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L^1-Approximation and Finding Solutions with Small Support

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Abstract In this paper, we study an interesting property of L^1-approximation. For many subspaces M, there exist $\alpha^*(M) > 0$ with the following property: if f vanishes off a set of measure at most $\alpha^*(M)$, then the zero function is a best L^1-approximant to f from M. We explain this phenomenon, provide estimates for $\alpha^*(M)$ in many cases, and present some open questions.

Keywords L^1-approximation · Nikolskii-type inequalities · Sparsest solutions · Best approximation · Minimal support

Mathematics Subject Classification (2000) 41A50 · 41A17

1 Introduction

For many subspaces M, there exist $\alpha^*(M) > 0$ with the following property: if f vanishes off a set of measure at most $\alpha^*(M)$, then the zero function is a best

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L^1-approximant to f from M. This relationship, between functions with small support and those whose best L^1-approximant from a given subspace is always the zero function, was first noted in the study of sparse representations (compressed sensing) in the ℓ_1^m setting. It is a relationship that is very L^1-norm dependent.

In Sect. 2, we explain the fundamentals of this relationship, starting with the characterization of best approximation from a linear subspace in the L^1-norm. We are then led to the definition of $\alpha^*(M)$, and discuss various basic properties thereof. In Sect. 3, we consider theoretical upper and lower bounds on $\alpha^*(M)$. Section 4 contains 18 specific examples of subspaces (or subsets), with lower bounds and sometimes upper bounds on the associated $\alpha^*(M)$. Finally, in Sect. 5, we examine three families of examples. The common feature of these examples is that M will have the property that all $m \in M$ with a fixed L^1-norm have the same distribution. This implies that we can explicitly calculate or characterize $\alpha^*(M)$.

2 L^1-Approximation and $\alpha^*(M)$

We start with some general results concerning L^1-approximation.

Let B be a set, Σ a σ-field of subsets of B, and ν a positive measure defined on Σ. Let $L^1(B,\nu)$ denote the usual space of real-valued functions with norm

$$\|f\|_1 := \int_B \big|f(x)\big|\,d\nu(x).$$

For $f \in L^1(B,\nu)$, we define its zero set

$$Z(f) := \big\{x : f(x) = 0\big\},$$

and its complement $N(f) := B\backslash Z(f)$. Note that $Z(f)$ and $N(f)$ are ν-measurable. In addition, for $f \in L^1(B,\nu)$, we set

$$\operatorname{sgn}\big(f(x)\big) := \begin{cases} 1, & f(x) > 0, \\ 0, & f(x) = 0, \\ -1, & f(x) < 0. \end{cases}$$

The following is the well-known elementary characterization of best approximation from linear subspaces in $L^1(B,\nu)$. This result goes back to James [13] and Kripke, Rivlin [16], see also Pinkus [30, Theorem 2.1].

Theorem 1 *Let M be a linear subspace of $L^1(B,\nu)$ and $f \in L^1(B,\nu)\backslash\overline{M}$. Then m^* is a best $L^1(B,\nu)$-approximant to f from M if and only if*

$$\left|\int_B m \operatorname{sgn}\big(f - m^*\big)\,d\nu\right| \le \int_{Z(f-m^*)} |m|\,d\nu$$

for all $m \in M$. In addition, if strict inequality holds for all $m \in M$, $m \neq 0$, then m^* is the unique best $L^1(B,\nu)$-approximant to f from M.

Thus, we see that the identically zero function is a best $L^1(B,\nu)$-approximant to f from the linear subspace M if and only if

$$\left|\int_B m \operatorname{sgn}(f)\,d\nu\right| \le \int_{Z(f)} |m|\,d\nu$$

for all $m \in M$, or equivalently,

$$\left|\int_{N(f)} m\ \mathrm{sgn}(f)\, d\nu\right| \le \int_{Z(f)} |m|\, d\nu$$

for all $m \in M$. In fact, the subspace property of M is not necessary. We have:

Proposition 2 *Let M be a homogeneous subset; i.e., $m \in M$ implies $cm \in M$ for all $c \in \mathbb{R}$. Then the zero function is a best $L^1(B, \nu)$-approximant to f from M if and only if*

$$\left|\int_{N(f)} m\ \mathrm{sgn}(f)\, d\nu\right| \le \int_{Z(f)} |m|\, d\nu$$

for all $m \in M$.

This is a simple consequence of the fact that the above is equivalent to the zero function being a best $L^1(B, \nu)$-approximant to f from each 1-dimensional subspace span$\{m\}$, with $m \in M$.

From Proposition 2, we easily obtain:

Proposition 3 *Let M be a homogeneous subset of $L^1(B, \nu)$. Let Z be any ν-measurable subset of B, and $N = B\backslash Z$. Then the zero function is a best $L^1(B, \nu)$-approximant from M to every $f \in L^1(B, \nu)$ that vanishes on Z if and only if*

$$\int_N |m|\, d\nu \le \int_Z |m|\, d\nu \tag{1}$$

for all $m \in M$.

