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On Chebyshev–Markov–Krein inequalities

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Abstract

We review the topic of Chebyshev–Markov–Krein inequalities, i.e. estimates for

$$\inf_{\nu\in V(\mu)} \int f\,\mathrm{d}\nu \quad \text{and} \quad \sup_{\nu\in V(\mu)} \int f\,\mathrm{d}\nu$$

where μ is a non-negative finite measure, and $V(\mu)$ is the set of all non-negative finite measures ν satisfying

$$\int u\,\mathrm{d}\nu = \int u\,\mathrm{d}\mu$$

for all $u \in U$, where U is a finite-dimensional subspace. For U a finite-dimensional T-space on $[a, b]$, we prove correct necessary and sufficient conditions for when a given non-negative function $f \in C[a, b]$ satisfies

$$\int_a^{\xi-} f\,\mathrm{d}\mu_\xi \le \int_a^{\xi-} f\,\mathrm{d}\nu \le \int_a^{\xi+} f\,\mathrm{d}\nu \le \int_a^{\xi+} f\,\mathrm{d}\mu_\xi$$

for every $\nu \in V(\mu)$ and all $\xi \in (a, b)$, where μ_ξ is the unique canonical representation in $V(\mu)$ containing the point ξ.

Keywords: Chebyshev–Markov–Krein inequalities; L^1-approximation; T-systems

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1. Introduction

Chebyshev, in [14], posed the following problem. Given real numbers $A < a < b < B$, a positive function f, and the values

$$\int_A^B f(x)\,\mathrm{d}x, \int_A^B xf(x)\,\mathrm{d}x, \ldots, \int_A^B x^m f(x)\,\mathrm{d}x,$$

find accurate bounds for

$$\int_a^b f(x)\,\mathrm{d}x.$$

His solution to this problem, via continued fractions, orthogonal polynomials and Gaussian quadrature, in the case of specific a, b and f, may be found in [14], without proof. A proof was provided ten years later by Markov and appeared in [9]. Krein, in his fascinating introduction to [7], gives a full history of this problem including details of later contributions by Chebyshev, Markov, Stieltjes, Posse, and others. See also the "Historical comments and notes on Chapter IV" in [8].

This problem has, over time, been generalized in various directions. One direction has led to a series of inequalities called "Čebyšev–Markov" inequalities in both [7,8], and "Markov–Krein" inequalities in [3]. (Krein's contribution, as found in [7], is significant.) In view of this, we will call these inequalities Chebyshev–Markov–Krein inequalities.

The above-mentioned texts approach these Chebyshev–Markov–Krein inequalities as they relate to T-systems and their associated canonical representations for a certain defined set of functions. This will be explained in Sections 3 and 4. In Section 2 we present an alternative approach to these inequalities based on semi-infinite optimization or one-sided L^1-approximation. We think that this is a natural approach to this problem as it generalizes, simplifies and, we hope, illuminates the ideas behind these inequalities.

In Section 3 we present basic facts concerning T-systems, T-spaces, canonical representations, convexity cones and the first of the classic Chebyshev–Markov–Krein inequalities. In Section 4, we discuss the second of these Chebyshev–Markov–Krein inequalities. We look, in detail, at necessary conditions for these inequalities to hold, generalizing what can be found in the above references. We then discuss the converse to these results. A converse result appears in [8, p. 137], but unfortunately it is incorrect.

2. One-sided L^1-approximation and generalized Chebyshev–Markov–Krein inequalities

Let K be a compact Hausdorff space, $C(K)$ denote the set of real-valued continuous functions on K, μ be a non-negative finite measure on K, and U be a finite-dimensional subspace of $C(K)$. Let $V(\mu)$ denote the set of all non-negative finite measures ν on K satisfying

$$\int_K u\,\mathrm{d}\nu = \int_K u\,\mathrm{d}\mu$$

for all $u \in U$. In other words, $V(\mu)$ is the set of all possible "representations" of μ on U by non-negative finite measures. What we call Chebyshev–Markov–Krein inequalities are estimates for

$$\inf_{\nu \in V(\mu)} \int_K f\,\mathrm{d}\nu \tag{2.1}$$

and

$$\sup_{\nu\in V(\mu)}\int_K f\,\mathrm{d}\nu \tag{2.2}$$

for a given $f \in C(K)$, or even for certain bounded, measurable and not necessarily continuous f defined on K.

In what follows we assume, for convenience, that U contains a function that is strictly positive on K. This is not totally necessary, but it makes the exposition significantly easier. Given $f \in C(K)$, set

$$\mathcal{U}_-(f) = \{u : u \in U, u \le f\}$$

and

$$\mathcal{U}_+(f) = \{u : u \in U, u \ge f\}.$$

The sets $\mathcal{U}_\pm(f)$ are convex and non-empty (by our assumption) subsets of U. The theory, as it relates to the two problems (2.1) and (2.2), is the same. Hence, let us consider (2.1). For each $u \in \mathcal{U}_-(f)$ and $\nu \in V(\mu)$ we have

$$\int_K u\,\mathrm{d}\mu = \int_K u\,\mathrm{d}\nu \le \int_K f\,\mathrm{d}\nu.$$

Thus

$$\sup_{u\in\mathcal{U}_-(f)}\int_K u\,\mathrm{d}\mu \le \inf_{\nu\in V(\mu)}\int_K f\,\mathrm{d}\nu. \tag{2.3}$$

Equality holds in (2.3). This fact may be found in many sources; see, e.g., [2], [3, p. 472], [4, Theorem 3], [5, Theorem 3], [6] and [12, Theorem 5.2]. Before stating this result, note that the solution to this problem also provides a solution to a best one-sided L^1-approximation problem in $C(K)$.

Consider the problem of characterizing best $L^1(K,\mu)$ approximations to f from $\mathcal{U}_-(f)$ or from $\mathcal{U}_+(f)$. Again, from symmetry considerations, we will consider the first problem. That is, we are interested in the problem

$$\inf_{u\in\mathcal{U}_-(f)}\|f-u\|_1 := \inf_{u\in\mathcal{U}_-(f)}\int_K |f-u|\,\mathrm{d}\mu.$$

Since $f - u \ge 0$ for all $u \in \mathcal{U}_-(f)$, the problem of finding, or characterizing, a $u \in \mathcal{U}_-(f)$ attaining the above infimum is equivalent to that of finding, or characterizing, a $u \in \mathcal{U}_-(f)$ attaining the supremum in

$$\sup_{u\in\mathcal{U}_-(f)}\int_K u\,\mathrm{d}\mu. \tag{2.4}$$

The values in these two problems differ, but the problem is one and the same. Set

$$\mathcal{P}_{\mathcal{U}_-(f)} = \{u^* : u^* \in \mathcal{U}_-(f), \|f-u^*\|_1 \le \|f-u\|_1 \text{ for all } u \in \mathcal{U}_-(f)\}.$$

That is, $\mathcal{P}_{\mathcal{U}_-(f)}$ is the set of best one-sided $L^1(K,\mu)$ approximants to f from below, and the set of $u \in \mathcal{U}_-(f)$ attaining the supremum in (2.4). Under our assumptions we have that $\mathcal{P}_{\mathcal{U}_-(f)}$

is always non-empty. A characterization of $u^* \in \mathcal{P}_{\mathcal{U}_-(f)}$ is given by the following; see e.g. [12, Theorem 5.2].

Theorem 2.1. *Let* $\dim U = n$. *Then under the above assumptions, we have* $u^* \in \mathcal{P}_{\mathcal{U}_-(f)}$ *if and only if* $u^* \in \mathcal{U}_-(f)$ *and there exist distinct points* $\{x_i\}_{i=1}^k$ *in* K, $1 \le k \le n$, *and positive numbers* $\{\lambda_i\}_{i=1}^k$ *for which*

(a) $(f - u^*)(x_i) = 0, \quad i = 1, \ldots, k$

(b) $\displaystyle\int_K u \, \mathrm{d}\mu = \sum_{i=1}^k \lambda_i u(x_i), \quad$ *all* $u \in U$.

