On a Recovery Problem

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Abstract. This paper concerns itself with the recovery of the coefficients, shifts and, where applicable, dilates of a given form

$$f(\mathbf{x}) = \sum_{j=1}^{m} c_j \, G(\mathbf{x} - \mathbf{t}_j)\,, \text{ or } f(\mathbf{x}) = \sum_{j=1}^{m} c_j \, g(\mathbf{a}_j \cdot \mathbf{x} - b_j)\,, \quad \mathbf{x} \in \mathbb{R}^n,$$

where f, G and g are known. That is, we provide a method that identifies the quantities c_j, $\mathbf{t}_j$, $\mathbf{a}_j$ and b_j. In some cases we can even find G given only f and knowing that f is of the above form.

§1. Introduction

The theme of this paper is the recovery of linear combinations of shifts and dilates of a prescribed function that generate a given function f. Sometimes we are also able to identify the function which is dilated and shifted. In the first case we assume that we are given a function $G : \mathbb{R}^n \to \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}$. The latter we know to be of the form

$$(1) \qquad f(\mathbf{x}) = \sum_{j=1}^{m} c_j \, G(\mathbf{x} - \mathbf{t}_j), \qquad \mathbf{x} \in \mathbb{R}^n \;,$$

for some unknown coefficient values $\{c_j\}_{j=1}^m \subset \mathbb{R}$ and shifts $\{\mathbf{t}_j\}_{j=1}^m \subset \mathbb{R}^n$. It is assumed that we do not know m but we know that $m \le M$ for some given $M \in \mathbb{N}$. Our problem is to identify or recover the coefficients and the shifts. We are able to determine these quantities for functions G which admit Fourier transforms. We are also able to recover G from f in some cases when G is essentially radially symmetric.

In the second problem we recover the $\{\mathbf{a}_j\}_{j=1}^m \subset \mathbb{R}^n$, $\{c_j\}_{j=1}^m \subset \mathbb{R}$ and $\{b_j\}_{j=1}^m \subset \mathbb{R}$, and know f to be of the form

$$(2) \qquad f(\mathbf{x}) = \sum_{j=1}^{m} c_j \, g(\mathbf{a}_j \cdot \mathbf{x} - b_j)\,, \qquad \mathbf{x} \in \mathbb{R}^n \;.$$

Here "$\cdot$" denotes the standard inner product and $g : \mathbb{R} \to \mathbb{R}$ again satisfies some additional constraints.

We may also consider this problem from another perspective. We know f and G, and wish to determine whether f and G are related as in (1). That is, we would like a method for deciding if f is a linear combination of M shifts and dilates of G. What we propose is to apply the method which we detail in this paper, and then check whether the resulting function agrees with f.

There are many applications of functions of the form (1) or (2) as approximation tools. One occurs when radial basis functions are used for the approximation of multivariate functions. In those approximation schemes, G is a rotationally invariant function and has therefore the specific form $G = \phi(\|\cdot\|)$, where $\|\cdot\|$ denotes the Euclidean norm. The function ϕ is called a radial basis function, and it is chosen in advance. Calling ϕ a radial basis function is a slight abuse of notation, because in fact ϕ is the *radial part* of the basis function $G(\cdot - \mathbf{t}_j)$ centered at $\mathbf{t}_j$, but we bow to convention in this matter.

Useful choices for ϕ are $\phi(r) = r$, $\phi(r) = \sqrt{r^2 + c^2}$, c a positive parameter, or $\phi(r) = \exp(-c^2 r^2)$, c again a parameter. Much work has been devoted to studying these approximation tools, because they were found experimentally to yield highly promising approximation results. As it turns out, radial symmetry is not always required to obtain good approximation results, and therefore some of the most recent analysis no longer requires radial symmetry, but studies the approximational efficacy of approximations of the form (1) where G is a general, n-variate function with global support as the radial basis functions above have. A recent review of these methods and their properties can be found in [3]. Another application, especially of (2), may be met in the theory of artificial neural networks. In that case g is often a "sigmoidal function" [2], and one views (2) as a single hidden layer feedforward network, the $\mathbf{a}_j$, c_j and b_j being various network parameters.

Our approach to (1) is simple conceptually but perhaps computationally expensive. We require G to be absolutely integrable, but often in fact absolutely integrable, because we must evaluate its Fourier transform or even derivatives thereof at points. We consider the Fourier transform of f in (1). It is the product of $\hat{G}$ times an exponential sum, i.e.,

$$\hat{f}(\boldsymbol{\omega}) = \sum_{j=1}^{m} c_j \, e^{-i\mathbf{t}_j \cdot \boldsymbol{\omega}} \, \hat{G}(\boldsymbol{\omega}), \quad \boldsymbol{\omega} \in I\!R^n \, .$$

In the simple set-up, where G is assumed to be prescribed, $\hat{G}$ is known and we may therefore restrict ourselves to considering an exponential sum of the form

$$h(\boldsymbol{\omega}) = \sum_{j=1}^{m} c_j \, e^{-i\mathbf{t}_j \cdot \boldsymbol{\omega}} \, , \quad \boldsymbol{\omega} \in I\!R^n \, , \tag{3}$$

(at least on the interior of the set where $\hat{G}$ does not vanish) whose unknown parameters are $\{c_j\}_{j=1}^m$ and $\{\mathbf{t}_j\}_{j=1}^m$.

