies Condition* A. *Let u^* be a best approximant to f from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$. If $f(0)-\beta_1\neq\|f-u^*\|$ and $f(0)-\alpha_1\neq-\|f-u^*\|$, then u^* is the unique best approximant to f from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$. In particular, if $\alpha_1\leqslant f(0)\leqslant\beta_1$, then the best approximant to f from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$ is unique.*

PROPOSITION 5.3 *Assume $(u_1,\ldots,u_n)$ satisfies Condition* B. *Let u^* be a best approximant to f from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$. If both $f(0)-\beta_1$ and $f(1)-\beta_n$ are not equal to $\|f-u^*\|$, and $f(0)-\alpha_1$ and $f(1)-\alpha_n$ are not equal to $-\|f-u^*\|$, then u^* is the unique best approximant to f from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$. In particular, if $\alpha_1\leqslant f(0)\leqslant\beta_1$ and $\alpha_n\leqslant f(1)\leqslant\beta_n$, then the best approximant to f from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$ is unique.*

Remark. In [14], Roulier & Taylor considered the above problem for the example $(1,x,\ldots,x^{n-1})$. They proved Proposition 5.2, but did not have Theorem 5.1 or any similar result. As such, their characterization was incomplete. Chalmers [2] improved their result but without obtaining the full characterization. Passow [10] considered the example $((1-x)^{n-1},x(1-x)^{n-2},\ldots,x^{n-1})$. In the case that $\alpha_i=0$ and $\beta_i=\infty$ (all i) and $f\geqslant0$, Passow proved the uniqueness of the best approximant from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$. However, the full characterization, as given by Theorem 5.1, was again lacking.

We will prove Proposition 5.2. The proof of Proposition 5.3 follows in a totally analogous manner.

Proof of Proposition 5.2. From Condition A, it not only follows that $\{u_{i_1},\ldots,u_{i_k}\}$ is a Haar system on $(0,1]$ for all $1\leqslant i_1<\cdots<i_k\leqslant n$, but also that it is a Haar system on $[0,1]$ for all $1=i_1<\cdots<i_k\leqslant n$.

Let $u^*=\sum_1^n a_i^*u_i$ be any best approximant to f from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$. From Theorem 2.3, there exist distinct points $\{x_j\}_1^r$, with $1\leqslant r\leqslant n+1$ and $0\leqslant x_1<\cdots<x_r\leqslant1$, and non-zero numbers $\{c_j\}_1^r$, satisfying

$$\text{(a)}\qquad (f-u^*)(x_j)=(\operatorname{sgn}c_j)\,\|f-u^*\|\quad(j=1,\ldots,r),$$

$$\text{(b)}\qquad \sum_{j=1}^r c_ju_i(x_j)\begin{Bmatrix}\geqslant0 & \text{if } b_i^*=1\\ =0 & \text{if } b_i^*=0\\ \leqslant0 & \text{if } b_i^*=-1\end{Bmatrix}\quad(i=1,\ldots,n).$$

Let $\tilde{u}=\sum_1^n\tilde{a}_iu_i$ be any other best approximant to f from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$. From

Proposition 2.4 we have

(c) $$u^*(x_j)=\tilde{u}(x_j) \quad (j=1,\ldots,r),$$

(d) $$a_i^*=\tilde{a}_i \quad \text{for all } i \notin I,$$

where $I=\{i:\sum_{j=1}^r c_j u_i(x_j)=0\}$. We shall determine when it is possible that $u^* \neq \tilde{u}$.

Case (i): $1 \in I$. From (d) and the definition of I it follows that $\sum_{i\in I} a_i^* u_i$ and $\sum_{i\in I} \tilde{a}_i u_i$ are best approximants to

$$f^*=f-\sum_{i\notin I} a_i^* u_i$$

from $U_I=\operatorname{span}\{u_i: i\in I\}$. Since $1\in I$, the space U_I is a Haar space on $[0,1]$, and therefore $a_i^*=\tilde{a}_i$ for $i\in I$. Thus $u^*=\tilde{u}$ and uniqueness holds.

Case (ii): $1\notin I$, $x_1>0$. As above, $\sum_{i\in I} a_i^* u_i$ and $\sum_{i\in I}\tilde{a}_i u_i$ are best approximants to f^* from U_I. Let $|I|$ denote the number of indices in I. From the definition of I,

$$\sum_{j=1}^r c_j u_i(x_j)=0 \quad (i\in I).$$

Since $x_1>0$, the matrix $[u_i(x_j)]_{i\in I,\, j=1}^{\ r}$ has rank $\min\{|I|, r\}$ and, because $c_j\neq 0$ (all j), it follows that $r>|I|$. However, from (c) and (d) we obtain

$$\sum_{i\in I}(a_i^*-\tilde{a}_i)u_i(x_j)=0 \quad (j=1,\ldots,r).$$

Since $r>|I|=\operatorname{rank}[u_i(x_j)]_{i\in I,\, j=1}^{\ r}$, this implies $a_i^*=\tilde{a}_i$ $(i\in I)$. Thus $u^*=\tilde{u}$ and uniqueness holds.