Equation (b) is what is called a *quadrature formula* for U. On the basis of Theorem 2.1 the equality in (2.3) is now easily proven. Namely,

Theorem 2.2. *Let* $f \in C(K)$ *and* $\{x_i\}_{i=1}^k$, *and* $\{\lambda_i\}_{i=1}^k$ *be as in Theorem 2.1. Then*

$$\inf_{\nu \in V(\mu)} \int_K f \, \mathrm{d}\nu = \sum_{i=1}^k \lambda_i f(x_i).$$

Proof. Let $u^* \in \mathcal{P}_{\mathcal{U}_-(f)}$ satisfy (a) and (b) of Theorem 2.1 with the above $\{x_i\}_{i=1}^k$ and $\{\lambda_i\}_{i=1}^k$. We apply, in order, (a) and (b) of Theorem 2.1, the fact that $\nu \in V(\mu)$, and the inequality $u^* \le f$ to obtain

$$\sum_{i=1}^k \lambda_i f(x_i) = \sum_{i=1}^k \lambda_i u^*(x_i) = \int_K u^* \, \mathrm{d}\mu = \int_K u^* \, \mathrm{d}\nu \le \int_K f \, \mathrm{d}\nu.$$

Thus for all $\nu \in V(\mu)$ we have

$$\sum_{i=1}^k \lambda_i f(x_i) \le \int_K f \, \mathrm{d}\nu.$$

The infimum is therefore attained on choosing ν to be the atomic measure with the positive weights λ_i at the points (nodes) x_i, $i = 1, \ldots, k$. $\square$

Thus, we see that to solve (2.1) it is both necessary and sufficient to construct a $u^* \in \mathcal{U}_-(f)$ satisfying (a) and (b) of Theorem 2.1. We, of course, have the similar result regarding the supremum

$$\sup_{\nu \in V(\mu)} \int_K f \, \mathrm{d}\nu.$$

Remark. In the above analysis we assumed that $f \in C(K)$. What happens if f is not continuous? It is not difficult to verify that Theorems 2.1 and 2.2 remain valid if f is bounded below and lower semicontinuous. (And the corresponding upper bound result will hold if f is bounded above and upper semicontinuous.)

Example 2.1. Let K be a convex set (with interior) in $\mathbb{R}^d$, and U be the $d + 1$-dimensional subspace of linear polynomials. Let μ be any non-negative finite measure with some support in the interior of K, i.e. $\mu(\text{int } K) > 0$. Let f be a convex function on K for which there is, for each interior point of K, exactly one supporting hyperplane that touches f therein, while there are no supporting hyperplanes for f that agree with f at any boundary point of K. Such functions exist. From Theorems 2.1 and 2.2 it therefore follows that there exists a point $x^* \in \text{int } K$ and $\lambda^* > 0$ such that

$$\int_K u \, d\mu = \lambda^* u(x^*)$$

for all $u \in U$, and

$$\lambda^* f(x^*) \leq \int_K f \, d\nu$$

for all non-negative finite measures $\nu \in V(\mu)$. The associated $u^* \in \mathcal{P}_{\mathcal{U}_-(f)}$ is given by the supporting hyperplane for f at the point x^*. Now, let g be any convex function on K. It then follows from Theorems 2.1 and 2.2 that we have

$$\lambda^* g(x^*) \leq \int_K g \, d\nu$$

for all non-negative finite measures $\nu \in V(\mu)$. That is, the same point and discrete measure solve the infimum problem for all convex g on K. Or, equivalently, for every convex function g on K a best one-sided $L^1(K, \mu)$ approximation from below from linear polynomials to g is given by the linear polynomial that agrees with g at x^*. (This problem can also be solved by other means.) The best one-sided $L^1(K, \mu)$ approximation from above from linear polynomials does not have the property that the points of the optimal discrete measure are the same for all convex g, except in the case $d = 1$ where the best one-sided $L^1(K, \mu)$ approximation from above from linear polynomials is given by the straight line that interpolates each g at its endpoints (independent also of the measure μ).

Example 2.2. Set $U = \Pi_2 = \text{span}\{1, t, t^2\}$ on $[-1, 1]$ with uniform measure. Let $f_c = t^2$ on $[-1, c)$ and 0 on $[c, 1]$. We will calculate the best one-sided L^1-approximation from below to every f_c from Π_2. For all $c \in [-1/7, 1]$ the polynomial $(-1 - 6t + 7t^2)/16$ is the best one-sided L^1-approximation to every such f_c. This polynomial interpolates to f_c at $-1/3$ (double zero) and at 1 and has a zero at $-1/7$. Note that there is a quadrature formula for Π_2 with nodes 1 and $-1/3$. For $c \in (-1, -1/3]$ the zero function is a best one-sided L^1-approximation to f_c since, as stated previously, there is a quadrature formula for Π_2 based on the nodes 1 and $-1/3$. At $c = -1/3$ there is also a best one-sided L^1-approximation given by $(3/4)(t - 1)(t + 1/3)$, i.e. the quadratic polynomial that is 0 at 1 and $-1/3$, and takes the value 1 at -1. Thus $\lambda(3/4)(t - 1)(t + 1/3)$ is also a best one-sided L^1-approximation for all $\lambda \in [0, 1]$ when $c = -1/3$. For $c \in (-1/3, -1/7)$ the situation is more complicated and depends critically upon c. We claim that

$$p_c(t) = \frac{4c(t-1)(c-t)}{(c-1)^2}$$

is the best one-sided L^1-approximation from below to f_c. It is readily verified that $p_c \leq 0$ on $[c, 1]$, and

$$t^2 - p_c(t) = \left(\frac{c+1}{c-1}\right)^2 \left(t - \frac{2c}{c+1}\right)^2$$

implying that $p_c \leq f_c$ on $[-1, 1]$. In addition, from the above, $f_c - p_c$ has a simple zero at 1 and c, and a double zero at $(2c)/(c + 1)$. (Note that for $c \in (-1/3, -1/7)$ we have

$-1 < (2c)/(c+1) < c$.) To prove that p_c is the best one-sided L^1-approximation from below to f_c from Π_2 it remains to verify the existence of a quadrature formula for Π_2 with nodes 1, c, $(2c)/(c+1)$, and positive weights. It can be verified that the associated weights are

$$\lambda_1 = \frac{2c(6c^2+c+1)}{3c(c-1)^2}, \qquad \lambda_2 = \frac{2(7c+1)}{3c(c-1)^2}, \qquad \lambda_3 = \frac{-2(c+1)^2(3c+1)}{3c(c-1)^2},$$

respectively. For $c \in (-1/3, -1/7)$ these coefficients are strictly positive.

We defer further examples to the following sections.

3. Moment theory for T-systems and the first Chebyshev–Markov–Krein inequalities

Most of the results of this section may be found in both [3,8]. We start with the definition of a Chebyshev system (abbreviated as a T-system since at one time—see the references—Cyrillic transliteration gave us a spelling of Chebyshev starting with a 'T').

Definition 3.1. The set of continuous functions $\{u_1, \ldots, u_n\}$ defined on $[a, b]$ is said to be a T-system if no nontrivial $u \in U = \text{span}\{u_1, \ldots, u_n\}$ vanishes at more than $n-1$ distinct points of $[a, b]$. We call U a T-space.

Note that with this definition, U is necessarily of dimension n, and every basis for U is a T-system. There are numerous equivalent definitions. One such is the following. Since we are assuming a connected domain of definition then $\{u_i\}_{i=1}^n$ is a T-system if and only if there exists an $\varepsilon \in \{-1, 1\}$ such that

$$\varepsilon U \begin{pmatrix} 1, \ldots, n \\ s_1, \ldots, s_n \end{pmatrix} := \varepsilon \det(u_i(s_\ell))_{i,\ell=1}^n > 0 \tag{3.1}$$

for every choice of $a \le s_1 < \cdots < s_n \le b$. If $\varepsilon = 1$ in the above, then we will say that $\{u_1, \ldots, u_n\}$ is a T^+-system. This is simply a normalization.

From Definition 3.1 one can obtain further information regarding zero counting of functions in T-spaces. Namely, interior zeros that are not sign changes may be counted as double zeros.

Definition 3.2. An isolated zero of $f \in C[a, b]$ at an interior point of (a, b) is said to be *nonnodal* if f does not change sign at that zero. All other zeros, including zeros at the endpoints, are called *nodal* zeros. Let $\widetilde{Z}(f)$ denote the number of zeros of f where nodal zeros are counted once and nonnodal zeros are counted twice.

Proposition 3.1. *Let U be an n-dimensional subspace of $C[a, b]$. Then U is a T-space if and only if for every nontrivial $u \in U$ we have $\widetilde{Z}(u) \le n-1$.*

What about a converse to Proposition 3.1? That is, given distinct $s_1, \ldots, s_\ell$ in $[a, b]$, and $t_1, \ldots, t_k$ in (a, b), with $\ell + 2k \le n-1$, does there exist a nontrivial $u \in U$ which has a nodal zero at each s_i and a nonnodal zero at each t_j? The answer is yes. While we do not need this general result, we will use the following. Firstly, let us define $\omega(t) = 2$ if $t \in (a, b)$ and $\omega(t) = 1$ if $t \in \{a, b\}$. Then we have:

Proposition 3.2. *Let U be an n-dimensional T-space in $C[a, b]$. Assume we are given $a \le s_1 < \cdots < s_k \le b$, where $\sum_{j=1}^k \omega(s_j) \le n-1$. Then there exists a nontrivial non-negative $u \in U$ satisfying $u(s_j) = 0$, $j = 1, \ldots, k$.*

Let μ denote a non-negative finite measure on $[a, b]$. If μ is a discrete measure, i.e.

$$\int_a^b f \,\mathrm{d}\mu = \sum_{j=1}^{k} \lambda_j f(s_j)$$

for all $f \in C[a, b]$ with $\lambda_j > 0$ and $a \le s_1 < \cdots < s_k \le b$, then we call the s_j *nodes* and the associated λ_j *weights*. The *index* of μ, that we will denote by $I(\mu)$, is defined as

$$I(\mu) := \sum_{j=1}^{k} \omega(s_j).$$

(In [3], the index is defined as half of the above quantity.)