Indeed, given $m \in M$, (1) follows from Proposition 2 by taking any $f \in L^1(B, \nu)$ with $Z(f) = Z$ and $\mathrm{sgn}(f) = \mathrm{sgn}(m)$ on N. Equation (1) is a sufficient but not necessary condition implying that the zero function is a best $L^1(B, \nu)$-approximant from M to a particular $f \in L^1(B, \nu)$.

Based on Proposition 3, it is natural to ask how large N might be for a given linear subspace M of $L^1(B, \nu)$. In Pinchasi, Pinkus [29], it is shown that if M is any finite-dimensional linear subspace of $L^1[0, 1]$ consisting of continuous functions, then for every $\varepsilon > 0$ there exists a subset $N \subset [0, 1]$ of Lebesgue measure at least $1/2 - \varepsilon$ such that (1) holds. (Note that if M contains the constant function, then N cannot have measure larger than $1/2$.) And, if n is fixed, and M is an n-dimensional linear subspace of $\mathbb{R}^m$ (with the usual ℓ_1^m-norm), then there exists a subset $N \subset \{1, \ldots, m\}$ of cardinality $(1/2 - o(1))m$ such that (1) holds.

When is the zero function a best $L^1(B, \nu)$-approximant from M to every $f \in L^1(B, \nu)$ that does not vanish on a set of measure at most $\alpha > 0$? It follows from Proposition 3 that we have:

Corollary 4 *Fix $\alpha > 0$, and let M be a homogeneous subset of $L^1(B, \nu)$. Then the zero function is a best $L^1(B, \nu)$-approximant from M to every $f \in L^1(B, \nu)$ with $\nu(N(f)) \le \alpha$ if and only if*

$$\int_N |m|\, d\nu \le \int_Z |m|\, d\nu,$$

or, equivalently,

$$2\int_N |m|\,d\nu \le \|m\|_1$$

for all $m \in M$ and all N such that $\nu(N) \le \alpha$. Thus, the zero function is a best $L^1(B,\nu)$-approximant from M to every $f \in L^1(B,\nu)$ that does not vanish on a set of measure at most $\alpha > 0$ if and only if

$$\sup_{m\in M}\ \sup_{\{N:\nu(N)\le\alpha\}} \frac{\int_N |m|\,d\nu}{\|m\|_1} \le \frac{1}{2}.$$

The quantity

$$|||f|||_\alpha := \sup_{\{N:\nu(N)\le\alpha\}} \int_N |f|\,d\nu$$

for $\alpha > 0$ is a norm (provided there are no atoms of measure strictly larger that α, otherwise it is a seminorm). We can thus restate Corollary 4 as:

Corollary 5 *Fix $\alpha > 0$, and let M be a homogeneous subset of $L^1(B,\nu)$. Then the zero function is a best $L^1(B,\nu)$-approximant from M to every $f \in L^1(B,\nu)$ with $\nu(N(f)) \le \alpha$ if and only if*

$$R_\alpha := \sup_{m\in M} \frac{|||m|||_\alpha}{\|m\|_1} \le \frac{1}{2}. \tag{2}$$

Moreover, if strict inequality holds in (2), then the zero function is the unique best $L^1(B,\nu)$-approximant from M to every such f.

Equivalently, (2) holds if and only if for every set of measure at most $\alpha > 0$ and every f that is zero off this set, there exists a continuous linear functional that attains its norm on f and annihilates M.

When R_α is strictly less than $1/2$, we actually have *strong uniqueness*, see Pinkus [30, p. 18] or Kroó, Pinkus [18].

Proposition 6 *Let M be a homogeneous subset, and assume that for a given $\alpha > 0$ we have*

$$\sup_{m\in M} \frac{|||m|||_\alpha}{\|m\|_1} = R_\alpha < \frac{1}{2}.$$

If $\nu(N(f)) \le \alpha$, then the zero function is the unique best $L^1(B,\nu)$-approximant from M to f, and

$$\|f-m\|_1 - \|f\|_1 \ge (1-2R_\alpha)\|m\|_1$$

for all $m \in M$.

The characterization of best $L^1(B,\nu)$-approximants was used to explicate and motivate Corollary 5. In fact, the previous two results can be both generalized and easily proven directly, as follows.

Let G be any real-valued function on M such that $G(0) = 0$ and $\|m\|_1 + G(m) > 0$ for all $m \in M$, $m \neq 0$. Consider the problem

$$\inf_{m\in M}\big\{\|f - m\|_1 + G(m)\big\}. \tag{3}$$

Theorem 7 *Fix $\alpha > 0$, and let M be a homogeneous subset of $L^1(B, \nu)$. Then*

$$\sup_{m\in M} \frac{|||m|||_\alpha}{\|m\|_1 + G(m)} \le \frac{1}{2}$$

if and only if the zero function is a solution of (3) for each f with $\nu(N(f)) \le \alpha$.