Our next section is devoted to studying just this problem whose univariate form is similar to a problem discussed by Draux [4, p. 585–595]. We call our problem an Hermite interpolation problem using exponential sums, and that is the subject of Section 2. In Section 3 we give answers to our principal questions outlined at the onset of this introduction, namely the recovery of (1) in Theorem 3 and of G if it is (asymptotically for large argument) radial (Theorem 4). In Section 4 the recovery of shifts and dilates in (2) is considered. Its main result is Theorem 6.

It should be immediately noted that without some a priori conditions on G (or g) neither problem (1) nor (2) is tractable. For example, in problem (1) if $n = 1$ and $G(x) = e^x$, then the knowledge that f is of the form

$$f(x) = \sum_{j=1}^{m} c_j \, G(x - t_j) = \Big(\sum_{j=1}^{m} c_j \, e^{-t_j} \Big) \, e^x$$

cannot in any way determine the $\{c_j\}_{j=1}^m$ and $\{t_j\}_{j=1}^m$. A similar example can be constructed for problem (2).

§2. Hermite Interpolation by Exponential and other Sums

Our aim in this section is to develop a method of identifying the c_j and $\mathbf{t}_j$ in (1) if we know f and G. To this end, suppose that G (and therefore f because m is finite) is integrable, and that we can determine $\hat{f}$ and $\hat{G}$. Since

$$\hat{f}(\boldsymbol{\omega}) = \Big(\sum_{j=1}^{m} c_j \, e^{-i\,\mathbf{t}_j \cdot \boldsymbol{\omega}} \Big) \, \hat{G}(\boldsymbol{\omega}) \,, \quad \boldsymbol{\omega} \in I\!R^n,$$

it follows that $\hat{f}/\hat{G}$ (on the set where $\hat{G} \neq 0$) is an exponential sum of the form

$$\frac{\hat{f}(\boldsymbol{\omega})}{\hat{G}(\boldsymbol{\omega})} = h(\boldsymbol{\omega}) = \sum_{j=1}^{m} c_j \, e^{-i\,\mathbf{t}_j \cdot \boldsymbol{\omega}} \;, \quad \boldsymbol{\omega} \in I\!R^n \,.$$

As such it is as differentiable as is necessary in a neighbourhood of a point $\boldsymbol{\omega}_0 \in I\!R^n$. Since we can shift $\hat{G}$ by $\boldsymbol{\omega}_0$, and absorb an exponential factor coming through the shift into G, there is no loss in generality if we assume $\boldsymbol{\omega}_0 = 0$. We assert that the c_j and $\mathbf{t}_j$ can be uniquely determined from the derivatives of h, up to some fixed order, at the origin. That is, we will prove that given at least an upper bound M on m, there is essentially a unique exponential sum of the form (3) which satisfies

$$\text{(4)} \qquad \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \boldsymbol{\omega}^{\boldsymbol{\alpha}}} \Big(h(\boldsymbol{\omega}) - \frac{\hat{f}(\boldsymbol{\omega})}{\hat{G}(\boldsymbol{\omega})} \Big)_{\boldsymbol{\omega}=\mathbf{0}} = 0, \qquad |\boldsymbol{\alpha}| \le 2M - 1,$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in Z\!\!Z_+^n$, $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$, and

$$\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \boldsymbol{\omega}^{\boldsymbol{\alpha}}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \omega_1^{\alpha_1} \cdots \partial \omega_n^{\alpha_n}} \,.$$

We will later in this paper show how this approach allows us to deal with sums of the form

$$\text{(5)} \qquad f(\mathbf{x}) = \sum_{j=1}^{m} c_j \, g(\mathbf{a}_j \cdot \mathbf{x}), \quad \mathbf{x} \in I\!R^n \,,$$

where the g which replaces the exponential is known in advance and, like the exponential, satisfies $g^{(\ell)}(0) \neq 0$ for all $\ell = 0, 1, \ldots, 2m-1$. We shall present a method to construct the $\mathbf{a}_j$ and c_j in a unique way to give (5).

Before dealing with the general multivariate problem we give a result for the univariate ($n = 1$) case. For this purpose, let E_M be the set of exponential polynomials of degree at most M. That is,

$$E_M = \left\{ p \,:\, p(x) = \sum_{k=1}^{m} q_k(x)\, e^{b_k\, x},\ b_k \in \mathbb{C},\ q_k \in P_{M-1},\ \sum_{k=1}^{m} (1 + \deg q_k) \le M \right\},$$

where P_{M-1} is the space of polynomials with complex coefficients of degree at most $M-1$. In other words, E_M is the set of all solutions of linear constant coefficient ordinary differential equations of order at most M ([1, p. 169]).