The measure μ is said to be *positive* (relative to U) provided that $\int_a^b u \,\mathrm{d}\mu > 0$ whenever u is a nontrivial non-negative function in U. For an n-dimensional T-space this simply means that the index of the points of support of μ is at least n, i.e. $I(\mu) \ge n$. A positive measure corresponds to an interior point of the moment space determined by the set of functions in U. We recall from Section 2 that we defined $V(\mu)$ to be the set of all non-negative finite measures satisfying

$$\int_a^b u \,\mathrm{d}\nu = \int_a^b u \,\mathrm{d}\mu$$

for all $u \in U$. In other words, $V(\mu)$ is the set of all possible "representations" of μ on U. A positive measure $\nu \in V(\mu)$ is said to be a *canonical representation* for μ if it is a discrete measure of index at most $n + 1$. It is said to be a *principal representation* for μ if it has index exactly n. The representations are also referred to as either *upper* or *lower*. They are called upper if they have a node at b. Otherwise they are called lower.

Two major results concerning $V(\mu)$ are the following:

Theorem 3.3. *Associated with every n-dimensional T-space and every positive measure μ there exist exactly two principal representations in $V(\mu)$, one upper and one lower.*

Theorem 3.4. *Let $\xi \in (a, b)$. Then associated with every n-dimensional T-space and every positive measure μ there is a unique canonical representation in $V(\mu)$ containing the node ξ.*

Many more facts concerning these representations and their nodes may be found in the previously mentioned references.

Associated with each n-dimensional T-space U is its *convexity cone* $\mathcal{C}(U)$. This is simply the set of functions f such that $\mathrm{span}\{U, f\}$ constitutes a WT (weak Chebyshev) space of dimension $n + 1$. (An m-dimensional subspace is a WT-space if no nontrivial function therein has more than $m - 1$ sign changes.) If $f \in \mathcal{C}(U)$, then $-f \in \mathcal{C}(U)$. We decompose $\mathcal{C}(U)$ into $\mathcal{C}^+(U)$ and $\mathcal{C}^-(U)$. We say that $f \in \mathcal{C}^+(U)$ if $\{u_1, \dots, u_n\}$ is a T^+-system (this is simply a normalization) and $\{u_1, \dots, u_n, f\}$ is a WT^+-system (this means that $\varepsilon = 1$ in (3.1) where the signs of the determinants are non-negative, rather than strictly positive). Alternatively, given $a \le s_1 < \cdots < s_k < b$ with $\sum_{j=1}^{k} \omega(s_j) = n$, there exists a $u \in U$ for which $(f - u)(s_j) = 0$, $j = 1, \dots, k$, and $f - u \ge 0$ on $[a, b]$. If the reverse inequality holds we say that $f \in \mathcal{C}^-(U)$. Note that $f \in \mathcal{C}^+(U)$ if and only if $-f \in \mathcal{C}^-(U)$.

The first set of classic Chebyshev–Markov–Krein inequalities are the following.

Theorem 3.5. *Let U be an n-dimensional T-space in $C[a,b]$, μ be a positive measure relative to U, and $f \in \mathcal{C}^+(U)$. Then for every $\nu \in V(\mu)$ we have*

$$\int_a^b f\,\mathrm{d}\mu_- \le \int_a^b f\,\mathrm{d}\nu \le \int_a^b f\,\mathrm{d}\mu_+$$

where μ_- and μ_+ are the lower and upper principal representations for μ, respectively.

In the above theorem the lower and upper bounds are uniquely attained by these principal representations if $\{u_1,\ldots,u_n,f\}$ is a T^+-system. When assuming that $\{u_1,\ldots,u_n,f\}$ is a WT^+-system we may lose the uniqueness. Generalizations of the above result to where U is a WT-space can be found in [10]; see also [3, Chapter III, Section 4].

The proof of this result, as presented in [7,3,8], essentially proceeds as follows. Firstly, prove the existence of the two principal representations. As $f \in \mathcal{C}^+(U)$, then construct using Proposition 3.2 with respect to the WT^+-space span$\{U,f\}$, the function $f-u^*$, which is non-negative, nontrivial and vanishes at the nodes (of index n) of the lower principal representation. Thus (a) and (b) of Theorem 2.1 hold and the result follows from Theorem 2.2. In a similar way we can see that the best one-sided $L^1(\mu)$ approximation from above from U to f is given by the u^{**} that interpolates to f at the nodes of the upper principal representation. If $f \in \mathcal{C}^-(U)$ then similar inequalities hold (with the inequalities reversed) simply from considering $-f$ in place of f. A variation on this proof is presented after the statement of Theorem 3.6. Further generalizations of this result may be found in [12, Appendix B], and references therein.

We could also prove Theorem 3.5 starting from Theorem 2.1 if $\{u_1,\ldots,u_n,f\}$ is a T^+-system, and in this way also get the existence of the principal representations for U. That is, assume that we have, as in (b) of Theorem 2.1, distinct points $a \le x_1 < \cdots < x_k \le b$ in $[a,b]$, $1 \le k \le n$, and positive numbers $\{\lambda_i\}_{i=1}^k$ for which

$$\int_a^b u\,\mathrm{d}\mu = \sum_{i=1}^k \lambda_i u(x_i) \tag{3.2}$$

for all $u \in U$. We first claim that $\sum_{i=1}^k \omega(x_i) \ge n$. For otherwise—see Proposition 3.2—there exists a nontrivial non-negative $u \in U$ satisfying $u(x_i)=0$, $i=1,\ldots,k$, contradicting (3.2). Assume $\{u_1,\ldots,u_n,f\}$ is a T^+-system, and let $u^* \in \mathcal{P}_{\mathcal{U}_-(f)}$. Then $f-u^* \ge 0$ on $[a,b]$ and $f-u^*$ vanishes at the x_i. Thus $\sum_{i=1}^k \omega(x_i) \le \widetilde{Z}(f-u^*) \le n$. This implies that $\sum_{i=1}^k \omega(x_i) = n$. Since we have that $\{u_1,\ldots,u_n,f\}$ is a T^+-system, this also implies that $x_k < b$. In other words the right-hand side of (3.2) is a lower principal representation for μ. It is not difficult to directly verify uniqueness of the principal representations; see e.g., [12, pp. 218–219].

In [8, pp. 135–6] there can be found the following converse to Theorem 3.5.

Theorem 3.6. *Let $\{u_1,\ldots,u_n\}$ be a T^+-system on $[a,b]$. Assume that, for a given $f =: u_{n+1} \in C[a,b]$, and for every positive measure μ relative to $U = \mathrm{span}\{u_1,\ldots,u_n\}$, the Chebyshev–Markov–Krein inequality holds. Namely for every $\nu \in V(\mu)$ we have*

$$\int_a^b f\,\mathrm{d}\mu_- \le \int_a^b f\,\mathrm{d}\nu$$

where μ_- is the lower principal representation for μ. Then

$$U\begin{pmatrix}1,\ldots,n+1\\ t_1,\ldots,t_{n+1}\end{pmatrix} \ge 0$$

for every $a < t_1 < \cdots < t_{n+1} < b$ *in the case where* n *is even, and*

$$U\begin{pmatrix}1,\ldots,n+1\\ t_1,\ldots,t_{n+1}\end{pmatrix} \geq 0$$

for every $a = t_1 < \cdots < t_{n+1} < b$ *if* n *is odd.*

A similar statement holds for the maximum value. In that case we only obtain conditions on determinants with points satisfying $t_{n+1} = b$, and also $t_1 = a$ if n is even.