Proof Assume

$$\sup_{m\in M} \frac{|||m|||_\alpha}{\|m\|_1 + G(m)} \le \frac{1}{2}.$$

Then $\nu(N) \le \alpha$ implies

$$2\int_N |m| \le \|m\|_1 + G(m),$$

which is equivalent to

$$\int_N |m| \le \int_{N^c} |m| + G(m).$$

For f that vanishes off N, and any $m \in M$,

$$\begin{aligned}\|f\|_1 + G(0) = \|f\|_1 = \int_N |f| &\le \int_N |f - m| + \int_N |m| \\ &\le \int_N |f - m| + \int_{N^c} |m| + G(m) = \|f - m\|_1 + G(m).\end{aligned}$$

Thus, $m = 0$ is a solution to (3).

Now, assume $m = 0$ is a solution to (3) for every f that vanishes off a set of measure at most α. Fix any $m^* \in M$, $m^* \neq 0$, and N with $\nu(N) \le \alpha$. Let $f = m^*$ on N and vanish off N. Since $m = 0$ is a solution to (3), it follows that

$$\|f\|_1 = \|f - 0\|_1 + G(0) \le \big\|f - m^*\big\|_1 + G\big(m^*\big);$$

i.e.,

$$\int_N \big|m^*\big| \le \int_{N^c} \big|m^*\big| + G\big(m^*\big),$$

which is equivalent to

$$2\int_N \big|m^*\big| \le \big\|m^*\big\|_1 + G\big(m^*\big),$$

implying

$$\frac{\int_N |m^*|}{\|m^*\|_1 + G(m^*)} \le \frac{1}{2}.$$

As this is valid for every set N of measure at most α, we have

$$\frac{|||m^*|||_\alpha}{\|m^*\|_1 + G(m^*)} \le \frac{1}{2}$$

for every $m^* \in M$. □

Consider, for example, $G(m) = \lambda\|m\|_1$, where $\lambda > -1$ (needed so that $\|m\|_1 + G(m) > 0$ for $m \in M$, $m \ne 0$). For $-1 < \lambda < 0$, we are looking at strong uniqueness; i.e., this is just a repeat of Proposition 6. The case $\lambda \ge 1$ is valueless, since

$$\|f\|_1 \le \|f - m\|_1 + \|m\|_1 \le \|f - m\|_1 + \lambda\|m\|_1$$

for every $m \in M$, and thus $m = 0$ always attains the above infimum. For $0 < \lambda < 1$, this result is of some interest. It shows us how, with the regularization term $\lambda\|m\|_1$, the associated α for which (3) holds grows with λ.

Of interest, given M, is to try to determine the largest α (if such exists) for which (2) holds. The main subject of this paper will be the study of the parameter

$$\alpha^*(M) = \sup\left\{\alpha : \sup_{m \in M} \frac{|||m|||_\alpha}{\|m\|_1} \le \frac{1}{2}\right\}.$$

It follows that if $\alpha < \alpha^*(M)$, and $f \in L^1(B, \nu)$ vanishes off a set of measure α, then the zero function is the best $L^1(B, \nu)$-approximant from M to f. Conversely, given any $\alpha > \alpha^*(M)$, there exists an $f \in L^1(B, \nu)$, vanishing off a set of measure α, for which the zero function is not a best $L^1(B, \nu)$-approximant from M to f.

If ν is a nonatomic measure (or a purely atomic measure with a finite number of atoms), then the above exterior supremum is a maximum. Easy examples show that this is not necessarily true in general.

We start the study of $\alpha^*(M)$ with a basic result. Recall that a subset $K \subset L^1(B, \nu)$ is uniformly integrable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_A |f|\,d\nu < \varepsilon$ for every $f \in K$ and every set $A \subseteq B$ satisfying $\nu(A) < \delta$.

In the examples of this paper, we consider only nonatomic measures. As such, and in order to avoid unnecessary explanation, we shall assume in what follows that ν is a nonatomic measure. However, these next results, with correct interpretation, also hold without this assumption.

Theorem 8 *Let M be a closed linear subspace of $L^1(B, \nu)$, and consider the following conditions*:

(i) *M is reflexive*,
(ii) *M does not contain a subspace isomorphic to ℓ_1*,
(iii) *the unit ball $B(M) = \{m : \|m\|_1 \le 1\}$ of M is uniformly integrable*,
(iv) $\alpha^*(M) > 0$.

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). *When ν is finite*, *all four conditions are equivalent.*

Remark It follows that if $M \subset L^1(B, \nu)$ is a finite-dimensional subspace, then $\alpha^*(M) > 0$, since every finite-dimensional space is reflexive. Note also that if M is a subspace of finite codimension, then $\alpha^*(M) = 0$, since the unit ball of a subspace M of finite codimension contains functions of arbitrarily small support.