Case (iii): $1\notin I$, $x_1=0$, $c_1 b_1^*<0$. Since $u_i(0)=\delta_{i,1}$ we have

$$\sum_{j=1}^r c_j u_i(x_j)=\sum_{j=2}^r c_j u_i(x_j) \quad (i=2,\ldots,n),$$

$$\sum_{j=1}^r c_j u_1(x_j)-c_1=\sum_{j=2}^r c_j u_1(x_j).$$

The condition $c_1 b_1^*<0$ together with $1\notin I$ implies that

$$c_1\sum_{j=1}^r c_j u_1(x_j)<0.$$

If $r=1$, then $c_1^2=c_1\cdot c_1 u_1(0)<0$, which is a contradiction. Thus $r\geqslant 2$, and we also have

$$b_1^*\sum_{j=2}^r c_j u_1(x_j)>0.$$

We now replace (a) and (b) by

(a′) $$(f-u^*)(x_j)=(\operatorname{sgn} c_j)\,\|f-u^*\| \quad (j=2,\ldots,r),$$

(b′) $$\sum_{j=2}^{r} c_j u_i(x_j)\begin{cases} \geqslant 0 & \text{if } b_i^*=1 \\ =0 & \text{if } b_i^*=0 \\ \leqslant 0 & \text{if } b_i^*=-1 \end{cases} \quad (i=1,\ldots,n).$$

The previous analysis proves that (a′) and (b′) are valid. Theorem 2.3 is therefore valid with (a′) and (b′), and we have reverted to case (ii). Thus $u^*=\tilde{u}$ and uniqueness holds.

Non-uniqueness can therefore only occur if $1\notin I$, $x_1=0$, and $c_1b_1^*>0$. (If $c_1b_1^*=0$, then $1\in I$.) If $c_1>0$ and $b_1^*>0$, then $f(0)-u^*(0)=\|f-u^*\|$, and $u^*(0)=a_1^*=\beta_1$. Thus $f(0)-\beta_1=\|f-u^*\|$. If $c_1<0$ and $b_1^*<0$, then $f(0)-u^*(0)=-\|f-u^*\|$, and $u^*(0)=a_1^*=\alpha_1$. Thus $f(0)-\alpha_1=-\|f-u^*\|$. □

We now return to the general case where $(u_1,\ldots,u_n)$ is a weak Descartes system on $[0,1]$. In Theorem 5.1 we noted the existence of a specific type of best approximant, but we did not characterize all best approximants. In this next theorem, using the properties of weak Descartes systems, we give an alternative form of Theorem 2.3. In the case $U(\boldsymbol{\alpha};\boldsymbol{\beta})=U_n$ (i.e. no constraints) this theorem was proved by Micchelli [9].

Some additional notation is needed. For $x_i\in[0,1]$ and $\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}$, we set

$$U\begin{pmatrix} i_1,\ldots,i_k \\ x_1,\ldots,x_k \end{pmatrix}=\det\,[u_{i_r}(x_s)]_{r,s=1}^{k},$$

$$U\begin{pmatrix} i_1, & \ldots & ,i_k \\ x_1,\ldots, & \hat{x}_j, & \ldots,x_{k+1} \end{pmatrix}=\det\,[u_{i_r}(x_s)]_{r=1}^{k}\,{}_{\substack{s=1 \\ s\neq j}}^{k+1}.$$

THEOREM 5.4 *Assume $(u_1,\ldots,u_n)$ is a weak Descartes system on $[0,1]$. Let $f\in C[0,1]\backslash U(\boldsymbol{\alpha};\boldsymbol{\beta})$. Then $u^*=\sum_1^n a_i^*u_i$ is a best approximant to f from $U(\boldsymbol{\alpha};\boldsymbol{\beta})$ if and only if there exist points $\{x_j\}_1^r$, with $0\leqslant x_1<\cdots<x_r\leqslant 1$ and $1\leqslant r\leqslant n+1$, and $\varepsilon\in\{-1,1\}$ such that the following conditions (A) and (B) hold.*

(A) $$(f-u^*)(x_j)=\varepsilon(-1)^{j-1}\,\|f-u^*\| \quad (j=1,\ldots,r).$$

(B) *There exist indices $i_0=0$, $1\leqslant i_1<\cdots<i_{r-1}\leqslant n$, and $i_r=n+1$ for which*