There are real differences between these various conditions, where we demand that certain endpoints be nodes. Take, for example, $U = \text{span}\{u_1\}$, where u_1 is the constant function. Then

$$U\begin{pmatrix}1,2\\ t_1,t_2\end{pmatrix} \geq 0$$

for all $a < t_1 < t_2 < b$ if and only if u_2 is a nondecreasing function on $[a, b]$. We have

$$U\begin{pmatrix}1,2\\ a,t_2\end{pmatrix} \geq 0$$

for all $t_2 \in (a, b]$ if and only if $u_2(a) = \min\{u_2(t) : t \in [a, b]\}$, while

$$U\begin{pmatrix}1,2\\ t_1,b\end{pmatrix} \geq 0$$

for all $t_1 \in [a, b)$ if and only if $u_2(b) = \max\{u_2(t) : t \in [a, b]\}$. Finally

$$U\begin{pmatrix}1,2\\ a,b\end{pmatrix} \geq 0$$

if and only if $u_2(a) \leq u_2(b)$.

A careful consideration of the analysis also shows that these conditions are sufficient for the associated Chebyshev–Markov–Krein inequality to hold. That is, we do not need or use the full convexity cone property in the case of the above minimum if n is odd, or in the case of the maximum, whether n is odd or even. We present a quick explanation of one of these cases.

We consider the lower bound and n odd. We wish to verify that the condition

$$U\begin{pmatrix}1,\ldots,n+1\\ t_1,\ldots,t_{n+1}\end{pmatrix} \geq 0$$

for every $a = t_1 < \cdots < t_{n+1} < b$ is sufficient for obtaining

$$\int_a^b u_{n+1}\, \mathrm{d}\mu_- \leq \int_a^b u_{n+1}\, \mathrm{d}\nu$$

for every $\nu \in V(\mu)$, where μ_- is the lower principal representation for μ.

As n is odd, say $n = 2k-1$, the lower principal representation for a given measure μ, positive relative to U, has nodes

$$a = \xi_1 < \xi_2 < \cdots < \xi_k < b.$$

We prove that there exists a $u^* \in U$ satisfying $u^* \leq u_{n+1}$ and $u^*(\xi_i) = u_{n+1}(\xi_i),\ i = 1,\ldots,k$. By Theorems 2.1 and 2.2, this is exactly what is needed.

Let $t_{2i-1} = \xi_i$, $i = 1, \ldots, k$, and $t_{2i} = \xi_{i+1} - \varepsilon$, $i = 1, \ldots, k-1$, for $\varepsilon > 0$, small. Thus the $\{t_1, \ldots, t_n\}$ are in strictly increasing order. The ratio

$$\frac{U\begin{pmatrix} 1, \ldots, n, n+1 \\ t_1, \ldots, t_n, t \end{pmatrix}}{U\begin{pmatrix} 1, \ldots, n \\ t_1, \ldots, t_n \end{pmatrix}}$$

is well-defined since, as $\{u_1, \ldots, u_n\}$ is a T^+-system, the denominator is strictly positive. Expanding the numerator by its last column, we have that it equals

$$u_{n+1}(t) - u_\varepsilon(t)$$

where $u_\varepsilon \in U$. Furthermore, since, by assumption, the numerator is non-negative when the points are arranged in increasing order, we have that

$$(-1)^{i+1}(u_{n+1}(t) - u_\varepsilon(t)) \geq 0, \quad t \in [t_i, t_{i+1}],\ i = 1, \ldots, n,$$

where $t_{n+1} = b$. That is, we can have $u_{n+1}(t) - u_\varepsilon(t) < 0$ only for t in $\cup_{i=2}^{k}[\xi_i - \varepsilon, \xi_i]$. It is easily shown that the u_ε are uniformly bounded. Thus, as $u_\varepsilon \in U$, and U is a finite-dimensional subspace, on a subsequence of $\varepsilon \to 0$ we have that u_ε uniformly tends to a $u^* \in U$. From the above, it follows that $u^* \leq u_{n+1}$ and $u^*(\xi_i) = u_{n+1}(\xi_i),\ i = 1, \ldots, k$.

The other cases are proven analogously.

4. The second Chebyshev–Markov–Krein inequalities and its converse

The statement of the second set of classic Chebyshev–Markov–Krein inequalities is more awkward. Hence we start with a definition that will we hope expedite the exposition.

Definition 4.1. We say that $\{u_1, \ldots, u_n, f\}$ satisfies Condition K on $[a, b]$ if:

(a) $\{u_1, \ldots, u_k\}$ is a T^+-system for $k = 1, \ldots, n$.
(b) f is strictly positive on $[a, b]$.
(c) $\{u_1, \ldots, u_k, f\}$ is a T^+-system for $k = 1, \ldots, n$.

We then have, from [7,3,8]:

Theorem 4.1. *Let $U = \text{span}\{u_1, \ldots, u_n\}$ and f be in $C[a, b]$ and assume that $\{u_1, \ldots, u_n, f\}$ satisfies Condition K thereon. Let μ be a positive measure relative to U, and $\xi \in (a, b)$. Then*

$$\int_a^{\xi-} f\, d\mu_\xi \leq \int_a^{\xi-} f\, d\nu \leq \int_a^{\xi+} f\, d\nu \leq \int_a^{\xi+} f\, d\mu_\xi$$

for every $\nu \in V(\mu)$, where μ_ξ is the (unique) canonical representation for μ containing the node ξ.

By $\int_a^{\xi-} f\, d\nu$ ($\int_a^{\xi+} f\, d\nu$) we mean the integral over the interval $[a, \xi)$ ($[a, \xi]$).

The method of proof is, of course, to show that one can construct the appropriate one-sided L^1-approximations. Let

$$f^{\xi-}(t) = \begin{cases} f(t), & a \leq t < \xi \\ 0, & \xi \leq t \leq b, \end{cases}$$

and

$$f^{\xi+}(t)=\begin{cases}f(t), & a\le t\le \xi\\ 0, & \xi<t\le b.\end{cases}$$

Recall that f is a strictly positive function. Thus, $f^{\xi-}$ is lower semicontinuous, while $f^{\xi+}$ is upper semicontinuous. Let $a\le \xi_1<\cdots<\xi_r\le b$ denote the nodes of the canonical representation with $\xi=\xi_\ell$ for some ℓ. The proof involves showing the existence of a $u^*\in U$ satisfying $u^*(\xi_i)=f^{\xi-}(\xi_i)$, $i=1,\ldots,r$, and $u^*\le f^{\xi-}$, and a $u^{**}\in U$ satisfying $u^{**}(\xi_i)=f^{\xi+}(\xi_i)$, $i=1,\ldots,r$, and $u^{**}\ge f^{\xi+}$. Additional conditions under which Theorem 4.1 holds may be found in [3, Sections 3 and 4 of Chapter III]. An important case is when f itself is a positive function in U. Theorem 4.2 in Chapter III of [3] provides conditions for when the desired result holds in this case. (It is not always true.) Alternatively, Theorem 4.1 is also valid in the case $f=u_m$ any $m\in\{1,\ldots,n\}$, if u_m is a non-negative function, $\{u_1,\ldots,u_k\}$ is a T^+-system for $k=1,\ldots,n$, and $\{u_1,\ldots,u_k,u_m\}$ is a T^+-system for $k=1,\ldots,m-1$. This can be proven directly, or can be regarded as belonging to the "boundary of Condition K". Thus, it is always valid for $f=u_1$ if $\{u_1,\ldots,u_k\}$ is a T^+-system for $k=1,\ldots,n$ (the case $k=1$ implying that u_1 is a positive function). It should also be noted that for every n-dimensional T-space U on $[a,b]$, there always exists a basis $\{v_1,\ldots,v_n\}$ for U such that $\{v_1,\ldots,v_k\}$ is a T^+-system on (a,b), for $k=1,\ldots,n$ (but not necessarily on $[a,b]$); see [15] for a proof and a history of this problem. An example where u^* is calculated can be found in [1].

The method of proof of Theorem 4.1 does not use Condition K, per se, but rather a consequence thereof. We state this consequence as Condition M, as it is, in fact, the relevant property and will also appear in the converse theorem.

To ease notation, we again set $f=:u_{n+1}$ and assume that u_{n+1} is a non-negative nontrivial function in $C[a,b]$. For each $\xi\in(a,b)$, we use the notation $u_{n+1}^{\xi\pm}$ as previously defined.

For $q\in\{1,\ldots,n+1\}$ and $a\le s_1<\cdots<s_{n+1}\le b$, we define

$$U^q\begin{pmatrix}1,\ldots,n+1\\ s_1,\ldots,s_{n+1}\end{pmatrix}=\begin{vmatrix}u_1(s_1) & \cdots & u_1(s_q) & u_1(s_{q+1}) & \cdots & u_1(s_{n+1})\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ u_n(s_1) & \cdots & u_n(s_q) & u_n(s_{q+1}) & \cdots & u_n(s_{n+1})\\ u_{n+1}(s_1) & \cdots & u_{n+1}(s_q) & 0 & \cdots & 0\end{vmatrix}.$$

That is, the last $n+1-q$ entries in the last row are set equal to 0.

