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ONE-SIDED L^1-APPROXIMATION

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ABSTRACT. Let U_n be an n-dimensional subspace of $C[0, 1]$. We prove that if $n \geq 2$, and U_n contains a function which is strictly positive on $(0, 1)$, then there exists an $f \in C[0, 1]$ which has more than one best one-sided L^1-approximation from U_n. We also characterize those U_n with the property that each $f \in C[0, 1]$ has a unique best one-sided $L^1(w)$-approximation from U_n with respect to every strictly positive continuous weight function w.

1. **Introduction**. In this paper we consider the problem of uniqueness of best one-sided L^1-approximations to continuous functions from a finite dimensional subspace. We prove two main results. To explain these results, some notation is needed.

U_n will denote a fixed n-dimensional subspace of $C[0, 1]$. A_n will be the set of $f \in C[0, 1]$ for which there exists a $u \in U_n$ satisfying $u(x) \leq f(x)$ for all $x \in [0, 1]$. Thus

$$A_n = \{f : f \in C[0, 1],\ \exists u \in U_n,\ u \leq f\}$$

If U_n contains a strictly positive function then $A_n = C[0, 1]$.

It is easily seen that for each $f \in A_n$ there exists a $u^* \in U_n$, $u^* \leq f$, for which

$$\|f - u^*\|_1 = \min\{\|f - u\|_1 : u \in U_n,\ u \leq f\} \tag{1.1}$$

where $\|f\|_1 = \int_0^1 |f(x)|\,dx$. Such a u^* we call a *best one-sided L^1-approximation to f from U_n*. Since we consider $u \leq f$, the minimum problem in (1.1) is seen to be equivalent to

$$\max\left\{\int_0^1 u(x)\,dx : u \in U_n,\ u \leq f\right\}. \tag{1.2}$$

We say that U_n is a *unicity space* for L^1 if for every $f \in A_n$ there exists a unique best one-sided L^1-approximation to f from U_n.

DeVore [1] proved that if U_n, $n \geq 2$, is a Tchebycheff space then U_n is not a unicity space for L^1. In Pinkus [3], this negative result was also shown to be valid if U_n, $n \geq 2$, is a subspace of splines with fixed knots. (Neither of these two results represented the main contents of these papers). Strauss [5] gave a series of necessary and

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sufficient conditions for U_n to be a unicity space for L^1. One of these equivalent conditions is the following:

THEOREM A (Strauss [5]). *U_n is a unicity space for L^1 if and only if for each $u \in U_n \backslash \{0\}$ there exists a $v \in U_n$ for which*

(1) $v(x) \leq |u(x)|$, *all* x,

(2) $\int_0^1 v(x)\,\mathrm{d}x > 0.$

On the basis of the above theorem, Strauss was able to prove that if U_n, $n \geq 2$, is a weak Tchebycheff space containing a strictly positive function, then U_n is not a unicity space for L^1. Our first result shows that Tchebycheff and weak Tchebycheff spaces are irrelevant in the above result. We prove

THEOREM 1. *If there exists a $u \in U_n$, $n \geq 2$, such that $u(x) > 0$ for all $x \in (0, 1)$, then U_n is not a unicity space for L^1.*

Note that for $n = 1$ it easily follows from (1.2) (or from Theorem A) that U_1 is a unicity space if and only if $\int_0^1 u(x)\,\mathrm{d}x \neq 0$ for $u \in U_1/\{0\}$.

As may be seen from Theorem A, the necessary and sufficient conditions given therein are generally very difficult to check for $n \geq 2$.

Let W denote the set of all continuous strictly positive functions on $[0, 1]$. For each $w \in W$, set

$$\|f\|_w = \int_0^1 |f(x)|w(x)\,\mathrm{d}x.$$

Paraphrasing the previous definitions we say that u^* is a best one-sided $L^1(w)$ approximation to $f \in A_n$ from U_n if $u^* \in U_n$, $u^* \leq f$, and

$$\|f - u^*\|_w = \min\{\|f - u\|_w : u \in U_n,\ u \leq f\}. \tag{1.3}$$

This problem is equivalent to

$$\max\left\{\int_0^1 u(x)w(x)\,\mathrm{d}x : u \in U_n,\ u \leq f\right\}. \tag{1.4}$$

We say that U_n is a unicity space for $L^1(w)$ if for every $f \in A_n$ (A_n does not depend on w) there exists a unique best one-sided $L^1(w)$ approximation to f from U_n. In general a best one-sided $L^1(w)$ approximation to f from U_n is w dependent.