(i) $$U\begin{pmatrix} i_1, & \ldots & ,i_{r-1} \\ x_1,\ldots, & \hat{x}_j, & \ldots,\ x_r \end{pmatrix}>0 \quad (j=1,\ldots,r)$$

and

(ii) *for $i_{k-1}<i<i_k$, where $1\leqslant k\leqslant r$, and defining*

$$m_i=U\begin{pmatrix} i_1,\ldots,i_{r-1},i \\ x_1,\ \ldots\ ,x_r \end{pmatrix},$$

we have that $m_i \neq 0$ implies

$$a_i^* = \begin{cases} \beta_i & \text{if } \varepsilon(-1)^{k-1} = 1, \\ \alpha_i & \text{if } \varepsilon(-1)^{k-1} = -1. \end{cases}$$

Remark. It is to be understood that, if $r = 1$, then (B)(i) is disregarded.

Proof. Assume (A) and (B) hold.

Case 1: $r = 1$. Set $c_1 = \varepsilon$. Then, from (A),

$$(f - u^*)(x_1) = (\operatorname{sgn} c_1)\, \|f - u^*\|.$$

Since $(u_1, \ldots, u_n)$ is a weak Descartes system, $m_i = u_i(x_1) \geqslant 0$. If $m_i \neq 0$ and $c_1 = \varepsilon = 1$, then (B)(ii) gives $a_i^* = \beta_i$, implying $b_i^* = 1$ and $c_1 u_i(x_1) > 0$. If $m_i \neq 0$ and $c_1 = \varepsilon = -1$, then $a_i^* = \alpha_i$, implying $b_i^* = -1$ and $c_1 u_i(x_1) < 0$. Thus (a) and (b) of Theorem 2.3 hold, and u^* is a best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$.

Case 2: $r > 1$. Let $i_1, \ldots, i_{r-1}$ be as in (B). There exists a vector $\mathbf{c} = (c_1, \ldots, c_r) \neq \mathbf{0}$ such that

$$\sum_{j=1}^{r} c_j u_{i_m}(x_j) = 0 \quad (m = 1, \ldots, r-1).$$

From (B)(i), we have $c_j c_{j+1} < 0$ $(j = 1, \ldots, r-1)$, and $\mathbf{c}$ is determined up to multiplication by a constant. We may therefore assume that $\operatorname{sgn} c_1 = \varepsilon$. Thus, from (A),

$$(f - u^*)(x_j) = (\operatorname{sgn} c_j)\, \|f - u^*\| \quad (j = 1, \ldots, r).$$

If $i_{k-1} < i < i_k$ and $m_i = 0$, then it follows that

$$\sum_{j=1}^{r} c_j u_i(x_j) = 0.$$

If $m_i \neq 0$, then it easily follows from the weak Descartes property and linear algebra that

$$(-1)^{k-1} \varepsilon \sum_{j=1}^{r} c_j u_i(x_j) > 0.$$

Thus, if $\sum_{j=1}^{r} c_j u_i(x_j) > 0$, then $\varepsilon(-1)^{k-1} > 0$ and (B)(ii) gives $a_i^* = \beta_i$, implying $b_i^* = 1$. If $\sum_{j=1}^{r} c_j u_i(x_j) < 0$, then $\varepsilon(-1)^{k-1} < 0$ and (B)(ii) gives $a_i^* = \alpha_i$, implying $b_i^* = -1$. Thus (a) and (b) of Theorem 2.3 hold, and u^* is a best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$.

It remains to prove the converse result, namely: if u^* is a best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, then u^* satisfies (A) and (B).

Since u^* is a best approximant to f there exist, from Theorem 2.3, points $\{x_j\}_1^r$, with $0 \leqslant x_1 < \cdots < x_r \leqslant 1$ and $1 \leqslant r \leqslant n+1$, and non-zero numbers $\{c_j\}_1^r$, satisfying

(a) $$(f-u^*)(x_j)=(\operatorname{sgn} c_j)\,\|f-u^*\| \quad (j=1,\ldots,r),$$

(b) $$\sum_{j=1}^{r} c_j u_i(x_j)\begin{cases}\geqslant 0 & \text{if } b_i^*=1\\ =0 & \text{if } b_i^*=0\\ \leqslant 0 & \text{if } b_i^*=-1\end{cases} \quad (i=1,\ldots,n).$$

Set $I=\{i:\sum_{j=1}^{r} c_j u_i(x_j)=0\}$.