Before proving Theorem 8, we need some preliminary results.

Lemma 9 *Let K be a weakly closed set in $L^1(B,\nu)$. Then K is weakly compact if and only if it is uniformly integrable and there are sets B_n with finite measure for which* $\lim_{n\to\infty}\int_{B\setminus B_n} f\,d\nu = 0$ *uniformly for $f\in K$.*

Proof See Dunford, Schwartz [7, Corollary IV.8.11] for the proof when ν is finite (and the uniformity condition is then clearly redundant). For the general case, see Dunford, Schwartz [7, Exercise IV.13.54]. □

The following lemma and theorem are due to Kadec, Pelczynski [15] (the indicator function of a set A is denoted by χ_A).

Lemma 10 *Let ν be a finite measure, and let $\{f_n\}$ be a bounded nonuniformly integrable sequence in $L^1(B,\nu)$. Then, there are a $\tau>0$, a subsequence $\{f_{n_k}\}$, and disjoint sets A_k such that* $\lim\int_{A_k}|f_{n_k}|\,d\nu=\tau$ *and such that the sequence $h_{n_k}=\chi_{A_k^c}f_{n_k}$ is weakly convergent.*

Theorem 11 *A close subspace of $L^1(B,\nu)$ is reflexive if and only if it does not contain a subspace isomorphic to l_1.*

Proof of Theorem 8 The equivalence of (i) and (ii) is Theorem 11. Since the unit ball of a reflexive space is weakly compact, Lemma 9 shows that (i) implies (iii) and that they are equivalent when ν is finite.

That (iii) implies (iv) is immediate (and does not depend on the finiteness of ν). Just choose any $\delta>0$ for which $\nu(A)<\delta$, $\|m\|\le 1$, and $m\in M$ imply that $\int_A|m|\,d\nu<1/2$, and it follows that $\alpha^*(M)\ge\delta$.

Finally, to prove that (iv) implies (ii) when ν is finite, assume that $B(M)$ contains a sequence $\{m_n\}$ that is not uniformly integrable. We shall show that $\alpha^*(M)=0$. By Lemma 10, there are a subsequence (which we assume, to simplify notation, is the original sequence), $\tau>0$, and disjoint sets A_j such that $\int_{A_j}|m_j|\,d\nu\to\tau$ and such that $h_j=\chi_{A_j^c}m_j$ is weakly convergent. Then $h_{2j+1}-h_{jn}$ converges weakly to 0, and it follows that there are convex combinations ϕ_n of $(h_{2j+1}-h_{2j})/2$ that converge in norm to zero; i.e., there are disjoint sets J_n of indices and coefficients λ_j^n, for $j\in J_n$, with $\sum_{j\in J_n}|\lambda_j^n|=1$ such that the $\phi_n=\sum_{j\in J_n}\lambda_j^n h_j$ satisfy $\|\phi_n\|\to 0$. Note that since the summands of $\psi_n=\sum_{j\in J_n}\lambda_j^n(m_j-h_j)$ are supported in the disjoint sets A_j, for $j\in J_n$, it follows that the $B_n=\bigcup_{j\in J_n}A_j$ are disjoint, and that $\|\psi_n\|=\int_{B_n}|\psi_n|\,d\nu\to\tau$.

Now fix $\varepsilon<\tau/6$, and choose n sufficiently large so that $\tau-\varepsilon<\|\psi_n\|<\tau+\varepsilon$, $\|\phi_n\|<\varepsilon$ and $\nu(B_n)<\varepsilon$. Then the function $F_n=\sum_{j\in J_n}\lambda_j^n m_j=\psi_n+\phi_n\in M$ satisfies $\|F_n\|\le\|\psi_n\|+\|\phi_n\|<\beta+2\varepsilon<2(\tau-2\varepsilon)$ by our choice of ε. Thus,

$$\int_{B_n}|F_n|\,d\nu\ge\int_{B_n}|\psi_n|\,d\nu-\int_{B_n}|\phi_n|\,d\nu\ge\int_{B_n}|\psi_n|\,d\nu-\|\phi_n\|>\tau-2\varepsilon>\frac{1}{2}\|F_n\|,$$

and $\nu(B_n)<\varepsilon$, which implies that $\alpha^*(M)<\varepsilon$. As ε was arbitrarily chosen, it follows that $\alpha^*(M)=0$. □

Remark The following examples show that (iii) or (iv) do not imply (ii) when ν is infinite. Let M be the subspace of $L^1(\mathbb{R})$ spanned by the functions $\chi_{[n,n+1]}$. Then M is isometric to l_1, yet $B(M)$ is uniformly integrable and $\alpha^*(M) = 1/2$. To obtain an example of a space isometric to l_1 with $\alpha^*(M) > 0$ and a nonuniformly integrable unit ball, fix a sequence $\delta_n \to 0$ and take the span of $f_n = \delta_n^{-1}\chi_{[n,n+\delta_n]} + \chi_{[n+\delta_n,n+1]}$.