It is easily seen that Theorems A and 1, and the examples previously mentioned, are valid in the case of best one-sided $L^1(w)$ approximation for any $w \in W$, fixed. The only change is that in Theorem A we must replace (2) by

$$(2')\ \int_0^1 v(x)w(x)\,\mathrm{d}x > 0.$$

Thus the conditions as given in Theorem A are w dependent. This is also the case in the problem of uniqueness for the two-sided $L^1(w)$ approximation problem (see e.g. Kroo [2], Pinkus [4]), while this is not the case for the corresponding one and two-sided approximation problem in the $L^\infty(w)$ norm, $\|f\|_{L^\infty(w)} = \max\{|f(x)|w(x) : 0 \le x \le 1\}$. (The Haar condition is w independent). Thus aside from the negative result of Theorem 1, it is rather difficult to ascertain whether a given U_n is a unicity space for $L^1(w)$ for a given $w \in W$. For one particular class of U_n, however, this question is readily answered. For $u \in U_n$, set $\operatorname{supp}(u) = \{x : u(x) \neq 0\}$. If U_n has a basis of functions $u_1, \dots, u_n$ for which $\operatorname{supp}(u_i) \cap \operatorname{sup}(u_j) = \Phi$, all $i \neq j$, then U_n is a unicity space for $L^1(w)$ if and only if $\int_0^1 u_i(x) w(x)\,dx \neq 0$, $i = 1, \dots, n$. This readily follows from (1.4). Here our n-dimensional problem has reduced to n 1-dimensional problems.

Since the conditions of Theorem A are difficult to check we might ask for conditions on U_n implying that U_n is a unicity space for $L^1(w)$ for *all* $w \in W$. Our second result deals with this problem.

THEOREM 2. *U_n is a unicity space for $L^1(w)$ for every $w \in W$ if and only if U_n has a basis of functions $u_1, \dots, u_n$ for which*

(1) $u_i(x) \ge 0$, $i = 1, \dots, n$, *all* x

(2) supp $(u_i) \cap$ supp $(u_j) = \Phi$, *all* $i \neq j$.

2. **Proof of Theorem 1**. Let $u_1 \in U_n$, $n \ge 2$, be such that $u_1(x) > 0$ for all $x \in (0, 1)$. Assume without loss of generality that $\int_0^1 u_1(x)\,dx = 1$. We may construct a basis for U_n, $\{u_1, u_2, \dots, u_n\}$ such that $\int_0^1 u_i(x)\,dx = 0$, $i = 2, \dots, n$. Let $V = \operatorname{span}\{u_2, \dots, u_n\}$.

For each $v \in V \backslash \{0\}$, set

$$J(v) = \{x : v(x) \le 0\}$$

and let $|J(v)|$ denote the Lebesgue measure of $J(v)$. Note that $J(v) = J(cv)$ for all $c > 0$. Since $\bar{V} = \{v : v \in V, \|v\|_\infty = 1\}$ is compact and equicontinuous, there exists a $v^* \in \bar{V}$ for which $|J(v^*)| \ge |J(v)|$ for all $v \in \bar{V} \subseteq U_n$.

Let $v_+^*(x) = \max\{v^*(x), 0\}$. Note that $v_+^* \not\equiv 0$ since $\int_0^1 v^*(x)\,dx = 0$. We claim that if $u \in U_n$, $u \le v_+^*$, then $\int_0^1 u(x)\,dx \le 0$. This will prove the theorem (from (1.2)) since $v_+^* \in C[0, 1], 0, v^* \le v_+^*$, and $0, v^* \in V$.

Every $u \in U_n$ is of the form $u = bu_1 + v$, where $b \in R$, $v \in V$. Furthermore, $\int_0^1 u(x)\,dx = b$. Assume that there exists a $u = bu_1 + v$, as above, with $b > 0$ and $u \le v_+^*$. For $x \in J(v^*)$, $v_+^*(x) = 0$ and $v(x) \le -bu_1(x)$. Since $u_1(x) > 0$ for all $x \in (0, 1)$, it follows that $v \not\equiv 0$ and $J(v) \supseteq J(v^*)$. Because $\int_0^1 v^*(x)\,dx = 0$, there exists an $x^* \in (0, 1)$ for which $v^*(x^*) = 0$, and v^* takes strictly positive values in every neighborhood of x^*. However $v(x^*) \le -bu_1(x^*) < 0$ and thus $J(v)$ contains a neighborhood of x^*, implying that $|J(v)| > |J(v^*)|$. This contradiction proves the theorem. □

REMARK. Both this proof and the theorem fail if we allow u_1 to vanish in $(0, 1)$. It is readily checked that $U_2 = \operatorname{span}\{(x - 1/2)^2, (x - 1/2)_+\}$ is a unicity space.