Among all choices of $\{x_j\}_1^r$ and $\{c_j\}_1^r$ as above, choose a set with r *minimal.* Among all sets with r minimal, choose a set for which $|I|$ is *maximal.*

Case 1: $r=1$. In this case there is essentially nothing to prove, since (A) and (B)(ii) are direct consequences of (a) and (b).

Case 2: $r>1$ and $|I|=n$. (This case may be found in Micchelli [9].) Since

$$\sum_{j=1}^{r} c_j u_i(x_j)=0 \quad (i=1,\ldots,n),$$

with r minimal and $c_j\neq 0$ (all j), we have

$$\operatorname{rank}\,[u_i(x_j)]_{i=1}^{n}\,{}_{j=1}^{r}=r-1.$$

Therefore $\boldsymbol{c}$ is determined up to multiplication by a constant, and there exist $1\leqslant i_1<\cdots<i_{r-1}\leqslant n$ satisfying

$$c_j=(-1)^{j-1}cU\begin{pmatrix} i_1, & \ldots & ,i_{r-1}\\ x_1,\ldots, & \hat{x}_j, & \ldots,x_r\end{pmatrix} \quad (j=1,\ldots,r),$$

where c is some non-zero constant. Since $(u_1,\ldots,u_n)$ is a weak Descartes system and $c_j\neq 0$ $(j=1,\ldots,r)$, we have

$$U\begin{pmatrix} i_1, & \ldots & ,i_{r-1}\\ x_1,\ldots, & \hat{x}_j, & \ldots,x_r\end{pmatrix}>0 \quad (j=1,\ldots,r)$$

and $c_j c_{j+1}<0$ $(j=1,\ldots,r-1)$. Further,

$$U\begin{pmatrix} i_1, & \ldots, & i_{r-1}, & i\\ x_1, & \ldots & & ,x_r\end{pmatrix}=0$$

for all i, from the rank condition. Thus (A) and (B) obtain.

Case 3: $r>1$ and $|I|<n$. Set $I=\{s_1,\ldots,s_k\}$, with $1\leqslant s_1<\cdots<s_k\leqslant n$ and $k=|I|$ (so that $k=0$ if $I=\varnothing$). We will prove that rank $[u_{s_i}(x_j)]_{i=1}^{k}\,{}_{j=1}^{r}=r-1$ (and therefore $k=|I|\geqslant r-1$).

Assume $k>0$. Since $\sum_{j=1}^{r} c_j u_{s_i}(x_j)=0$ $(i=1,\ldots,k)$ and $c_j\neq 0$, it follows that rank $[u_{s_i}(x_j)]_{i=1}^{k}\,{}_{j=1}^{r}\leqslant r-1$. Assume rank $[u_{s_i}(x_j)]_{i=1}^{k}\,{}_{j=1}^{r}<r-1$. There then exists $\boldsymbol{d}=(d_1,\ldots,d_r)\in\mathbb{R}^r$, linearly independent of $\boldsymbol{c}=(c_1,\ldots,c_r)$, such that

$$\sum_{j=1}^{r} d_j u_{s_i}(x_j)=\sum_{j=1}^{r} c_j u_{s_i}(x_j)=0 \quad (i=1,\ldots,k).$$

2 For $i \notin I$, set

$$c_i' = \sum_{j=1}^{r} c_j u_i(x_j), \qquad d_i' = \sum_{j=1}^{r} d_j u_i(x_j).$$

For any $\delta \in \mathbb{R}$,

$$\sum_{j=1}^{r} (c_j - \delta d_j) u_i(x_j) = \begin{cases} 0 & (i \in I), \\ c_i' - \delta d_i' & (i \notin I). \end{cases}$$

Since $c_j \neq 0$ $(j = 1, \ldots, r)$ and $c_i' \neq 0$ $(i \notin I)$, then, by definition, there exists $\delta_0 \in \mathbb{R} \backslash \{0\}$ for which

$$c_j(c_j - \delta_0 d_j) \geqslant 0 \quad (j = 1, \ldots, r), \qquad c_i'(c_i' - \delta_0 d_i') \geqslant 0 \quad (i \notin I),$$

while at least one of the $c_j - \delta_0 d_j$ $(j = 1, \ldots, r)$ and $c_i' - \delta_0 d_i'$ $(i \notin I)$ is zero. However, this then contradicts either the minimality of r or the maximality (with this r) of $|I|$. From this contradiction, it follows that

$$\operatorname{rank} [u_{s_i}(x_j)]_{i=1}^{k}{}_{j=1}^{r} = r - 1.$$

If $k = 0$, then a contradiction ensues from the above argument. Therefore, as in Case 2, $\mathbf{c}$ is determined up to multiplication by a non-zero constant by the conditions on I, and there exist indices $1 \leqslant i_1 < \cdots < i_{r-1} \leqslant n$, with $\{i_1, \ldots, i_{r-1}\} \subseteq \{s_1, \ldots, s_k\} = I$, for which

$$c_j = (-1)^j c U\begin{pmatrix} i_1, & \ldots & , i_{r-1} \\ x_1, \ldots, \hat{x}_j, \ldots, x_r \end{pmatrix} \quad (j = 1, \ldots, r).$$

The constant $c \neq 0$ is uniquely determined by the value $\sum_{j=1}^{r} c_j u_i(x_j)$ for any $i \notin I$.