Assume now that we are given values $\{d_j\}_{j=0}^{2M-1}$ and we wish to construct a $p \in E_M$ satisfying

$$p^{(j)}(0) = d_j, \quad j = 0, 1, \ldots, 2M-1\,. \tag{6}$$

This cannot be done for every arbitrary choice of $\{d_j\}_{j=0}^{2M-1}$. As an example, take $M = 1$. Thus

$$E_1 = \{ a\, e^{bx} \mid a, b \in \mathbb{C} \}\,.$$

Here $d_0 = 0$ and $d_1 \neq 0$ lead to a contradiction. There does not exist a $p \in E_1$ satisfying $p(0) = 0$, $p'(0) \neq 0$.

The general condition for existence and uniqueness of a solution is the following. Let D be the $M \times (M+1)$ Hankel matrix

$$D = (d_{i+j})_{i=0}^{M-1}\,{}_{j=0}^{M}\,.$$

We assume that if rank $D = m$, then the submatrix

$$\widetilde{D} = (d_{i+j})_{i,j=0}^{m-1} \tag{7}$$

is non-singular. We call this condition the "D **rank condition**". We can now state and prove (in a constructive way) the following theorem.

Theorem 1. *Suppose that the $\{d_j\}_{j=0}^{2M-1}$ satisfy the D rank condition. Then there is a unique $p \in E_M$ such that (6) holds. Furthermore, if $m =$ rank D, then $p \in E_m \subseteq E_M$.*

Proof: Let $m =$ rank D and

$$B_x = \begin{pmatrix} d_0 & d_1 & \cdots & d_{m-1} & d_m \\ d_1 & d_2 & \cdots & d_m & d_{m+1} \\ \vdots & \vdots & & \vdots & \vdots \\ d_{m-1} & d_m & \cdots & d_{2m-2} & d_{2m-1} \\ 1 & x & \cdots & x^{m-1} & x^m \end{pmatrix}.$$

As a function of x, $\det B_x$ is a polynomial of degree at most m. It follows from the rank D condition that $\deg\det B_x = m$. Let $b_1, b_2, \ldots, b_r$ be its zeros with multiplicities $\mu_1, \mu_2, \ldots, \mu_r$, respectively. Therefore we have $\sum_{j=1}^{r} \mu_j = m$. Set

$$p(x) = \sum_{k=1}^{r} q_k(x)\, e^{b_k\, x} \ ,$$

where the $q_k \in P_{\mu_k - 1}$ are chosen so that (6) holds for j restricted to $0, 1, \ldots, m-1$. This linear problem always has a unique solution.

We claim that the remaining conditions of (6) hold as well. We will prove this only in the case when all of $\det B_x$'s zeros are simple and distinct. The idea of the proof is the same in the general case.

Let

$$D_j = (d_j, d_{j+1}, d_{j+2}, \ldots, d_{j+m}), \quad j = 0, 1, \ldots, 2M-1-m \ .$$

These $2M - m$ vectors span a vector space, call it V, in $\mathbb{R}^{m+1}$. By the rank D condition, the $D_0, D_1, \ldots, D_{m-1}$ are linearly independent, and the $D_m, D_{m+1}, \ldots, D_M$ must each be a linear combination of the $D_0, D_1, \ldots, D_{m-1}$. It follows from an induction argument, considering the first m columns and the last column (which contains D_j, $j \geq M+1$) of the matrix D, and using the rows $j - M + 1, \ldots, j - M + 1 + m$, that D_j, $j \geq M + 1$, is also a linear combination of the $D_0, D_1, \ldots, D_{m-1}$. In particular, $\dim V = m$. Since

$$\det B_{b_k} = 0, \quad k = 1, 2, \ldots, m\, ,$$

it is true that the vectors

$$\hat{B}_k = (1, b_k, b_k^2, \ldots, b_k^m) \in V, \quad k = 1, \ldots, m\ .$$

By a Vandermonde determinant argument, the vectors $\{\hat{B}_k\}_{k=1}^m$ are linearly independent and therefore also span V. Furthermore the vectors obtained by considering their first m components are also linearly independent.

Recall that $\mu_1 = \mu_2 = \cdots = \mu_r = 1$. For any p of the form

$$p(x) = \sum_{k=1}^{m} q_k\, e^{b_k\, x} \tag{8}$$

we have

$$p^{(j)}(0) = \sum_{k=1}^{m} q_k\, b_k^j\, , \quad j = 0, 1, \ldots, 2M-1\ .$$

The q_k were chosen so that (6) holds for $j = 0, 1, \ldots, m-1$. We use an induction argument to show that this remains true for all applicable j. Suppose we have (6) for $j = 0, 1, \ldots \ell + m - 1$, $\ell \geq 0$. Recall that $D_\ell \in \text{span}\ \{\hat{B}_1, \ldots, \hat{B}_m\}$. For the first m components, it is true that

$$D_\ell = \sum_{k=1}^{m} q_k (b_k)^\ell\, \hat{B}_k\, .$$

Equality must then also hold for the last coordinate. As such we obtain

$$d_{\ell+m} = \sum_{k=1}^{m} q_k (b_k)^{\ell+m} = p^{(\ell+m)}(0)\,.$$

This advances the induction and proves the existence. The uniqueness follows from the easily proven fact that if p and q from E_M satisfy

$$(p-q)^{(j)}(0) = 0, \quad j = 0, 1, \ldots, 2M-1\,,$$

then $p - q = 0$. ■

Remark: Theorem 1 proves the sufficiency of the rank D condition for interpolation from E_M. In fact this same condition is also necessary. We do not prove this here as we shall not use it.