Definition 4.2. We will say that the function $u_{n+1}\in C[a,b]$ satisfies *Condition M* with respect to the n-dimensional T^+-system $\{u_1,\ldots,u_n\}$ and the points $a\le s_1<\cdots<s_{n+1}\le b$ and $1\le q\le n+1$ if

$$(-1)^{n+1+q}U^q\begin{pmatrix}1,\ldots,n+1\\ s_1,\ldots,s_{n+1}\end{pmatrix}\ge 0.$$

We say that u_{n+1} satisfies the strong Condition M if strict inequality holds in the above.

Condition M, unlike Condition K, does not depend on the choice of a bases for U. Note that the case $q=1$ actually implies that u_{n+1} is a non-negative function.

What is proven in [3, pp. 33–36] is that Condition K implies the strong Condition M for all $a\le s_1<\cdots<s_{n+1}\le b$ and all $1\le q\le n+1$. And it is this strong Condition M that is used to prove the existence of the desired functions u^* and u^{**}.

Before considering the exact connection between Condition M and these Chebyshev–Markov–Krein inequalities, we will study Condition M in further detail. The first important and surprising result is the following.

Proposition 4.2. *Assume u_{n+1} satisfies Condition M with respect to the n-dimensional T^+-system $\{u_1, \ldots, u_n\}$ and all points $a \le s_1 < \cdots < s_{n+1} = b$ for all $q \in \{1, \ldots, n\}$. Then u_{n+1} satisfies Condition M with respect to the n-dimensional T^+-system $\{u_1, \ldots, u_n\}$ and all points $a \le s_1 < \cdots < s_{n+1} \le b$ for all $q \in \{1, \ldots, n\}$.*

Proof. It is easily verified that the result holds for $q = 1$. This follows from the fact that

$$(-1)^{n+2} U^1 \begin{pmatrix} 1, \ldots, n+1 \\ s_1, \ldots, s_{n+1} \end{pmatrix} = u_{n+1}(s_1) U \begin{pmatrix} 1, \ldots, n \\ s_2, \ldots, s_{n+1} \end{pmatrix}$$

for every choice of $a \le s_1 < \cdots < s_{n+1} \le b$. Thus Condition M with $s_{n+1} = b$ and $q = 1$, and the fact that $\{u_1, \ldots, u_n\}$ is a T^+-system imply that u_{n+1} is a non-negative function, which in turn implies that Condition M holds for $q = 1$ and for all $a \le s_1 < \cdots < s_{n+1} \le b$.

The general case is proven as follows. Let A be an $(n + 1) \times (n + 2)$ matrix. The following determinantal identity is well-known, and may be easily proven as a consequence of Sylvester's determinant identity; see e.g., [13, p. 5]:

$$A \begin{pmatrix} 1, \ldots, n+1 \\ 1, \ldots, n+1 \end{pmatrix} A \begin{pmatrix} 1, \ldots, n \\ 2, \ldots, n, n+2 \end{pmatrix} = A \begin{pmatrix} 1, \ldots, n+1 \\ 1, \ldots, n, n+2 \end{pmatrix} A \begin{pmatrix} 1, \ldots, n \\ 2, \ldots, n+1 \end{pmatrix} - A \begin{pmatrix} 1, \ldots, n+1 \\ 2, \ldots, n+2 \end{pmatrix} A \begin{pmatrix} 1, \ldots, n \\ 1, \ldots, n \end{pmatrix}.$$

Let $a \le s_1 < \cdots < s_{n+1} < b$ and set $s_{n+2} = b$. Define the $(n + 1) \times (n + 2)$ matrix A by

$$A = \begin{bmatrix} u_1(s_1) & \cdots & u_1(s_q) & u_1(s_{q+1}) & \cdots & u_1(s_{n+2}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_n(s_1) & \cdots & u_n(s_q) & u_n(s_{q+1}) & \cdots & u_n(s_{n+2}) \\ u_{n+1}(s_1) & \cdots & u_{n+1}(s_q) & 0 & \cdots & 0 \end{bmatrix}.$$

Assume $1 \le q \le n$ and apply the previous determinantal identity to obtain

$$\begin{aligned} &U^q \begin{pmatrix} 1, \ldots, n+1 \\ s_1, \ldots, s_{n+1} \end{pmatrix} U \begin{pmatrix} 1, \ldots, n \\ s_2, \ldots, s_n, s_{n+2} \end{pmatrix} \\ &\quad = U^q \begin{pmatrix} 1, \ldots, n+1 \\ s_1, \ldots, s_n, s_{n+2} \end{pmatrix} U \begin{pmatrix} 1, \ldots, n \\ s_2, \ldots, s_{n+1} \end{pmatrix} \\ &\qquad - U^{q-1} \begin{pmatrix} 1, \ldots, n+1 \\ s_2, \ldots, s_{n+2} \end{pmatrix} U \begin{pmatrix} 1, \ldots, n \\ s_1, \ldots, s_n \end{pmatrix}. \end{aligned}$$

By assumption, since $s_{n+2} = b$, we have

$$(-1)^{n+1+q} U^q \begin{pmatrix} 1, \ldots, n+1 \\ s_1, \ldots, s_n, s_{n+2} \end{pmatrix} \ge 0$$

and

$$(-1)^{n+1+q-1} U^{q-1} \begin{pmatrix} 1, \ldots, n+1 \\ s_2, \ldots, s_{n+2} \end{pmatrix} \ge 0.$$

As $\{u_1, \dots, u_n\}$ is a T^+-system on $[a, b]$ the three determinants containing only the rows $\{1, \dots, n\}$ are strictly positive. Thus

$$(-1)^{n+1+q} U^q \begin{pmatrix} 1, \dots, n+1 \\ s_1, \dots, s_{n+1} \end{pmatrix} \geq 0. \quad \square$$

Remark. The above result does not hold for $q = n + 1$, as may be seen by the example given after the statement of Theorem 3.6.

We could also have demanded less in our definition of Condition M, as is attested to by the following.

Proposition 4.3. *Assume $\{u_1, \dots, u_n\}$ is a T^+-system on $[a, b]$, u_{n+1} is a non-negative nontrivial function in $C[a, b]$, and $q \in \{1, \dots, n\}$. Assume there exists a $\delta_q \in \{-1, 1\}$ for which*

$$\delta_q\, U^q \begin{pmatrix} 1, \dots, n+1 \\ s_1, \dots, s_{n+1} \end{pmatrix} \geq 0$$

for all $a \leq s_1 < \cdots < s_{n+1} \leq b$. Then $\delta_q = (-1)^{n+1+q}$.

Proof. Consider the function

$$G^q(t) = \begin{vmatrix} u_1(s_1) & \cdots & u_1(s_q) & u_1(t) & u_1(s_{q+2}) & \cdots & u_1(s_{n+1}) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_n(s_1) & \cdots & u_n(s_q) & u_n(t) & u_n(s_{q+2}) & \cdots & u_n(s_{n+1}) \\ u_{n+1}(s_1) & \cdots & u_{n+1}(s_q) & 0 & 0 & \cdots & 0 \end{vmatrix},$$

where $s_q \in (a, b)$ is chosen such that $u_{n+1}(s_q) > 0$. Note that

$$G^q(s_{q+1}) = U^q \begin{pmatrix} 1, \dots, n+1 \\ s_1, \dots, s_{n+1} \end{pmatrix}$$

for any $s_{q+1} \in (s_q, s_{q+2})$.

Now,

$$G^q(s_q) = \begin{vmatrix} u_1(s_1) & \cdots & u_1(s_q) & u_1(s_q) & \cdots & u_1(s_{n+1}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_n(s_1) & \cdots & u_n(s_q) & u_n(s_q) & \cdots & u_n(s_{n+1}) \\ u_{n+1}(s_1) & \cdots & u_{n+1}(s_q) & 0 & \cdots & 0 \end{vmatrix}.$$

Expanding the above by the last row, and since the qth and $(q + 1)$st columns are equal except for their last components, we have

$$G^q(s_q) = (-1)^{n+1+q}\, u_{n+1}(s_q)\, U \begin{pmatrix} 1, \dots, n \\ s_1, \dots, s_q, s_{q+2}, \dots, s_{n+1} \end{pmatrix}.$$

Since $\{u_1, \dots, u_n\}$ is a T^+-system, we have

$$U \begin{pmatrix} 1, \dots, n \\ s_1, \dots, s_q, s_{q+2}, \dots, s_{n+1} \end{pmatrix} > 0$$

for all choices of strictly increasing points. In addition, $u_{n+1}(s_q) > 0$ and therefore $(-1)^{n+1+q} G^q(s_q) > 0$. Since $G^q \in C[a, b]$ we obtain $(-1)^{n+1+q} G^q(s_q + \varepsilon) > 0$ for $\varepsilon > 0$, small. Thus $\delta_q = (-1)^{n+1+q}$. $\square$

From continuity and connectedness we also have:

Corollary 4.4. *Assume* $\{u_1, \dots, u_n\}$ *is a* T^+*-system on* $[a, b]$, u_{n+1} *is a strictly positive function in* $C[a, b]$, *and* $q \in \{1, \dots, n\}$. *Assume*

$$U^q \begin{pmatrix} 1, \dots, n+1 \\ s_1, \dots, s_{n+1} \end{pmatrix} \neq 0$$

for all $a \leq s_1 < \cdots < s_{n+1} \leq b$. *Then*

$$(-1)^{n+1+q} U^q \begin{pmatrix} 1, \dots, n+1 \\ s_1, \dots, s_{n+1} \end{pmatrix} > 0$$

for all $a \leq s_1 < \cdots < s_{n+1} \leq b$.