What is the connection between this theoretical L^1-approximation problem and the subject of sparse representations (compressed sampling)? Consider the following model. Let V be a linear space, and let $L : L^1(B, \nu) \to V$ be a linear operator with kernel M; i.e., $Lm = 0$ for all $m \in M$. Assume that $Lf = v$ and f vanishes off a set of measure smaller than $\alpha^*(M)$. Then

$$\inf_{\{h:Lh=v\}} \|h\|_1 = \inf_{m\in M} \|f - m\|_1 = \|f\|_1,$$

and f uniquely attains this infimum. Thus, there cannot exist two distinct solutions to $Lh = v$ that vanish off sets of measure smaller than $\alpha^*(M)$. In other words, if among the solutions h of $Lh = v$ there exists a solution that vanishes off a set of measure at most α for some $\alpha < \alpha^*(M)$, then it is the unique such solution, and it is obtained by solving the problem

$$\inf_{\{h:Lh=v\}} \|h\|_1.$$

The theory of sparse representations deals with exactly this problem in the discrete setting, i.e., when L is an $n \times m$ matrix. The interested reader may consult Elad [8], and the references therein.

Remark We consider in this paper real-valued functions and spaces. Many of these results are also valid in the complex-valued setting.

3 Lower and Upper Bounds for $\alpha^*(M)$

In this section, we consider theoretical lower and upper bounds on $\alpha^*(M)$. Unfortunately, there do not seem to be many of either.

There is clearly no strictly positive lower bound on $\alpha^*(M)$ valid even for all 1-dimensional M. Indeed, if $M = \mathrm{span}\{m\}$ and $\nu(N(m)) < 2\varepsilon$, then necessarily $\alpha^*(M) < \varepsilon$. However, for many classic examples, lower bounds do exist. The following elementary result will prove surprisingly useful.

Proposition 12 *Assume that* $M \subseteq L^p(B, \nu)$ *for some* $p \in (1, \infty]$. *Define*

$$A_p := \sup_{m\in M} \frac{\|m\|_p}{\|m\|_1},$$

and assume that $A_p < \infty$. *Then*

$$\alpha^*(M) \geq \frac{1}{(2A_p)^{p'}},$$

where, *as usual*, $1/p + 1/p' = 1$.

Proof Hölder's inequality gives, for each $\alpha > 0$,

$$|||m|||_\alpha \le \alpha^{1/p'} \|m\|_p.$$

Thus,

$$\frac{|||m|||_\alpha}{\|m\|_1} \le \frac{\alpha^{1/p'} \|m\|_p}{\|m\|_1},$$

and

$$\sup_{m \in M} \frac{|||m|||_\alpha}{\|m\|_1} \le \sup_{m \in M} \frac{\alpha^{1/p'} \|m\|_p}{\|m\|_1} = \alpha^{1/p'} A_p.$$

Hence,

$$\sup_{m \in M} \frac{|||m|||_\alpha}{\|m\|_1} \le \frac{1}{2},$$

whenever $\alpha^{1/p'} A_p \le 1/2$, implying that

$$\alpha^*(M) \ge \frac{1}{(2A_p)^{p'}}.$$

□

Nikolskii-type inequalities are inequalities of the form

$$\|m\|_p \le C_{p,q} \|m\|_q$$

for a given class of functions, where $\| \cdot \|_p$ and $\| \cdot \|_q$ are the usual L^p and L^q norms, respectively, see, e.g., Nikolskii [27]; Szegő, Zygmund [32]; Timan [34]; and Milovanović, Mitrinović, Rassias [23]. Note that $A_p = C_{p,1}$ for the class of functions M. Thus, Nikolskii-type inequalities have immediate consequences for our problem. Numerous Nikolskii-type inequalities may be found in the literature. We list some of these inequalities and their consequences in Sect. 4.

Lower bounds on $\alpha^*(M)$ can also be obtained, under suitable conditions on the subspace M and/or the domain B, via other inequalities. Two such conditions (both stronger than Nikolskii-type inequalities) are Bernstein–Markov inequalities (see Proposition 13(ii)) and Remez inequalities (see Proposition 14).

Let B be a compact metric space, and recall that a subset $A \subset C(B)$ is said to be equicontinuous if there is a continuous function $\omega(\varepsilon) > 0$, defined for $0 < \varepsilon \le \operatorname{diam}(B)$, with $\lim_{\varepsilon \to 0^+} \omega(\varepsilon) = 0$ so that $d(x, y) < \varepsilon$ implies $|f(x) - f(y)| < \omega(\varepsilon)$ for all $f \in A$. Such a function $\omega(\varepsilon)$ is called a *modulus of continuity* for A.