Corollary 2. *Let p be of the form*

$$p(x) = \sum_{k=1}^{m} a_k \, e^{b_k \, x}$$

with distinct b_k's and nonzero a_k's. Then the matrix in (7) is nonsingular, and the b_k's are the distinct zeros of det B_x, for $k = 1, 2, \ldots, m$.

Proof: Based on Theorem 1, we need only show that det $\widetilde{D} \neq 0$. We claim that

$$\det \widetilde{D} = \prod_{k=1}^{m} a_k \prod_{1 \le i < j \le m} (b_j - b_i)^2\,.$$

This then implies that det $\widetilde{D} \neq 0$ by virtue of the conditions in the statement of the corollary. The desired result follows from the Vandermonde formula and from the easily verified identity

$$\widetilde{D} = A \cdot B$$

where $A = (a_j \, b_j^{i-1})_{i,j=1}^{\,m}$ and $B = (b_i^{j-1})_{i,j=1}^{\,m}$. ■

§3. A Recovery Problem with Shifts

We are now in a position to prove our first result which applies to the general multivariate case in (1).

Theorem 3. *Assume f and G are given, $G \in L^1(\mathbb{R}^n)$, and f is of the form (1) for some $m \le M$. Further assume that $\hat{G}$ is nonzero in a neighbourhood of a point $\boldsymbol{\omega}_0 \in \mathbb{R}^n$. Then we can uniquely determine the c_j and $\mathbf{t}_j$.*

Proof: We may assume without loss of generality that $\boldsymbol{\omega}_0 = 0$, for otherwise we can replace $G(\mathbf{x})$ by $e^{-i\boldsymbol{\omega}_0\cdot\mathbf{x}}G(\mathbf{x})$ and thus $\hat{G}$ by $\hat{G}(\cdot - \boldsymbol{\omega}_0)$. We know that

$$\frac{\hat{f}(\boldsymbol{\omega})}{\hat{G}(\boldsymbol{\omega})} = h(\boldsymbol{\omega}) = \sum_{j=1}^{m} c_j\, e^{-i\mathbf{t}_j\cdot\boldsymbol{\omega}}, \quad \boldsymbol{\omega} \in \mathbb{R}^n\ , \tag{9}$$

and that this is well-defined and sufficiently differentiable, at least in a neighbourhood of the origin (we shall require no more). The c_j and $\mathbf{t}_j$ are to be identified. We assume in the above representation that the c_j are nonzero, and the $\mathbf{t}_j$ are pairwise distinct. Otherwise we would rewrite the sum with a reduced m. For the moment we also assume that the c_j are distinct.

Given a non-zero vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$ we consider the directional derivative

$$D_{\boldsymbol{\gamma}} = \gamma_1 \frac{\partial}{\partial\omega_1} + \cdots + \gamma_n \frac{\partial}{\partial\omega_n}\,.$$

For each non-negative integer k

$$D^k_{\boldsymbol{\gamma}} h(\mathbf{0}) = \sum_{j=1}^{m} c_j(-i\,\boldsymbol{\gamma}\cdot\mathbf{t}_j)^k\,. \tag{10}$$

Assume that we have determined m. If the $\{(\boldsymbol{\gamma}\cdot\mathbf{t}_j)\}_{j=1}^m$ are all distinct, then we may apply Theorem 1 to obtain their values (and those of the $\{c_j\}_{j=1}^m$). From Theorem 1 and Corollary 2, it follows that the $\{(\boldsymbol{\gamma}\cdot\mathbf{t}_j)\}_{j=1}^m$ are distinct if and only if the associated Hankel matrix (7) has full rank m. That is, given $\boldsymbol{\gamma}$ we have a reasonable method of checking whether we can obtain the values $\{(\boldsymbol{\gamma}\cdot\mathbf{t}_j)\}_{j=1}^m$. If we can find, for n linearly independent vectors $\{\boldsymbol{\gamma}_{\hat{r}}\}_{\hat{r}=1}^n$, the values

$$\{(\boldsymbol{\gamma}_{\hat{r}}\cdot\mathbf{t}_j)\}_{\hat{r}=1}^{n}\,{}_{j=1}^{m},$$

then we can uniquely determine the $\{\mathbf{t}_j\}_{j=1}^m$.

Let $\{\boldsymbol{\gamma}_r\}_{r=1}^s$, where $s = \binom{m}{2}(n-1)+n$, be vectors in $\mathbb{R}^n$ in generic position, i.e., every n of them are linearly independent. We claim that given any m distinct vectors $\{\mathbf{t}_j\}_{j=1}^m$ there exist n of the $\boldsymbol{\gamma}_{\hat{r}}$, say $r_1, \ldots, r_n$, for which the

$$\{(\boldsymbol{\gamma}_{r_\ell}\cdot\mathbf{t}_j)\}_{j=1}^m$$

are distinct for each $\ell = 1, \ldots, n$. This may be proven as follows. For each pair (i,j), $1 \le i < j \le m$, we have $(\boldsymbol{\gamma}_r\cdot\mathbf{t}_i) = (\boldsymbol{\gamma}_r\cdot\mathbf{t}_j)$ for at most $n-1$ of the vectors $\{\boldsymbol{\gamma}_r\}_{r=1}^s$, because $\mathbf{t}_i - \mathbf{t}_j \ne \mathbf{0}$. Since there are $\binom{m}{2}$ such pairs (i,j), the desired result follows.