Since $(u_1, \ldots, u_n)$ is a weak Descartes system, and $c_j \neq 0$ $(j = 1, \ldots, r)$, we obtain

$$U\begin{pmatrix} i_1, & \ldots & , i_{r-1} \\ x_1, \ldots, \hat{x}_j, \ldots, x_r \end{pmatrix} > 0 \quad (j = 1, \ldots, r),$$

and $c_j c_{j+1} < 0$ $(j = 1, \ldots, r-1)$. Set $\varepsilon = \operatorname{sgn} c_1$. Then (A) and (B)(i) hold. It remains to prove (B)(ii). Let

$$m_i = U\begin{pmatrix} i_1, & \ldots, & i_{r-1}, i \\ x_1, & \ldots & , x_r \end{pmatrix}$$

and assume $i_{k-1} < i < i_k$, where $i_0 = 0$ and $i_r = n + 1$. If $m_i = 0$, then

$$\sum_{j=1}^{r} c_j u_i(x_j) = 0,$$

i.e. $i \in I$. If $m_i \neq 0$, then, with the above choice of ε, it follows from the weak Descartes property and linear algebra that

$$\varepsilon(-1)^{k-1} \sum_{j=1}^{r} c_j u_i(x_j) > 0.$$

If $\sum_{j=1}^{r} c_j u_i(x_j) > 0$, then from (b) we have $b_i^* = 1$ and therefore $a_i^* = \beta_i$. If $\sum_{j=1}^{r} c_j u_i(x_j) < 0$, then from (b) we have $b_i^* = -1$ and therefore $a_i^* = \alpha_i$.

The theorem is proved. □

Remark. Note that Proposition 2.4 also remains valid. If $\tilde{u} = \sum_1^n \tilde{a}_i u_i$ is any other best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, then $u^*(x_j) = \tilde{u}(x_j)$ $(j = 1, \ldots, r)$ and, if $m_i \neq 0$ in (B)(ii), then $a_i^* = \tilde{a}_i$.

6. Splines

One of the main motivating examples for the study of weak Descartes systems is splines. Splines also satisfy many additional properties, and from these properties it is possible to obtain characterizations of best approximants different from that given by Theorem 5.4. In Theorem 6.2, we give a characterization which is closer in spirit to Theorem 4.2. First we give some explanations concerning splines.

We will deal with the space of splines of degree $m-1$ with s simple fixed knots in $[0, 1]$: $0 < \xi_1 < \cdots < \xi_s < 1$. This space has dimension $n = m + s$, and we will denote it by $S_{m,s}$. One basis for this space is

$$(1, x, \ldots, x^{m-1}, (x-\xi_1)_+^{m-1}, \ldots, (x-\xi_s)_+^{m-1}),$$

where

$$(x-\xi)_+^{m-1} = \begin{cases} (x-\xi)^{m-1} & \text{if } x \geq \xi, \\ 0 & \text{if } x < \xi. \end{cases}$$

The above basis is a weak Descartes system on $[0, 1]$ (where we will always assume $m \geq 2$ so that the splines are continuous functions). However, we choose to deal with a different basis for $S_{m,s}$.

Let $\xi_{1-m} < \cdots < \xi_{-1} < \xi_0 = 0$ and $1 = \xi_{s+1} < \cdots < \xi_{s+m}$ be $2m$ additional points. By u_i we denote the B-spline of order m (degree $m-1$) with $\operatorname{supp} u_i \subseteq (\xi_{i-m}, \xi_i)$ $(i = 1, \ldots, n)$, where $\operatorname{supp} u = \{x : u(x) \neq 0\}$ is the support of u. By this we mean that u_i is a spline of degree $m-1$ with simple knots $\{\xi_j\}_{1-m}^{s+m}$ which identically vanishes outside (ξ_{i-m}, ξ_i). Such splines are known to be determined up to multiplication by a constant and vanish nowhere in (ξ_{i-m}, ξ_i) (see e.g. de Boor [1]). We therefore assume that $u_i(x) > 0$ for all $x \in (\xi_{i-m}, \xi_i)$. Now, on $[0, 1]$,