Let $B \subset \mathbb{R}^d$ be convex and compact with nonempty interior, and let ν be the Lebesgue measure on B. If M is a linear subspace of $C(B)$ consisting of functions differentiable in the interior of B, then the *Bernstein–Markov Factor* of M is

$$b(M) := \sup_{m \in M} \frac{\|m'\|_\infty}{\|m\|_\infty}.$$

(Here m' stands for the gradient of m, and $\|m'\|_\infty$ is the sup of the ℓ_2^d-norm of m'.)

Proposition 13 *Let M be a subspace of $C(B)$ of dimension > 1. Let $B \subset \mathbb{R}^d$ be convex and compact with nonempty interior, and let ν be the Lebesgue measure on B.*

(i) *Assume that the unit ball of* M, *under the uniform norm*, *is equicontinuous with modulus of continuity* $\omega(\varepsilon)$. *Then there is a constant* $C > 0$, *depending only upon* B, *so that*

$$\alpha^*(M) \geq C \max_{t\in(0,1]} (1-t)\big(\omega^{-1}(t)\big)^d \geq \frac{C}{2}\big(\omega^{-1}(1/2)\big)^d.$$

(ii) *Assume*, *in addition*, *that the functions in* M *are differentiable in the interior of* B. *Then there is a constant* $C > 0$, *depending only upon* B, *such that*

$$\alpha^*(M) \geq \frac{C}{b(M)^d}.$$

Proof We shall use the simple geometric observation that there is a constant $c > 0$, depending only upon B, so that for any ball $B(y,\varepsilon)$ centered at some point $y \in B$ and of radius $0 < \varepsilon \leq \operatorname{diam}(B)$, we have

$$\nu\big(B(y,\varepsilon)\cap B\big) \geq c\varepsilon^d.$$

(i) We shall show that

$$A_\infty = \sup_{m\in M} \frac{\|m\|_\infty}{\|m\|_1} \leq \frac{1}{c(1-t)(\omega^{-1}(t))^d}$$

for every $t \in (0,1]$. The result then follows from Proposition 12, with $C = c/2$. Since $\dim M > 1$, there exists a $\widetilde{m} \in M$ with $\|\widetilde{m}\|_\infty = 1$ which vanishes at some point in B. Thus, the range of ω includes $(0,1]$ and therefore the value t. Let $m \in M$ satisfy $\|m\|_\infty = 1$, and let $y \in B$ be such that $|m(y)| = 1$. Taking $\varepsilon = \omega^{-1}(t)$, we obtain that $|m(z)| \geq 1-t$ whenever $z \in B(y,\varepsilon)\cap B$. Thus,

$$\begin{aligned}\|m\|_1 &\geq \int_{B(y,\varepsilon)\cap B} |m|\,d\nu \geq (1-t)\nu\big(B(y,\varepsilon)\cap B\big) \geq (1-t)\,c\,\varepsilon^d \\ &= c\,(1-t)\,\big(\omega^{-1}(t)\big)^d.\end{aligned}$$

(ii) By the Mean Value theorem every $m \in M$ with $\|m\|_\infty = 1$ satisfies

$$\big|m(x)-m(y)\big| \leq \|y-x\|\,\big\|m'\big\|_\infty \leq b(M)\|y-x\|.$$

It follows that M satisfies the conditions of (i) with $\omega(\varepsilon) \leq b(M)\varepsilon$ for $0 < \varepsilon \leq \operatorname{diam}(B)$, and $\omega^{-1}(1/2) \geq \frac{1}{2b(M)}$. □

Remark (i) The convexity of B was used twice in the proof of Proposition 13. It was used to obtain the lower estimate on the measure of balls centered in B, and used in the application of the Mean Value theorem in part (ii). These properties can also be ensured by suitable geometric conditions for more general subsets of $\mathbb{R}^d$ and for more general compact metric spaces. For example, the Mean Value theorem can be similarly applied for subsets of $\mathbb{R}^d$ for which there is a constant $C > 0$ so that any two points $x, y \in B$ can be connected by a differentiable curve whose length is bounded by $C\|x-y\|$.

(ii) The estimates in Proposition 13 may fail when $\dim M = 1$ because m need not vanish on B. An extreme example of this is when M consists of the constant functions.

Let B be a compact subset of $\mathbb{R}^d$, and ν the Lebesgue measure on B. The *Remez Factor* of a subspace M of $C(B)$ is given by:

$$r_B(M;\delta) := \sup\left\{\frac{\|m\|_{C(B)}}{\|m\|_{C(B_\delta)}} : m \in M,\ B_\delta \subseteq B,\ \nu(B_\delta) \geq (1-\delta)\nu(B)\right\}.$$

Inequalities for Remez factors imply Nikolski-type inequalities. We prove the following result.