It thus remains to find m. Since $m \le M$ for some given M, we can take

$$s = \binom{M}{2}(n-1) + n$$

in the above. The maximum rank of the associated Hankel matrices will be m.

At the end of the first paragraph of this proof we assumed that the c_j are distinct. Why is this assumption necessary, and how can we overcome it? It is necessary for the following reason. Assuming that the $\{(\boldsymbol{\gamma}_r \cdot \mathbf{t}_j)\}_{j=1}^m$ are distinct, the method of Theorem 1 gives us the $\{(\boldsymbol{\gamma}_r \cdot \mathbf{t}_j)\}_{j=1}^m$ and the $\{c_j\}_{j=1}^m$. If, for example, $c_1 = c_2 = c$, then we know that all of the values $(\boldsymbol{\gamma}_r \cdot \mathbf{t}_j)$, $r, j = 1, 2$ go with the value c. However we have no way of knowing how to pair the value $(\boldsymbol{\gamma}_1 \cdot \mathbf{t}_1)$ with $(\boldsymbol{\gamma}_2 \cdot \mathbf{t}_1)$, and the value $(\boldsymbol{\gamma}_1 \cdot \mathbf{t}_2)$ with $(\boldsymbol{\gamma}_2 \cdot \mathbf{t}_2)$. (This is of course necessary for recovering the $\mathbf{t}_j$.) We could just as easily have paired $(\boldsymbol{\gamma}_1 \cdot \mathbf{t}_1)$ with $(\boldsymbol{\gamma}_2 \cdot \mathbf{t}_2)$, and $(\boldsymbol{\gamma}_1 \cdot \mathbf{t}_2)$ with $(\boldsymbol{\gamma}_2 \cdot \mathbf{t}_1)$. This problem does not arise if the $\{c_j\}_{j=1}^m$ are distinct.

One way to overcome this problem is to calculate the $\{c_j\}_{j=1}^m$. (Assuming that we have determined m, this can always be done.) If some c_j are equal, go back to h and shift it by $\widetilde{\boldsymbol{\omega}}$. Since

$$h(\boldsymbol{\omega} - \widetilde{\boldsymbol{\omega}}) = \sum_{j=1}^{m} \left(c_j e^{i\mathbf{t}_j \cdot \widetilde{\boldsymbol{\omega}}} \right) e^{-i\mathbf{t}_j \cdot \boldsymbol{\omega}}, \quad \boldsymbol{\omega} \in I\!R^n ,$$

we have a new problem totally equivalent to our old problem, but with altered c_j. In general, this will provide us with distinct coefficients. Since we can readily test whether the resulting new coefficients are distinct, we regard this problem as overcome. (Another method of dealing with this problem is to check all possible combinations – these are finite in number assuming the $(\boldsymbol{\gamma}_r \cdot \mathbf{t}_j)_{j=1}^m$ are distinct for each $r = 1, \ldots, n$ – and then see which of the resulting functions agrees with h.) ■

Remark: Assume that we know m, and that the $\{c_j\}_{j=1}^m$ are distinct. If, for each $i = 1, \ldots, n$, the ith components t_{ji} of the $\mathbf{t}_j$ are distinct, then we do not need all the mixed derivatives as in (10). In this case it suffices to consider only the pure partial derivatives $\frac{\partial^k}{\partial x_i^k}$. Even in the case where not all the $\{t_{ji}\}_{j=1}^m$ are distinct, we still obtain their values. However in this case we have trouble identifying how often a particular t_{ji} has occurred, and what are the values of the associated c_j. (When this happens we will obtain appropriate sums of the c_j, rather than the c_j themselves.) If the c_j and every possible partial sum of the c_j are all distinct, then it is possible to easily unravel the resulting data. In general the situation is rather more complicated. One can try all possible (finite) assignations of the $\{t_{ji}\}_{j=1}^m{}_{i=1}^n$, solve for the $\{c_j\}_{j=1}^n$ (if possible; whenever that is not possible, the chosen assignment cannot be correct), and then check whether the resulting function agrees with h. This does not seem to be an efficient method (see also the assumptions at the beginning of this remark), but does have the distinct advantage of only needing knowledge of the pure partial derivatives.

We now again consider (1). Moreover we now seek to identify not only the c_j and $\mathbf{t}_j$, but also G. This we do by finding the asymptotics of the Fourier transform of G at zero, thus giving us the required derivatives at zero.