The above arguments do not apply, and are not valid, when $q = n + 1$.

Condition M might, at first glance, seem to be a bit odd. However similar phenomena may be found in the literature. For example, if A is an $m \times m$ totally positive matrix (all minors are non-negative) and we alter A to A^q by setting the last $m - q$ elements of the last row of A to be zero, then in fact

$$(-1)^{m+q} \det A^q \geq 0$$

(see [13, p. 30]). Thus, if $\{u_1, \dots, u_{n+1}\}$ is a Descartes system, i.e. $\{u_{i_1}, \dots, u_{i_r}\}$ is a T^+-system for every $1 \leq i_1 < \cdots < i_r \leq n + 1$ and every $r = 1, \dots, n + 1$, then (strong) Condition M holds for all $a \leq s_1 < \cdots < s_{n+1} \leq b$ and all $q \in \{1, \dots, n\}$. This is a case where Condition K also holds.

Example 4.1. On any interval $[a, b]$ where $0 \leq a < b$ the functions $\{1, t, \dots, t^{n-1}\}$ form a Descartes system (aside from a degeneracy at $t = 0$). This means that the associated matrix U, as defined above, is totally positive. One consequence thereof is that $f(t) = t^k$ satisfies Condition M with respect to U for all $k \in \mathbb{Z}_+$. (Compare this to Example 2.2.)

Example 4.2. Assume $[a, b] = [0, 1]$, $U = \text{span}\{1, t\}$, and $f(t) = 1$. Let μ be any non-negative measure satisfying

$$\int_0^1 1 \, d\mu = 1$$

and

$$\int_0^1 t \, d\mu = c$$

where $0 < c < 1$. The fact that $c \in (0, 1)$ guarantees that μ is a positive measure relative to U. It is readily verified that the canonical representation for μ containing $\xi \in (0, 1)$ is given by

$$d\mu_\xi = \frac{\xi - c}{\xi} \delta_0 + \frac{c}{\xi} \delta_\xi$$

for $\xi \geq c$, and

$$d\mu_\xi = \frac{1 - c}{1 - \xi} \delta_\xi + \frac{c - \xi}{1 - \xi} \delta_1$$

for $\xi \leq c$, where δ_y represents the unit point measure at y. (For $\xi = c$ these formulas give the lower principal representation for μ, while for $\xi = 0$ or $\xi = 1$ we get the upper principal representation for μ.) It therefore follows from Theorem 4.1 that

$$\min_{\nu \in V(\mu)} \int_0^{\xi-} 1 \, d\nu = \int_0^{\xi-} 1 \, d\mu_\xi = \begin{cases} 0, & 0 < \xi \le c \\ \dfrac{\xi - c}{\xi}, & c \le \xi < 1, \end{cases}$$

and

$$\max_{\nu \in V(\mu)} \int_0^{\xi+} 1 \, d\nu = \int_0^{\xi+} 1 \, d\mu_\xi = \begin{cases} \dfrac{1-c}{1-\xi}, & 0 < \xi \le c \\ 1, & c \le \xi < 1. \end{cases}$$

We can now show the exact connection between Condition M and the result in Theorem 4.1.

We will prove direct and converse directions for both the lower and upper bounds that are more general than those found in the above-mentioned texts. Furthermore, the converse theorem as stated in [8, p. 137], and in [11], is incorrect.

Theorem 4.5. *Assume $U = \text{span}\{u_1, \ldots, u_n\}$ is an n-dimensional T-space in $C[a, b]$, and $\{u_1, \ldots, u_n\}$ is a T^+-system. Let μ be a positive measure relative to U. Assume u_{n+1} satisfies Condition M with respect to the $\{u_1, \ldots, u_n\}$ and any set of points $a \le s_1 < \cdots < s_{n+1} \le b$, and all $q \in \{1, \ldots, n\}$. Then for each $\xi \in (a, b)$*

$$\int_a^{\xi-} u_{n+1} \, d\mu_\xi \le \int_a^{\xi-} u_{n+1} \, d\nu$$

for every $\nu \in V(\mu)$, where μ_ξ is the (unique) canonical representation for μ containing the node ξ.

Proof. Let $a \le \xi_1 < \cdots < \xi_r \le b$ denote the nodes of the canonical representation for μ_ξ with $\xi = \xi_\ell$. We wish to prove the existence of a $u^* \in U$ satisfying $u^* \le u_{n+1}^{\xi-}$ and $u^*(\xi_i) = u_{n+1}^{\xi-}(\xi_i)$, $i = 1, \ldots, r$. From Theorems 2.1 and 2.2, this then proves our theorem. Note that if $\ell = 1$, then we simply set $u^* = 0$. Hence, we can assume that $1 < \ell \le r$. Furthermore, as $\xi \in (a, b)$, if $\ell = r$ then we have $\xi_r < b$.

As μ_ξ is a canonical representation, we have $I(\mu_\xi) = n$ or $n + 1$. If $I(\mu_\xi) = n$, then we will add another node at an endpoint. In the case where μ_ξ has a node at each endpoint, then we will add a node at a and consider it as a double node. That is, we will assume that $I(\mu_\xi) = n + 1$. We add to the node set $\{\xi_i\}_{i=1}^r$ the values $\{\xi_i + \varepsilon\}$ for all $i \in \{1, \ldots, r\}$ satisfying $a < \xi_i < b$, except for $i = \ell$. (If there is a double node at a, then we consider the two values a and $a + \varepsilon$.) We call this new set of n strictly increasing values $\{t_i\}_{i=1}^n$.

Consider the ratio

$$\frac{U^{\xi-}\begin{pmatrix} 1, \ldots, n, n+1 \\ t_1, \ldots, t_n, t \end{pmatrix}}{U\begin{pmatrix} 1, \ldots, n \\ t_1, \ldots, t_n \end{pmatrix}}.$$

It is well-defined since, as $\{u_1, \ldots, u_n\}$ is a T^+-system, the denominator is strictly positive. Expanding the numerator by its last column, we have that it equals

$$u_{n+1}^{\xi-}(t) - u_\varepsilon(t)$$

where $u_\varepsilon \in U$. Let us consider the sign of this function.

Assume $\xi = \xi_\ell = t_k$. For $t \in [t_i, t_{i+1}]$ with $i \geq k$, i.e. $t \geq \xi$, we are considering a determinant of the form U^q with $q = k - 1$. Rearranging the columns in increasing order of the points, i.e. moving the column with the value t a total of $n - i$ positions to its left, implies, from Condition M, that

$$(-1)^{i+k} U^{\xi-} \begin{pmatrix} 1, \ldots, n, n+1 \\ t_1, \ldots, t_n, t \end{pmatrix} \geq 0, \quad t \in [t_i, t_{i+1}],\ i \geq k.$$

And similarly, for $t \in [t_i, t_{i+1}]$ with $i < k$, i.e. $t \leq \xi$. But here we are considering a determinant of the form U^q with $q = k$. Therefore

$$(-1)^{i+k+1} U^{\xi-} \begin{pmatrix} 1, \ldots, n, n+1 \\ t_1, \ldots, t_n, t \end{pmatrix} \geq 0, \quad t \in [t_i, t_{i+1}],\ i < k.$$

(Note that we always have $q \leq k \leq n$.)

From the above two inequalities it is readily verified that

$$u_{n+1}^{\xi-}(t) - u_\varepsilon(t) < 0$$

can only hold in the intervals $(\xi_i, \xi_i + \varepsilon)$. It is easily shown that the u_ε are uniformly bounded. Thus, as $u_\varepsilon \in U$, and U is a finite-dimensional subspace, on a subsequence of $\varepsilon \to 0$ we have that u_ε uniformly tends to a $u^* \in U$. The limit function u^* satisfies $u^* \leq u_{n+1}^{\xi-}$ and $u^*(\xi_i) = u_{n+1}^{\xi-}(\xi_i),\ i = 1, \ldots, k$. We have constructed the desired $u^* \in U$. $\square$

Remark. Note that there is no demand that Condition M holds for the case $q = n + 1$.