REMARK. The proofs of the negative results considered in the introduction, due to DeVore, Pinkus, and Strauss, all used quadrature formulae. If there exist m distinct points $\{x_i\}_{i=1}^m$ in $[0, 1]$, and strictly positive numbers $\{\lambda_i\}_{i=1}^m$ with $1 \leq m \leq n - 1$, for which

$$\int_0^1 u(x)\,dx = \sum_{i=1}^m \lambda_i u(x_i)$$

for all $u \in U_n$, then non-uniqueness may be proven as follows. Let $u^* \in U_n \backslash \{0\}$ be such that $u^*(x_i) = 0$, $i = 1, \ldots, m$. Such a u^* exists since $m \leq n - 1$. Furthermore from the quadrature formula

$$\int_0^1 u^*(x)\,dx = 0.$$

As above, assume $u \leq u_+^*$, $u \in U_n$. Then

$$\int_0^1 u(x)\,dx = \sum_{i=1}^m \lambda_i u(x_i) \leq \sum_{i=1}^m \lambda_i u_i^*(x_i) = 0.$$

This implies the non-uniqueness. As was pointed out to us by G. Jameson, if we assume that there exists a $u \in U_n$ for which $u(x) > 0$ for all $x \in [0, 1]$, then by convexity-type arguments there exists a quadrature formula of the above form with m points, $1 \leq m \leq n - 1$. We may therefore apply the above quadrature formula argument if U_n, $n \geq 2$, contains a strictly positive function on $[0, 1]$.

3. **Proof of Theorem 2**. Our proof of Theorem 2 very much depends upon the following proposition which was proved in Pinkus [4]. The proof given therein is a functional analytic proof. We here reprove the result by an "elementary" and more constructive method.

PROPOSITION 3.1. *Let V_m be an m-dimensional ($m < \infty$) subspace of $C[0, 1]$ with the property that there does not exist a $v \in V_m \backslash \{0\}$ satisfying $v(x) \geq 0$ for all $x \in [0, 1]$. Then there exists a $w \in W$ for which*

$$\int_0^1 v(x)w(x)\,dx = 0$$

for all $v \in V_m$.

We prove the proposition via a series of lemmas.

LEMMA 3.2. *For V_m as above, there exist k points ($k < \infty$), $\{x_i\}_{i=1}^k$ such that if $v \in V_m$ and $v(x_i) \geq 0, i = 1, \ldots, k$, then $v \equiv 0$.*

PROOF. Follows from a compactness argument. □

Let $V_m = \operatorname{span}\{v_1, \ldots, v_m\}$, and set $\mathbf{v}_i = (v_i(x_1), \ldots, v_i(x_k)) \in R^k$, $i = 1, \ldots, m$. $(\cdot, \cdot)$ will denote the usual vector inner product.

LEMMA 3.3. *Let V_m be as in the statement of the proposition, and let $J \subseteq \{1, \ldots, m\}$. Then there exists a $\boldsymbol{w}^J \in R^k$, $(\boldsymbol{w}^J)_j > 0$, $j = 1, \ldots, k$, such that $(\boldsymbol{v}_i, \boldsymbol{w}^J) > 0$ for $i \in J$, and $(\boldsymbol{v}_i, \boldsymbol{w}^J) < 0$, for $i \notin J$.*

PROOF. Let V denote the $m \times k$ matrix $V = (v_i(x_j))_{i=1, j=1}^{m \;\; k}$. From Lemma 3.2 there does not exist a vector $\boldsymbol{a} \in R^m \backslash \{0\}$ for which $\boldsymbol{a}V \geq \boldsymbol{0}$. This also implies that rank $V = m \leq k$.