Theorem 4. *Let f and G satisfy the conditions of the previous theorem. Suppose that*

$\sum_{j=1}^{m} c_j = 1$ and f can be written as

$$f(\mathbf{x}) = \sum_{j=1}^{k} a_j \,\|\mathbf{x}\|^{-\beta_j}\, g_j(\mathbf{x}) + h(\mathbf{x})\,. \tag{11}$$

This is to be valid for real coefficients a_j, $\beta_1 < \beta_2 < \cdots < \beta_k$, and g_j is, for each j, either $e^{i\boldsymbol{\gamma}_j\cdot\mathbf{x}}$ (any $\boldsymbol{\gamma}_j \in I\!R^n$) or $\log\|\mathbf{x}\|$. Further assume that $\beta_k \geq 2m+n-1$ and that $h : I\!R^n \to I\!R$ satisfies $|h(\mathbf{x})| = O\big((1+\|\mathbf{x}\|)^{-n-2m}\big)$ and the moment conditions

$$\int_{I\!R^n} h(\mathbf{x})\mathbf{x}^{\boldsymbol{\alpha}}\, d\mathbf{x} = 0, \qquad |\boldsymbol{\alpha}| \leq 2m-1.$$

Then, not only c_j and $\mathbf{t}_j$, but also G can be identified from f in (1).

Proof: It is a consequence of the proof of Theorem 3 that c_j and $\mathbf{t}_j$ can be found from just knowing the values of the left-hand side of (10). Here again, we take $\boldsymbol{\omega}_0 = \mathbf{0}$. Once we know the values of the c_j and $\mathbf{t}_j$, then from (9) we may obtain $\hat{G}$, and thus G.

Next we are using the fact that f and G have the same asymptotics at ∞ because $m < \infty$ and $\sum_{j=1}^{m} c_j = 1$. This implies that when f has the form (11), G must have that form too, perhaps with a different h, call it H, that still satisfies the same moment conditions. Concretely, if f has the form (11), $G(\mathbf{x}-\mathbf{t}_j)$ must be of the form

$$\sum_{\ell=1}^{k} a_\ell \,\|\mathbf{x}-\mathbf{t}_j\|^{-\beta_\ell}\, g_\ell(\mathbf{x}-\mathbf{t}_j) + H(\mathbf{x}-\mathbf{t}_j),$$

where H satisfies the same *decay* properties as h does. Therefore the expression

$$\sum_{j=1}^{m} c_j \sum_{\ell=1}^{k} a_\ell \,\|\mathbf{x}-\mathbf{t}_j\|^{-\beta_\ell}\, g_\ell(\mathbf{x}-\mathbf{t}_j) + H(\mathbf{x}-\mathbf{t}_j) - \sum_{j=1}^{k} a_j \,\|\mathbf{x}\|^{-\beta_j}\, g_j(\mathbf{x}) = h(\mathbf{x})$$

must vanish when integrated against $\mathbf{x}^{\boldsymbol{\alpha}}$ for the appropriate range of $\boldsymbol{\alpha}$. By a change of variables, however, that integral can be transformed to

$$\int_{I\!R^n} \Big(\sum_{j=1}^{m} c_j(\mathbf{x}+\mathbf{t}_j)^{\boldsymbol{\alpha}}\Big)\Big(\sum_{\ell=1}^{k} a_\ell \,\|\mathbf{x}\|^{-\beta_\ell}\, g_\ell(\mathbf{x}) + H(\mathbf{x})\Big) - \mathbf{x}^{\boldsymbol{\alpha}} \sum_{j=1}^{k} a_j \,\|\mathbf{x}\|^{-\beta_j}\, g_j(\mathbf{x})\, d\mathbf{x} = 0.$$

Now, the linear independence of monomials of different degrees and the fact that the c_j sum to one imply the assertion that H satisfies the moment conditions.

According to [5, p. 530], this now implies that $\hat{G}$ has the following asymptotic expansion at zero. The expansion is

$$\sum_{j=1}^{k} \widetilde{g}_j(\boldsymbol{\omega})\, \|\boldsymbol{\omega}\|^{\beta_j - n} + O(\|\boldsymbol{\omega}\|^{2m}), \quad \|\boldsymbol{\omega}\| \to 0\,,$$

where $\widetilde{g}_j(\boldsymbol{\omega})\,\|\boldsymbol{\omega}\|^{\beta_j-n}$ is the generalized Fourier transform of $a_j\,\|\mathbf{x}\|^{-\beta_j}\,g_j(\mathbf{x})$, the O term coming from the moment conditions on h which imply that $|\hat{h}(\boldsymbol{\omega})| = O(\|\boldsymbol{\omega}\|^{2m})$ for small argument. The $\widetilde{g}_j$ are known, bounded, smooth functions that can be found in [5, p. 530f.].

As $\beta_k - n \geq 2m-1$, the above asymptotic expansion suffices to find all the necessary derivatives at 0. Hence, c_j and $\mathbf{t}_j$ can be determined. ■

We remark that the condition $\sum_{j=1}^m c_j = 1$ is only a restriction in so far as it requires that the sum of the c_j be nonzero. If it has a nonzero value other than one, it can be absorbed into the G.