As we now show, these same sufficient conditions are necessary if the above Chebyshev–Markov–Krein lower bounds are to hold for every $\xi \in (a, b)$ and all positive measures relative to U.

Theorem 4.6. *Let $\{u_1, \ldots, u_n\}$ be a T^+-system on $[a, b]$. Assume that for a given $u_{n+1} \in C[a, b]$, for each $\xi \in (a, b)$, and for every positive measure μ relative to $U = \operatorname{span}\{u_1, \ldots, u_n\}$ we have that*

$$\int_a^{\xi-} u_{n+1}\, \mathrm{d}\mu_\xi \leq \int_a^{\xi-} u_{n+1}\, \mathrm{d}\nu \tag{4.1}$$

for every $\nu \in V(\mu)$, where μ_ξ is the unique canonical representation for μ containing the node ξ. Then u_{n+1} satisfies Condition M with respect to the n-dimensional T^+-system $\{u_1, \ldots, u_n\}$ for every choice of points $a \leq s_1 < \cdots < s_{n+1} \leq b$, and all $q \in \{1, \ldots, n\}$.

Proof. This proof is a variation on the method of proof in [8, pp. 135–136], for the above Theorem 3.6.

Choose any set of $n + 1$ strictly increasing points $a \leq s_1 < \cdots < s_{n+1} \leq b$. Set

$$a_i = U \begin{pmatrix} 1, \ldots, n \\ s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n+1} \end{pmatrix}, \quad i = 1, \ldots, n+1.$$

Then

$$\sum_{i=1}^{n+1} (-1)^i a_i u(s_i) = 0, \tag{4.2}$$

for all $u \in U$. As $\{u_1, \ldots, u_n\}$ is a T^+-system we have $a_i > 0,\ i = 1, \ldots, n+1$. Expanding the determinant U^q, at the points $\{s_i\}_{i=1}^{n+1}$, by its last row we also have

$$U^q \begin{pmatrix} 1, \ldots, n+1 \\ s_1, \ldots, s_{n+1} \end{pmatrix} = (-1)^{n+1} \sum_{i=1}^{q} (-1)^i a_i u_{n+1}(s_i).$$

Thus the fact that u_{n+1} satisfies Condition M with respect to the n-dimensional T^+-system $\{u_1, \ldots, u_n\}$ for the choice of points $a \le s_1 < \cdots < s_{n+1} \le b$, and $q \in \{1, \ldots, n\}$, is equivalent to

$$(-1)^q \sum_{i=1}^{q} (-1)^i a_i u_{n+1}(s_i) \ge 0, \tag{4.3}$$

for the choice of points $a \le s_1 < \cdots < s_{n+1} \le b$, and $q \in \{1, \ldots, n\}$. We will prove (4.3).

We start with the case of n odd, i.e. $n = 2k - 1$. Choose any set of $n + 1 = 2k$ strictly increasing points in $[a, b]$,

$$a \le s_1 < \cdots < s_{2k} \le b.$$

From (4.2) we have

$$\sum_{i=1}^{k} a_{2i-1} u(s_{2i-1}) = \sum_{i=1}^{k} a_{2i} u(s_{2i}) \tag{4.4}$$

for all $u \in U$.

Choose $\xi = s_{q+1}$ for some $q \in \{1, \ldots, n\}$, where $a < \xi = s_{q+1} < b$. Assume, for the moment, that $q = 2\ell - 1$ is odd. Let μ_ξ denote the non-negative discrete measure given by

$$\int_a^b u \,\mathrm{d}\mu_\xi = \sum_{i=1}^{k} a_{2i} u(s_{2i}).$$

Note that since $\xi \in \{s_2, \ldots, s_{2k}\}$ and $a < s_2 < \cdots < s_{2k} \le b$, we have that $n = 2k - 1 \le I(\mu_\xi) \le 2k = n + 1$. Therefore, μ_ξ is a positive measure relative to U, and is also a canonical representation containing the node ξ. Thus, from (4.1),

$$\int_a^{\xi-} u_{n+1} \,\mathrm{d}\mu_\xi \le \int_a^{\xi-} u_{n+1} \,\mathrm{d}\nu$$

for every $\nu \in V(\mu_\xi)$, and in particular, from (4.4), we obtain

$$\sum_{i=1}^{\ell-1} a_{2i} u_{n+1}(s_{2i}) \le \sum_{i=1}^{\ell} a_{2i-1} u_{n+1}(s_{2i-1}),$$

which is simply

$$(-1)^q \sum_{i=1}^{q} (-1)^i a_i u_{n+1}(s_i) \ge 0,$$

since $q = 2\ell - 1$ odd. This is (4.3).

Now, assume that q is even, i.e. $q = 2\ell$, and $\xi = s_{2\ell+1}$. Let μ_ξ denote the non-negative discrete measure given by

$$\int_a^b u \, \mathrm{d}\mu_\xi = \sum_{i=1}^{k} a_{2i-1} u(s_{2i-1}).$$

We follow exactly the same analysis as above. The difference is that we obtain the inequality

$$\sum_{i=1}^{\ell-1} a_{2i-1} u_{n+1}(s_{2i-1}) \le \sum_{i=1}^{\ell-1} a_{2i} u_{n+1}(s_{2i}),$$

and thus

$$(-1)^q \sum_{i=1}^{q} (-1)^i a_i u_{n+1}(s_i) \ge 0,$$

for q even. This proves (4.3).

Let us now consider the case where n is even, i.e. $n = 2k$. As above, choose any set of $n+1 = 2k+1$ strictly increasing points in $[a, b]$,

$$a \le s_1 < \cdots < s_{2k+1} \le b.$$

From (4.2) we have

$$\sum_{i=1}^{k+1} a_{2i-1} u(s_{2i-1}) = \sum_{i=1}^{k} a_{2i} u(s_{2i})$$

for all $u \in U$.

The proof now proceeds in essentially the same manner as previously. Choose $\xi = s_{q+1}$ for some $q \in \{1, \ldots, n\}$, where $a < \xi = s_{q+1} < b$. Assume, for the moment, that $q = 2\ell - 1$ is odd. Then we proceed exactly as above since the index associated with the points $\{s_2, \ldots, s_{2k}\}$ is $n = 2k$. That is, the associated measure μ_ξ given by

$$\int_a^b u \, \mathrm{d}\mu_\xi = \sum_{i=1}^{k} a_{2i} u(s_{2i})$$

is a canonical (in fact principal) representation. Thus we obtain in this case

$$(-1)^q \sum_{i=1}^{q} (-1)^i a_i u_{n+1}(s_i) \ge 0,$$

for q odd.

However, if q is even then the index associated with the points $\{s_1, \ldots, s_{2k+1}\}$ can be $n, n+1$ or $n+2$. The associated measure is not canonical when the index is $n+2$. We must therefore exclude this case. But this is exactly the case when both $a < s_1$ and $s_{2k+1} < b$. It therefore follows that when n is even we obtain

$$(-1)^q \sum_{i=1}^{q} (-1)^i a_i u_{n+1}(s_i) \ge 0,$$

for q even, only for those $a \le s_1 < \cdots < s_{2k+1} \le b$ with $s_1 = a$ and/or $s_{2k+1} = b$. This implies that in this case we have that u_{n+1} satisfies Condition M with respect to the n-dimensional T^+-system $\{u_1, \ldots, u_n\}$ for every choice of points $a \le s_1 < \cdots < s_{n+1} \le b$, and all $q \in \{1, \ldots, n\}$, except that if q is even we must restrict ourselves to $a \le s_1 < \cdots < s_{2k+1} \le b$ with $s_1 = a$ and/or $s_{2k+1} = b$. We now apply Proposition 4.2 to eliminate this restriction. □

The same arguments as above prove the following results for the upper bound. The difference between the upper and lower bound is that in the upper bound we demand that $u^{**} \ge u_{n+1}^{\xi+}$ and, in particular, $u^{**}(\xi) = u_{n+1}(\xi)$.

Theorem 4.7. *Assume $U = \operatorname{span}\{u_1, \ldots, u_n\}$ is an n-dimensional T-space in $C[a, b]$, and $\{u_1, \ldots, u_n\}$ is a T^+-system. Let μ be a positive measure relative to U. Assume that for $q \in \{1, \ldots, n\}$ we have that u_{n+1} satisfies Condition M with respect to the $\{u_1, \ldots, u_n\}$ and any set of points $a \le s_1 < \cdots < s_{n+1} \le b$. Assume, in addition, that if $q = n + 1$ and n is odd, then u_{n+1} satisfies Condition M with respect to the $\{u_1, \ldots, u_n\}$ and any set of points $a \le s_1 < \cdots < s_{n+1} \le b$, while if $q = n + 1$ and n is even, then u_{n+1} satisfies Condition M with respect to the $\{u_1, \ldots, u_n\}$ and any set of points satisfying $a = s_1 < \cdots < s_{n+1} \le b$. Then for each $\xi \in (a, b)$*

$$\int_a^{\xi+} u_{n+1}\, d\nu \le \int_a^{\xi+} u_{n+1}\, d\mu_\xi$$

for every $\nu \in V(\mu)$ where μ_ξ is the (unique) canonical representation for μ containing the node ξ.