Proposition 14 *Let M be a linear subspace of $C(B)$, with B, ν, and $r_B(M;\delta)$ as above. Then*

$$\alpha^*(M) \geq \sup_{\{\delta:0<\delta<1\}} \frac{\delta\nu(B)}{2r_B(M;\delta)}.$$

Proof Let $m \in M$ be such that $\|m\|_1 = 1$, and fix $\delta \in (0,1)$. Set $Q(m;\delta) = \{x : |m(x)| \geq 1/(\delta\nu(B))\}$. Then

$$1 = \|m\|_1 = \int_B \big|m(x)\big|\,d\nu(x) \geq \int_{Q(m;\delta)} \big|m(x)\big|\,d\nu(x) \geq \frac{\nu(Q(m;\delta))}{\delta\nu(B)};$$

hence,

$$\nu\big(B\backslash Q(m;\delta)\big) = \nu(B) - \nu\big(Q(m;\delta)\big) \geq (1-\delta)\nu(B).$$

As $\|m\|_{C(B\backslash Q(m;\delta))} \leq 1/(\delta\nu(B))$, the definition of $r_B(M;\delta)$ gives

$$\|m\|_{C(B)} \leq r_B(M;\delta)\|m\|_{C(B\backslash Q(m;\delta))} \leq \frac{r_B(M;\delta)}{\delta\nu(B)},$$

which implies (the Nikolskii-type inequality)

$$A_\infty \leq \frac{r_B(M;\delta)}{\delta\nu(B)}.$$

□

Remark Assume that $\nu(B)$ is finite and, to simplify notation, that $\nu(B) = 1$. Analogous to the Remez factor with respect to the $C(B)$ norm, one can also define the Remez factor with respect to the L^1 norm by

$$r_B^1(M;\delta) := \sup\left\{\frac{\|m\|_{L^1(B)}}{\|m\|_{L^1(B_\delta)}} : m \in M,\ B_\delta \subseteq B,\ \nu(B_\delta) \geq 1-\delta\right\}.$$

The L^1 Remez factor is closely related to the modulus of uniform integrability of the unit ball of M. Passing to complements, we can rewrite $r_B^1(M;\delta)$ as

$$\sup\left\{\frac{\|m\|_{L^1(B)}}{\|m\|_{L^1(N^c)}} : m \in M,\ N \subset B,\ \nu(N) \leq \delta\right\}$$
$$= 1 + \sup\left\{\frac{\|m\|_{L^1(N)}}{\|m\|_{L^1(N^c)}} : m \in M,\ N \subset B,\ \nu(N) \leq \delta\right\},$$

where N^c is the complement to N in B. Rewriting

$$\alpha^*(M) = \sup\left\{\alpha : \sup_{m\in M} \frac{|||m|||_\alpha}{\|m\|_1} \leq \frac{1}{2}\right\}$$

as the largest α for which

$$\sup\left\{\frac{\|m\|_{L^1(N)}}{\|m\|_{L^1(N^c)}} : m \in M,\ N \subseteq B,\ \nu(N) \le \alpha\right\} \le 1,$$

it follows that $\alpha^*(M)$ is the largest $\alpha > 0$ for which

$$r_B^1(M;\alpha) \le 2.$$

Unfortunately, we have found no Remez factors with respect to the L^1 norm that have proved relevant here.

We now consider upper bounds on $\alpha^*(M)$. If the M_n are a nested sequence of n-dimensional subspaces that are fundamental, i.e., for which

$$\lim_{n\to\infty} \min_{m\in M_n} \|f-m\|_1 = 0 \tag{4}$$

for all $f \in L^1(B,\nu)$, then necessarily $\lim_{n\to\infty}\alpha^*(M_n) = 0$. Indeed, $\alpha^*(M_n)$ is a nonincreasing function of n, and if $\alpha^*(M_n) \ge c > 0$ for all n, then (4) cannot hold for any f with $\nu(N(f)) < c$. The converse need not hold, as may be easily verified.

Certain basic properties associated with good approximating subspaces imply small upper bounds on $\alpha^*(M_n)$.

We recall that an n-dimensional subspace M_n of $C[a,b]$ is said to be a weak Tchebycheff (WT)-system on $[a,b]$ if every $m \in M_n$ has at most $n-1$ sign changes on $[a,b]$. That is, there does not exist an $m \in M_n$ and points $a \le x_1 < \cdots < x_{n+1} \le b$ for which $m(x_i)m(x_{i+1}) < 0$, $i = 1,\dots,n$.

Proposition 15 *Let ν be a finite nonatomic positive measure on $[a,b]$ and M_n an n-dimensional weak Tchebycheff (WT)-system on $[a,b]$. Then*

$$\alpha^*(M_n) \le \frac{\nu([a,b])}{n+1}.$$

Proof By the Hobby–Rice theorem, see, e.g., Pinkus [30, p. 208], there exist n points $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$ such that

$$\sum_{i=0}^{n}(-1)^i \int_{x_i}^{x_{i+1}} m(x)\,d\nu(x) = 0 \tag{5}$$

for all $m \in M_n$.