We further remark that the radial asymptotic behaviour is salient because it enables us to relate the form of G to an asymptotic expansion of $\hat{G}$ at zero. This approach resembles the univariate Abelian and Tauberian theorems, e.g. in Widder [7, Chap. V].

§4. Recovery with Dilation and Shifts

In this section we address ourselves to resolving recovery problems where f is of the form (2). That is,

$$f(\mathbf{x}) = \sum_{j=1}^m c_j g(\mathbf{a}_j\cdot\mathbf{x} - b_j)\,, \qquad \mathbf{x}\in \mathbb{R}^n,$$

for some given g and unknown $\{\mathbf{a}_j\}_{j=1}^m \subset \mathbb{R}^n$, $\{c_j\}_{j=1}^m \subset \mathbb{R}$ and $\{b_j\}_{j=1}^m \subset \mathbb{R}$.

Note that if we know that the b_j are all zero, then the analysis of Theorem 3 suffices.

Proposition 5. *Let g be C^{2m-1} in a neighbourhood of the origin, and $g^{(k)}(0) \neq 0$, $k = 0, 1, \ldots, 2m-1$. Assume that f and g are given and satisfy*

$$f(\mathbf{x}) = \sum_{j=1}^m c_j g(\mathbf{a}_j\cdot\mathbf{x})$$

for some unknown non-zero c_j, distinct $\mathbf{a}_j$, and $m \leq M$. Then we can uniquely determine the c_j and the $\mathbf{a}_j$.

Proof: For each given $\boldsymbol{\gamma}\in\mathbb{R}^n$, and integer k, $1 \leq k \leq 2m-1$,

$$D^k_{\boldsymbol{\gamma}} f(\mathbf{0}) = \sum_{j=1}^m c_j(\mathbf{a}_j\cdot\boldsymbol{\gamma})^k g^{(k)}(0)\,.$$

Since we know $D^k_{\boldsymbol{\gamma}} f(\mathbf{0})$ and $g^{(k)}(0) (\neq 0)$ for all applicable k, we have the values of

$$\sum_{j=1}^m c_j(\mathbf{a}_j\cdot\boldsymbol{\gamma})^k\,, \qquad k = 0, 1, \ldots, 2m-1\,,$$

for any $\boldsymbol{\gamma}\in\mathbb{R}^n$. We can now apply the analysis of Theorem 3. ■

Remark: Assume we apply the method of finding the $\mathbf{t}_j$ as described in the Remark following Theorem 3 (with the appropriate restrictions and assumptions). That is, we only use the pure partial derivatives and make no use of the mixed or directional derivatives. Then using the method of proof of Proposition 5, we can find the t_{ji} and c_j in the more general

$$f(\mathbf{x}) = \sum_{j=1}^{m} c_j G(t_{j1}x_1, t_{j2}x_2, \ldots, t_{jn}x_n)$$

whenever $G \in C^{2m-1}(\mathbb{R}^n)$ and $\frac{\partial^k}{\partial x_i^k} G(\mathbf{0}) \neq 0$, $i = 1, \ldots, n$, $k = 0, 1, \ldots, 2m-1$.

The addition of the translates b_j significantly complicates matters. In what follows we will, for ease of exposition, assume that m is, a priori, known.

Theorem 6. *Let $g \in C^{2m-1}(\mathbb{R})$ and $g^{(k)} \in L^1(\mathbb{R})$, $k = 0, 1, \ldots, 2m-1$. Further assume that $\widehat{g^{(k)}}(0) \neq 0$, $k = 0, 1, \ldots, 2m-1$. We are given f and g and know that they satisfy*

$$f(\mathbf{x}) = \sum_{j=1}^{m} c_j g(\mathbf{a}_j \cdot \mathbf{x} - b_j)\,, \qquad \mathbf{x} \in \mathbb{R}^n\,.$$

We assume that the $\{c_j\}_{j=1}^m$ are unknown and non-zero, the $\{b_j\}_{j=1}^m$ are unknown, and the $\{\mathbf{a}_j\}_{j=1}^m$ in $\mathbb{R}^n$ are, for $n \geq 2$, unknown pairwise linearly independent. If $n = 1$ we only demand that the $\mathbf{a}_j = a_j$ be distinct and non-zero. Then we can uniquely determine the unknown parameters.

Proof: Assume for the moment that $n \geq 2$. For any $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, and $0 \leq k \leq 2m-1$,

$$(D_{\boldsymbol{\alpha}}^{2m-1-k} D_{\boldsymbol{\gamma}}^{k} f)(\mathbf{0}) = \sum_{j=1}^{m} c_j (\mathbf{a}_j \cdot \boldsymbol{\alpha})^{2m-1-k} (\mathbf{a}_j \cdot \boldsymbol{\gamma})^k g^{(2m-1)}(-b_j). \tag{12}$$

On the assumption that $\mathbf{a}_j \cdot \boldsymbol{\alpha} \neq 0$, $j = 1, \ldots, m$,

$$(D_{\boldsymbol{\alpha}}^{2m-1-k} D_{\boldsymbol{\gamma}}^{k} f)(\mathbf{0}) = \sum_{j=1}^{m} \left[c_j (\mathbf{a}_j \cdot \boldsymbol{\alpha})^{2m-1} g^{(2m-1)}(-b_j) \right] \left[\frac{(\mathbf{a}_j \cdot \boldsymbol{\gamma})}{(\mathbf{a}_j \cdot \boldsymbol{\alpha})} \right]^k .$$