Theorem 4.8. *Let $\{u_1, \ldots, u_n\}$ be a T^+-system on $[a, b]$. Assume that for a given $u_{n+1} \in C[a, b]$, for each $\xi \in (a, b)$, and for every positive measure μ relative to $U = \operatorname{span}\{u_1, \ldots, u_n\}$ we have that*

$$\int_a^{\xi+} u_{n+1}\, d\nu \le \int_a^{\xi+} u_{n+1}\, d\mu_\xi$$

for every $\nu \in V(\mu)$, where μ_ξ is the unique canonical representation for μ containing the node ξ. Then u_{n+1} satisfies Condition M with respect to the n-dimensional T^+-system $\{u_1, \ldots, u_n\}$ for every choice of points $a \le s_1 < \cdots < s_{n+1} \le b$, and all $q \in \{1, \ldots, n\}$. If $q = n + 1$ and n is odd, then u_{n+1} satisfies Condition M with respect to the $\{u_1, \ldots, u_n\}$ and any set of points $a \le s_1 < \cdots < s_{n+1} \le b$, while if $q = n + 1$ and n is even, then u_{n+1} satisfies Condition M with respect to the $\{u_1, \ldots, u_n\}$ only for the set of points satisfying $a = s_1 < \cdots < s_{n+1} \le b$.

Remark. Note that there is, in Theorems 4.5–4.8, no a priori demand that u_{n+1} be a non-negative function. However from Condition M with $q = 1$ it follows that u_{n+1} is a non-negative function on $[a, b]$. What exactly the other demands of Condition M imply as regards u_{n+1} is unclear.

Remark. In Theorems 4.7 and 4.8 there is the additional demand that does not appear in Theorems 4.5 and 4.6, namely that Condition M must hold with $q = n + 1$ either for all $a \le s_1 < \cdots < s_{n+1} \le b$, or only for $a = s_1 < \cdots < s_{n+1} \le b$. As an example of the difference, with or without this condition with $q = n + 1$, consider $U = \Pi_1 = \operatorname{span}\{1, t\}$ on $[a, b]$. If u_3 is positive and increasing, then it satisfies Condition M with respect to Π_1 for the points $a \le s_1 < s_2 < s_3 \le b$ and for $q = 1$ and $q = 2$. For each $\xi \in (a, b)$, u^* is the linear polynomial satisfying $u^*(\xi) = 0$ and $u^*(a) = u_3(a)$. It is easily shown that there exist u_3 that are positive and increasing, but where the best L^1 one-sided approximation from above from Π_1 to $u_3^{\xi+}$ does not interpolate to $u_3^{\xi+}$ at ξ. The additional conditions of Theorems 4.7 and 4.8, namely that Condition M holds with respect to Π_1 for the points $a = s_1 < s_2 < s_3 \le b$ and $q = 3$, is equivalent to the demand that $(u_3(t) - u_3(a))/(t - a)$ be a nondecreasing function on $[a, b]$.

It is of interest to ask the same questions for the parallel problems of

$$\inf_{\nu \in V(\mu)} \int_{\xi_+}^{b} f \, d\nu$$

and

$$\sup_{\nu \in V(\mu)} \int_{\xi_-}^{b} f \, d\nu,$$

where $\xi \in (a, b)$. The analysis is very much the same. We state the analogue of Condition M as Condition M'.

For $p \in \{1, \ldots, n+1\}$ and $a \le s_1 < \cdots < s_{n+1} \le b$, we define

$$U_p \begin{pmatrix} 1, \ldots, n+1 \\ s_1, \ldots, s_{n+1} \end{pmatrix} = \begin{vmatrix} u_1(s_1) & \cdots & u_1(s_p) & u_1(s_{p+1}) & \cdots & u_1(s_{n+1}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_n(s_1) & \cdots & u_n(s_p) & u_n(s_{p+1}) & \cdots & u_n(s_{n+1}) \\ 0 & \cdots & 0 & u_{n+1}(s_{p+1}) & \cdots & u_{n+1}(s_{n+1}) \end{vmatrix}.$$

That is, the first p entries in the last row are set equal to 0.

Definition 4.3. We will say that the function $u_{n+1} \in C[a, b]$ satisfies *Condition* M' with respect to the n-dimensional T^+-system $\{u_1, \ldots, u_n\}$ and the points $a \le s_1 < \cdots < s_{n+1} \le b$ and $0 \le p \le n$ if

$$(-1)^{n+p} U_p \begin{pmatrix} 1, \ldots, n+1 \\ s_1, \ldots, s_{n+1} \end{pmatrix} \ge 0.$$

There is an inherent symmetry between Condition M and Condition M'. It is that if $u \in U$, then u satisfies Condition M for all $a \le s_1 < \cdots < s_{n+1} \le b$ and all $1 \le q \le n+1$ if and only if it satisfies Condition M' for all $a \le s_1 < \cdots < s_{n+1} \le b$ and all $0 \le p \le n$.

Note that the analogue of Condition K in this setting is:

Definition 4.4. We say that $\{u_1, \ldots, u_n, f\}$ satisfies Condition K' on $[a, b]$ if:

(a′) $\{u_1, \ldots, u_k\}$ is a T^+-system for $k = 1, \ldots, n$.
(b′) f is strictly positive on $[a, b]$.
(c′) $\{f, u_1, \ldots, u_k\}$ is a T^+-system for $k = 1, \ldots, n$.

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References

[1] J. Bustamante, J.M. Quesada, R. Martínez-Cruz, One-sided L_1 approximation to the Heaviside and sign functions, J. Approx. Theory 164 (2012) 791–802.
[2] K. Isii, The extrema of probability determined by generalized moments (I) bounded random variables, Ann. Inst. Statist. Math. 12 (1960) 119–133.
[3] S. Karlin, W.J. Studden, Tchebycheff Systems: With Applications in Analysis and Statistics, Interscience, New York, 1966.
[4] J.H.B. Kemperman, The general moment problem, a geometric approach, Ann. Math. Statist. 39 (1968) 93–122.

[5] J.H.B. Kemperman, On the role of duality in the theory of moments, in: Semi-Infinite Programming and Applications, Austin, TX, 1981, in: Lecture Notes in Econ. and Math. Systems, vol. 215, Springer, Berlin, New York, 1983, pp. 63–92.

[6] J.H.B. Kemperman, Geometry of the moment problem, in: Moments in Mathematics, San Antonio, TX, 1987, in: Proc. Sympos. Appl. Math., vol. 37, AMS, Providence, 1987, pp. 16–53.

[7] M.G. Kreĭn, The ideas of P.L. Čebyšev and A.A. Markov in the theory of limiting values of integrals and their further development, Uspekhi Mat. Nauk 6 (1951) 3–120. in English transl., AMS Translations 12 (2) (1959) 1–121.

[8] M.G. Kreĭn, A.A. Nudel'man, The Markov Moment Problem and Extremal Problems, in: Transl. Math. Monographs, vol. 50, AMS, Providence, 1977, Originally published in Russian in 1973.

[9] A. Markoff, Démonstration de certaines inégalités de M. Tchébychef, Math. Ann. 24 (1884) 172–180.

[10] C.A. Micchelli, A. Pinkus, Moment theory for weak Chebyshev systems with applications to monosplines, quadrature formulae and best one-sided L^1 approximation by spline functions with fixed knots, SIAM J. Math. Anal. 8 (1977) 206–230.

[11] A.A. Nudel'man, On functions whose integrals satisfy Chebyshev–Markov inequalities, Dopov. Akad. Nauk Ukr. RSR 1965 (1965) 691–695.

[12] A. Pinkus, On L^1-Approximation, in: Cambridge Tracts in Mathematics, vol. 93, Cambridge University Press, 1989.

[13] A. Pinkus, Totally Positive Matrices, in: Cambridge Tracts in Mathematics, vol. 181, Cambridge University Press, 2010.

[14] P.L. Tchébychef, Sur les valeurs limites des intégrales, J. Math. Pures Appl. (2) 19 (1874) 19–34. Also in Oeuvres de P.L. Tchébychef, Volume 2, 183–185, Chelsea, New York, 1961.

[15] R.A. Zalik, On transforming a Tchebycheff system into a complete Tchebycheff system, J. Approx. Theory 20 (1977) 220–222.