Let $\boldsymbol{e}^i$ denote the ith unit vector in R^k. Set $A = \{\boldsymbol{a}V : \boldsymbol{a} \in R^m\}$ and $B = \{\sum_{i=1}^k \lambda_i \boldsymbol{e}^i : \lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1\}$. The sets A and B are closed convex subsets of R^k. Furthermore, by assumption, $A \cap B = \emptyset$. Thus there exists a strictly separating hyperplane, i.e., a $\boldsymbol{c} \in R^k$, $c_0 \in R$ for which

$$(3.1) \qquad (\boldsymbol{a}V, \boldsymbol{c}) < c_0, \quad \text{all} \quad \boldsymbol{a} \in R^m$$

$$(3.2) \qquad (\boldsymbol{x}, \boldsymbol{c}) > c_0, \quad \text{all} \quad \boldsymbol{x} \in B.$$

From (3.1) it follows that $V\boldsymbol{c} = \boldsymbol{0}$, and $c_0 > 0$. Thus from (3.2) we also obtain $(\boldsymbol{c})_j > 0, j = 1, \ldots, k$.

For given $J \subseteq \{1, \ldots, m\}$ there exists, since rank $V = m \leq k$, a vector $\boldsymbol{b}^J \in R^k$ for which $(\boldsymbol{v}_i, \boldsymbol{b}^J) > 0$ for $i \in J$ and $(\boldsymbol{v}_i, \boldsymbol{b}^J) < 0$ for $i \notin J$. Set $\boldsymbol{w}^J = \boldsymbol{c} + \epsilon \boldsymbol{b}^J$, where $\epsilon > 0$ is sufficiently small so that $(\boldsymbol{c} + \epsilon \boldsymbol{b}^J)_j > 0$, $j = 1, \ldots, k$. This proves the lemma. $\square$

LEMMA 3.4. *Let V_m be as in the statement of the proposition. Let $J \subseteq \{1, \ldots, m\}$. Then there exists a $w^J \in W$ for which*

$$\int_0^1 v_i(x) w^J(x)\, \mathrm{d}x = \begin{cases} >0, & i \in J \\ <0, & i \notin J \end{cases}.$$

PROOF. Simply smooth the atomic measures corresponding to the vectors $\boldsymbol{w}^J$ of Lemma 3.3. $\square$

PROOF OF PROPOSITION 3.1. For each $J \subseteq \{1, \ldots, m\}$ let $w^J \in W$ be as given in Lemma 3.4. Set $c_i^J = \int_0^1 v_i(x) w^J(x)\, \mathrm{d}x$, $i = 1, \ldots, m$, and $\boldsymbol{c}^J = (c_1^J, \ldots, c_m^J)$. Let C denote the convex hull of the $\{\boldsymbol{c}^J : J \subseteq \{1, \ldots, m\}\}$. Each quadrant of R^m contains a vector of C in its interior. Therefore $\boldsymbol{0} \in C$. Thus there exists a convex combination w of the w^J for which $\int_0^1 v_i(x) w(x)\, \mathrm{d}x = 0$, $i = 1, \ldots, m$. By construction $w \in W$. $\square$

REMARK. This proof actually shows that given any dense linear subset of $C[0, 1]$ there exists a $w \in W$, which is also in this dense linear subset, and which satisfies the conditions of the proposition. Thus, for example, w may be taken to be a polynomial.

The following proposition is used in the proof of Theorem 2.

PROPOSITION 3.5. *Assume that U_n is a unicity space for $L^1(w)$ for every $w \in W$. Given any $n - 1$ distinct points $\{x_i\}_{i=1}^{n-1}$, there exists a non-negative $u \in U_n \backslash \{0\}$ for which $u(x_i) = 0$, $i = 1, \ldots, n - 1$.*

PROOF. The proof is by induction on the number of points. We prove that given any k distinct points $\{x_i\}_{i=1}^k$, $0 \le k \le n-1$, there exists a non-negative $u \in U_n \backslash \{0\}$ for which $u(x_i) = 0$, $i = 1, \ldots, k$. This statement for $k = 0$ simply says that U_n contains a non-negative, non-trivial function. If this is not the case, then by Proposition 3.1 there exists a $w \in W$ for which $\int_0^1 u(x)w(x)\,dx = 0$ for all $u \in U_n$. From the form (1.4), this immediately implies that U_n is not a unicity space, contradicting our hypothesis. Thus U_n contains a non-negative non-trivial function.