$$S_{m,s} = \operatorname{span}\{u_1, \ldots, u_n\}$$

and $(u_1, \ldots, u_n)$ is a weak Descartes system thereon. In fact, it is well known (see e.g. de Boor [1]) that, for any given $1 \leq i_1 < \cdots < i_r \leq n$ and $0 \leq x_1 < \cdots < x_r \leq 1$,

$$U\begin{pmatrix} i_1, \ldots, i_r \\ x_1, \ldots, x_r \end{pmatrix} \geq 0,$$

with strict inequality if and only if $x_j \in \operatorname{supp} u_{i_j} = (\xi_{i_j - m}, \xi_{i_j})$ $(j = 1, \ldots, r)$.

In preparation for the main result of this section, we prove the following.

Proposition 6.1 *Let $1 \leq i_1 < \cdots < i_{r-1} \leq n$ and $0 \leq x_1 < \cdots < x_r \leq 1$, and assume* *that*

(*) $$U\begin{pmatrix} i_1, & \dots & , i_{r-1} \\ x_1, \dots, \hat{x}_j, \dots, x_r \end{pmatrix} > 0 \quad (j = 1, \dots, r).$$

Set

$$m_k = U\begin{pmatrix} i_1, & \dots, i_{r-1}, k \\ x_1, & \dots & , x_r \end{pmatrix}.$$

Then the following results hold.

(i) *If* $i_1 < k < i_{r-1}$ *and* $k \notin \{i_1, \dots, i_{r-1}\}$, *then* $m_k \neq 0$.

(ii) *If* $1 \leq k < p < i_1$ *and* $m_k \neq 0$, *then* $m_p \neq 0$.

(iii) *If* $i_{r-1} < p < k \leq n$ *and* $m_k \neq 0$, *then* $m_p \neq 0$.

Proof. From (*), it follows that

(a) $$x_1 \in \operatorname{supp} u_{i_1},$$

(b) $$x_j \in \operatorname{supp} u_{i_{j-1}} \cap \operatorname{supp} u_{i_j} \quad (j = 2, \dots, r-1),$$

(c) $$x_r \in \operatorname{supp} u_{i_{r-1}}.$$

Assume $i_1 < k < i_{r-1}$, with $k \notin \{i_1, \dots, i_{r-1}\}$. Then $i_{t-1} < k < i_t$ for some $t \in \{2, \dots, r-1\}$. Now $m_k \neq 0$ if and only if

(a′) $$x_j \in \operatorname{supp} u_{i_j} \quad (j = 1, \dots, t-1),$$

(b′) $$x_t \in \operatorname{supp} u_k,$$

(c′) $$x_j \in \operatorname{supp} u_{i_{j-1}} \quad (j = t+1, \dots, r).$$

Obviously (a′) and (c′) follow from (a), (b), and (c). Since $t \in \{2, \dots, r-1\}$, we have $x_t \in \operatorname{supp} u_{i_{t-1}} \cap \operatorname{supp} u_{i_t}$ from (b). Since $i_{t-1} < k < i_t$, it is easily checked that $\operatorname{supp} u_{i_{t-1}} \cap \operatorname{supp} u_{i_t} \subset \operatorname{supp} u_k$. Thus (b′) holds and $m_k \neq 0$, proving (i).

To prove (ii), assume $m_k \neq 0$. Thus

(a″) $$x_1 \in \operatorname{supp} u_k$$

(b″) $$x_j \in \operatorname{supp} u_{i_{j-1}} \quad (j = 2, \dots, r).$$

Therefore $m_p \neq 0$ if and only if $x_1 \in \operatorname{supp} u_p$. But $x_1 \in \operatorname{supp} u_k \cap \operatorname{supp} u_{i_1} \subset \operatorname{supp} u_p$ since $k < p < i_1$. Thus (ii) is proved. Result (iii) is proved similarly. □

We can now state the main result of this section, which, in the case $U(\boldsymbol{\alpha}; \boldsymbol{\beta}) = U_n$, was proved by both Rice [12] and Schumaker [16] (see also Micchelli [9]).