Thus, if $\mathbf{a}_j \cdot \boldsymbol{\alpha} \neq 0$ and $g^{(2m-1)}(-b_j) \neq 0$, then we can determine, by the method of proof of Theorem 3, the vectors

$$\frac{\mathbf{a}_j}{(\mathbf{a}_j \cdot \boldsymbol{\alpha})}\,.$$

Here we use the fact that for $n \geq 2$ the $\mathbf{a}_j$ are pairwise linearly independent. As such the above ratios are always distinct vectors. This determines the $\mathbf{a}_j$, up to multiplication by a non-zero constant, for those j such that $g^{(2m-1)}(-b_j) \neq 0$. However it is possible that $g^{(2m-1)}(-b_j) = 0$ for some j. The function g and thus $g^{(2m-1)}$ are known. Furthermore we can also shift $\mathbf{0}$ to any $\mathbf{y}$ by substituting $\mathbf{x} + \mathbf{y}$ for $\mathbf{x}$. Using these facts we will assume that in this way we have determined

$$\mathbf{a}_j = \widetilde{d}_j \widetilde{\mathbf{a}}_j\,, \qquad j = 1, \ldots, m\,,$$

for some fixed $\widetilde{\mathbf{a}}_j \in I\!\!R^n \backslash \{\mathbf{0}\}$ and unknown $\widetilde{d}_j \in I\!\!R \backslash \{0\}$, and all $j = 1, \ldots, m$. (If g is of finite support, this may prove to be impractical for some values of b_j.)

Choose $\mathbf{z} \in I\!\!R^n$ such that $\mathbf{a}_j \cdot \mathbf{z} \neq 0$, $j = 1, \ldots, m$. Thus

$$h(t) = f(t\,\mathbf{z}) = \sum_{j=1}^{m} c_j g(d_j t - b_j)$$

where the $d_j = \widetilde{d}_j(\widetilde{\mathbf{a}}_j \cdot \mathbf{z})$ is unknown, but non-zero. If $n = 1$, then all the above is not needed, and we start the analysis from here. For $0 \leq k \leq 2m - 1$,

$$\widehat{h^{(k)}}(\omega) = \sum_{j=1}^{m} c_j \frac{d_j^k}{|d_j|} e^{\frac{-ib_j\omega}{d_j}} \widehat{g^{(k)}}(\frac{\omega}{d_j}) \,.$$

Thus

$$\widehat{h^{(k)}}(0) = \sum_{j=1}^{m} c_j \frac{d_j^k}{|d_j|} \widehat{g^{(k)}}(0) \,.$$

The $\widehat{h^{(k)}}(0)$ and $\widehat{g^{(k)}}(0)$ are known, and $\widehat{g^{(k)}}(0) \neq 0$. The c_j are non-zero and m is known. If the d_j are distinct we can directly apply the result of Theorem 1 to obtain the $c_j/|d_j|$ and d_j (and thus the c_j). If the d_j are not distinct, and this can be discerned from the method of determining the d_j, (i.e., the associated matrix will have rank $< m$), then we should alter the $\mathbf{z}$. As such we now assume that we have determined the c_j and $\mathbf{a}_j$, $j = 1, \ldots, m$.

It remains to determine the b_j, $j = 1, \ldots, m$. Let $\mathbf{z}$ be as above. Thus

$$h(t) = \sum_{j=1}^{m} c_j g(d_j t - b_j)$$

with *known* c_j and d_j. We assume, as above, that the d_j are distinct and non-zero. Now

$$\widehat{h^{(k)}}(\omega) = \sum_{j=1}^{m} c_j \frac{d_j^k}{|d_j|} e^{\frac{-ib_j\omega}{d_j}} \widehat{g^{(k)}}(\frac{\omega}{d_j}) \,,$$

and thus

$$\frac{\partial}{\partial\omega}\widehat{h^{(k)}}(0) = \sum_{j=1}^{m} c_j \frac{d_j^{k-1}}{|d_j|}(-ib_j)\widehat{g^{(k)}}(0) + \sum_{j=1}^{m} c_j \frac{d_j^{k-1}}{|d_j|} \frac{\partial}{\partial\omega}\widehat{g^{(k)}}(0) \,.$$

Since $\frac{\partial}{\partial\omega}\widehat{h^{(k)}}(0)$, $\widehat{g^{(k)}}(0)$, $\frac{\partial}{\partial\omega}\widehat{g^{(k)}}(0)$, c_j and d_j are all known, and $\widehat{g^{(k)}}(0) \neq 0$, $k = 0, 1, \ldots, m-1$, we therefore have the values of

$$\frac{c_j d_j^{k-1}}{|d_j|} b_j \,, \qquad k = 0, 1, \ldots, m-1 \,.$$

Since the c_j are non-zero and the d_j are distinct and non-zero, the b_j can be uniquely determined from the associated square non-singular linear system. ■

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