We now use induction. Assume the result is valid for $k-1$, $0 \le k \le n-1$. Let $\{x_i\}_{i=1}^k$ by any k distinct points and assume that there does not exist a non-negative $u \in U_n \backslash \{0\}$ for which $u(x_i) = 0$, $i = 1, \ldots, k$. By the induction hypothesis there exist non-negative $u_1, \ldots, u_k \in U_n \backslash \{0\}$ which satisfy $u_i(x_j) = 0$, $i \ne j$; $i, j = 1, \ldots, k$. By assumption $u_i(x_i) \ne 0$. Thus we can assume that $u_i(x_j) = \delta_{ij}$, $i, j = 1, \ldots, k$.

Set

$$M = \{u : u \in U_n,\ u(x_i) = 0,\ i = 1, \ldots, k\}.$$

M is a subspace of U_n, and since $k \le n-1$, $\dim M \ge n-k > 0$. Furthermore the $u_1, \ldots, u_k$ are linearly independent and not in M. Thus $\dim M = n-k$. By assumption M does not contain a non-negative non-trivial function. From Proposition 3.1 there exists a $w \in W$ for which $\int_0^1 u(x)w(x)\,dx = 0$ for all $u \in M$. Let $u^* \in M \backslash \{0\}$, and set $u_+^*(x) = \max\{u^*(x), 0\}$. Then $u_+^* \in C[0,1]$ and $u_+^* \not\equiv 0$. We claim that if $u \in U_n$ satisfies $u \le u_+^*$, then $\int_0^1 u(x)w(x)\,dx \le 0$. If this is true, then 0 and u^* are two one-sided best $L^1(w)$ approximations to u_+^*, contradicting the unicity assumption of the proposition.

Let $u \in U_n$, $u \le u_+^*$. Then $u = \bar{u} + \sum_{i=1}^k u(x_i)u_i$, where $\bar{u} \in M$. Since $u_+^*(x_i) = 0$, it follows that $u(x_i) \le 0$, $i = 1, \ldots, k$. Thus

$$\int_0^1 u(x)w(x)\,dx = \sum_{i=1}^k u(x_i) \int_0^1 u_i(x)w(x)\,dx \le 0. \qquad \square$$

PROOF OF THEOREM 2. If U_n has a basis of functions which satisfy conditions (1) and (2), then it easily follows that U_n is a unicity space for $L^1(w)$ for every $w \in W$. We therefore assume that U_n is a unicity space for $L^1(w)$ for every $w \in W$ and construct a basis of functions which satisfy (1) and (2).

Let $y_1, \ldots, y_n$ be any n distinct points for which $u(y_i) = 0$, $i = 1, \ldots, n$, $u \in U_n$, implies $u \equiv 0$. By Proposition 3.5 there exist non-negative $u_1, \ldots, u_n \in U_n \backslash \{0\}$ satisfying $u_i(y_j) = 0$, $i \ne j$; $i, j = 1, \ldots, n$. If $u_i(y_i) = 0$, then $u_i \equiv 0$. We may therefore assume that $u_i(y_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. The $\{u_i\}_{i=1}^n$ form a basis of functions for U_n which satisfy (1). We claim that they also satisfy (2).

Assume that there exists a $y \in [0,1]$ and $j, k \in \{1, \ldots, n\}$, $j \ne k$, such that $u_j(y)$, $u_k(y) > 0$. Obviously $y \notin \{y_1, \ldots, y_n\}$. From Proposition 3.5 there exists a non-negative $u^* \in U_n \backslash \{0\}$ for which $u^*(y) = 0$ and $u^*(y_i) = 0$, $i = 1, \ldots, n$; $i \ne j, k$. Now $u^* = \sum_{i=1}^n u^*(y_i)u_i$. Since $u^*(y_i) = 0$, $i = 1, \ldots, n$; $i \ne j, k$, we have $u^* = u^*(y_j)u_j + u^*(y_k)u_k$. However $u^* \ge 0$, $u_j(y)$, $u_k(y) > 0$, and

$0 = u^*(y) = u^*(y_j)u_j(y) + u^*(y_k)u_k(y)$. Thus $u^*(y_j) = u^*(y_k) = 0$. Therefore $u^*(y_i) = 0$, $i = 1, \ldots, n$ which implies that $u^* = 0$. This contradiction proves the theorem. $\square$

REMARK. Theorem 2 is also valid if U_n is an n-dimensional subspace of $C(B)$, where B is any compact Hausdorff space.

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