THEOREM 6.2 *Let* $u_1, \dots, u_n$ *be as above, and assume that* $f \in C[0, 1] \backslash U(\boldsymbol{\alpha}; \boldsymbol{\beta})$. *Then* $u^* = \sum_1^n a_i^* u_i$ *is a best approximant to f from* $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ *if and only if there exists an interval* $[\xi_j, \xi_{j+t}]$, *with* $0 \leq j \leq s+1-t$ *and* $0 \leq t \leq s+1$, *for which the following conditions* (i) *and* (ii) *hold:*

(i) $(b_{j+1}^*, \dots, b_{j+t+m-1}^*) = \mathbf{0}$ *implies that* $f - u^*$ *has at least* $t + m$ *points of equioscillation on* $[\xi_j, \xi_{j+t}]$;

(ii) $(b_{j+1}^*, \dots, b_{j+t+m-1}^*) \neq \mathbf{0}$ *implies that* $f - u^*$ *has at least*

$$S^+(b_{j+1}^*, \dots, b_{j+t+m-1}^*) + 1$$

points of equioscillation on $[\xi_j, \xi_{j+t}]$ *and, if it has exactly this number, then the orientation of the points of equioscillation and of the vector* $(b^*_{j+1}, \ldots, b^*_{j+t+m-1})$ *agree.*

In addition, there exists an interval $[\xi_j, \xi_{j+t}]$ *satisfying* (i) *or* (ii), *and such that every best approximant* $\tilde{u} \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ *to f satisfies* $\tilde{u} = u^*$ *thereon.*

Remark. Theorem 6.2 holds for our particular basis for $S_{m,s}$. There exist, as previously indicated, different choices of bases which are also weak Descartes systems. For most such choices, similar (but not necessarily identical) results will obtain. We have chosen to consider this B-spline basis since it is both theoretically and computationally more useful.

Proof. Assume $u^* = \sum_1^n a_i^* u_i \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ satisfies (i) or (ii). We will prove that u^* is a best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$.

On the interval $[\xi_j, \xi_{j+t}]$,

$$u^* = \sum_{i=1}^{n} a_i^* u_i = \sum_{i=j+1}^{j+t+m-1} a_i^* u_i$$

since $\operatorname{supp} u_i = (\xi_{i-m}, \xi_i)$ $(i = 1, \ldots, n)$. Assume $t > 0$. On $[\xi_j, \xi_{j+t}]$, the ordered set $\{u_i\}_{j+1}^{j+t+m-1}$ is a weak Descartes system of dimension $t + m - 1$. Our result is then a direct consequence of Theorem 5.1 thereon. Assume $t = 0$, i.e. $[\xi_j, \xi_{j+t}] = \{\xi_j\}$. We must be in case (ii) with $b^*_{j+1} = \cdots = b^*_{j+m-1} \in \{-1, 1\}$. If $b^*_{j+1} = \cdots = b^*_{j+m-1} = 1$, then, for any $u \in U(\boldsymbol{\alpha}; \boldsymbol{\beta})$,

$$u(\xi_j) = \sum_{i=1}^{n} a_i u_i(\xi_j) = \sum_{i=j+1}^{j+m-1} a_i u_i(\xi_j) \leqslant \sum_{i=j+1}^{j+m-1} \beta_i u_i(\xi_j) = \sum_{i=1}^{n} a_i^* u_i(\xi_j) = u^*(\xi_j).$$

Thus

$$\|f - u\| \geqslant (f - u)(\xi_j) \geqslant (f - u^*)(\xi_j) = \|f - u^*\|.$$

A similar inequality holds in the case $b^*_{j+1} = \cdots = b^*_{j+m-1} = -1$.

We now assume that $u^* = \sum_1^n a_i^* u_i$ is a best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$ and construct an interval $[\xi_j, \xi_{j+t}]$ satisfying (i) or (ii).

Let $\{x_j\}_1^r$, with $0 \leqslant x_1 < \cdots < x_r \leqslant 1$, be as given in Theorem 5.4. Define $[\xi_j, \xi_{j+t}]$ by $\xi_j \leqslant x_1 < \xi_{j+1}$ and $\xi_{j+t-1} < x_r \leqslant \xi_{j+t}$.

Case 1: $r = 1$. Here we will differentiate between the possible cases $t = 0$ and $t = 1$. Note that $t = 0$ if and only if $x_1 = \xi_j$ for some $j \in \{0, \ldots, s+1\}$.

(a) $t = 0$: Here $x_1 = \xi_j \in [0, 1]$. For all i, we have $u_i(x_1) \geqslant 0$ while $u_i(x_1) > 0$ if and only if $i \in \{j+1, \ldots, j+m-1\}$. From Theorem 5.4, if

$$(f - u^*)(x_1) = \|f - u^*\|,$$

then $a_i^* = \beta_i$ and hence $b_i^* = 1$ for all $i \in \{j+1, \ldots, j+m-1\}$. If

$$(f - u^*)(x_1) = -\|f - u^*\|,$$

then $a_i^* = \alpha_i$ and hence $b_i^* = -1$ for all $i \in \{j+1, \ldots, j+m-1\}$. In either of these cases, (ii) is satisfied.

(b) $t = 1$: Here $x_1 \in (\xi_j, \xi_{j+1})$ for some $j \in \{0, \ldots, s\}$. As above, $u_i(x_1) > 0$ if and only if $i \in \{j+1, \ldots, j+m\}$. We now follow, exactly, the reasoning of the previous Case 1(a).

Case 2: $r > 1$. From Theorem 5.4(B)(i), we have

$$U\begin{pmatrix} i_1, & \dots & , i_{r-1} \\ x_1, \dots, & \hat{x}_p, \dots & , x_r \end{pmatrix} > 0 \quad (p = 1, \dots, r).$$

Therefore

$$x_1 \in \operatorname{supp} u_{i_1}, \qquad x_p \in \operatorname{supp} u_{i_{p-1}} \cap \operatorname{supp} u_{i_p} \quad (p = 2, \dots, r-1), \qquad x_r \in \operatorname{supp} u_{i_{r-1}}.$$

From our definition of ξ_j and ξ_{j+t}, we obtain

$$i_1 - m \leq j \leq i_1 - 1, \qquad i_{r-1} - (m-1) \leq j + t \leq i_{r-1}.$$

Thus $j + 1 \leq i_1 \leq i_{r-1} \leq j + t + m - 1$.

(a) $b_i^* = 0$ $(i = j+1, \dots, j+t+m-1)$: We first claim that $i_1, \dots, i_{r-1}$ are consecutive integers. If they are not consecutive, there then exists some k, with $i_1 < k < i_{r-1}$, for which $k \notin \{i_1, \dots, i_{r-1}\}$. From Proposition 6.1, $m_k \neq 0$. From Theorem 5.4, this implies that $b_k^* \neq 0$. But $k \in \{j+1, \dots, j+t+m-1\}$, contradicting our assumption. Thus $i_1, \dots, i_{r-1}$ are consecutive integers.

Now assume that $j + 1 < i_1$. Since $\xi_j \leq x_1 < \xi_{j+1}$, we must have $x_1 \in \operatorname{supp} u_{j+1}$. It is now readily deduced that $m_{j+1} \neq 0$, implying $b_{j+1}^* \neq 0$. This contradicts our assumption, and therefore $i_1 = j + 1$. We similarly show that $i_{r-1} = j + t + m - 1$. Therefore $(i_1, \dots, i_{r-1}) = (j+1, \dots, j+t+m-1)$ implying that $r = t + m$. Thus (i) holds.

(b) $(b_{j+1}^*, \dots, b_{j+t+m-1}^*) \neq \mathbf{0}$: Recall that $j + 1 \leq i_1 \leq i_{r-1} \leq j + t + m - 1$. If $r = t + m$, i.e. $(i_1, \dots, i_{r-1}) = (j+1, \dots, j+t+m-1)$, then

$$S^+(b_{j+1}^*, \dots, b_{j+t+m-1}^*) + 1 \leq t + m - 1 < r$$

and we are finished. ($f - u^*$ equioscillates on at least r points on $[\xi_j, \xi_{j+t}]$.) We therefore assume that $r < t + m$. Set

$$m_k = U\begin{pmatrix} i_1, & \dots, i_{r-1}, & k \\ x_1, & \dots & , x_r \end{pmatrix}.$$

If $j + 1 \leq k \leq j + t + m - 1$, with $k \notin \{i_1, \dots, i_{r-1}\}$, then, from Proposition 6.1 and the fact that $x_1 \in \operatorname{supp} u_{j+1}$ and $x_r \in \operatorname{supp} u_{j+t+m-1}$, it follows that $m_k \neq 0$. Thus, from Theorem 5.4, setting $i_0 = j$ and $i_r = j + t - m$, and letting $i_{p-1} < i < i_p$ and $i_{q-1} < k < i_q$ for some $p, q \in \{1, \dots, r\}$, we obtain

$$b_i^* b_k^* (-1)^{p-q} = 1.$$

Therefore,

$$S^+(b_{j+1}^*, \dots, b_{j+t+m-1}) + 1 \leq r.$$

If equality holds, then it follows from (A) and (B)(ii) of Theorem 5.4 that the orientation of the r points of equioscillation and the (strong) orientation of $(b_{j+1}^*, \dots, b_{j+t+m-1}^*)$ must agree. This proves the converse result.

If $\tilde{u}$ is also a best approximant to f from $U(\boldsymbol{\alpha}; \boldsymbol{\beta})$, then it follows from the remark after the proof of Theorem 5.4 that $\tilde{u} = u^*$ on $[\xi_j, \xi_{j+t}]$. □

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