Fix j so that

$$\nu\big([x_j,x_{j+1}]\big) \le \frac{\nu([a,b])}{n+1}.$$

By Zielke [36, Lemma 4.1], there is an $m \in M_n$, $m \ne 0$, that weakly changes sign at all the x_i in (a,b) except for x_j and x_{j+1}. That is, $(-1)^i \operatorname{sgn} m(x) \ge 0$ for $x \in [x_i,x_{i+1}]$, $i \ne j$, while $(-1)^j \operatorname{sgn} m(x) \le 0$ for $x \in [x_j,x_{j+1}]$. From (5) it therefore follows that

$$\int_{x_j}^{x_{j+1}} \big|m(x)\big|\,d\nu(x) = \int_{[a,b]\setminus[x_j,x_{j+1}]} \big|m(x)\big|\,d\nu(x).$$

As m cannot vanish identically on either $[x_j, x_{j+1}]$ or $[a,b]\backslash[x_j, x_{j+1}]$, we have

$$\alpha^*(M_n) \le \nu\big([x_j, x_{j+1}]\big) \le \frac{\nu([a,b])}{n+1}. \qquad \square$$

From the above proof we have the more exact:

Corollary 16 *Let ν be a finite nonatomic positive measure on $[a,b]$, and let M_n be an n-dimensional weak Tchebycheff (WT)-system on $[a,b]$. Let $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$ be the associated Hobby–Rice points. Then*

$$\alpha^*(M_n) \le \min_{0\le i\le n} \nu\big([x_i, x_{i+1}]\big).$$

We will use both Proposition 15 and Corollary 16 in the next section.

4 Examples

In this and the next section we provide estimates on $\alpha^*(M)$ for various specific M.

4.1 Trigonometric Polynomials, Functions of Exponential Type and More

Example 1 Let $B = (-\pi, \pi]$, and set

$$\|f\|_p = \left(\int_{-\pi}^{\pi} \left|f(x)\right|^p dx\right)^{1/p}$$

for $p \in [1,\infty)$ with the usual definition of $\|f\|_\infty$. Let $\mathcal{T}_n$ denote the space of trigonometric polynomials of degree n. From Ibragimov [12]; Timan [34, p. 229]; see also DeVore, Lorentz [5, p. 102]; and Milovanović, Mitrinović, Rassias [23, p. 497]; we have the Nikolskii-type inequalities

$$\|T\|_p \le \left(\frac{2nr+1}{2\pi}\right)^{\frac{1}{q}-\frac{1}{p}} \|T\|_q$$

for every $T \in \mathcal{T}_n$, where r is the least integer $\ge q/2$. (The correct asymptotics with a worse constant may be found in Nikolskii [27], and in Jackson [14] for $p = \infty$ and $q = 1$.) Taking $p = \infty$ and $q = 1$ gives

$$\|T\|_\infty \le \left(\frac{2n+1}{2\pi}\right)\|T\|_1.$$

In fact, a better bound was obtained by Taikov [33]; namely,

$$\|T\|_\infty \le \left(\frac{c_n n}{2\pi}\right)\|T\|_1,$$

where $c_n \in (1.078, 1.16) + o(1)$. Bounds on c_n have been improved upon, see Gorbachev [10] and references therein. Thus, $A_\infty \le (c_n n)/(2\pi)$, implying, by Proposition 12, the lower bound

$$\alpha^*(\mathcal{T}_n) \ge \frac{\pi}{c_n n}.$$

It is known (and may be easily verified) that the $2n+2$ equally spaced points on $[-\pi,\pi]$ satisfy the Hobby–Rice theorem for $\mathcal{T}_n$. As $\mathcal{T}_n$ is of dimension $2n+1$, this implies by Proposition 15 that

$$\alpha^*(\mathcal{T}_n) \le \frac{2\pi}{2n+2} = \frac{\pi}{n+1}.$$

Thus,

$$\frac{\pi}{2n+1} \le \frac{\pi}{c_n n} \le \alpha^*(\mathcal{T}_n) \le \frac{\pi}{n+1}.$$

Example 2 Let $B=[-\pi,\pi]^d$, and

$$\|f\|_p = \left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|f(x_1,\dots,x_d)\right|^p dx_1\cdots dx_d\right)^{1/p}$$

for $p\in[1,\infty)$ with the usual definition of $\|f\|_\infty$. Let K be any finite subset of $\mathbb{Z}^d$, and let $|K|$ denote the cardinality (number of points) in K. In Nessel, Wilmes [25], it is proven that for each $T\in\mathcal{T}_K=\operatorname{span}\{\exp(ik\cdot x):k\in K\}$, we have

$$\|T\|_p \le \left(\frac{|K|}{(2\pi)^d}\right)^{\frac{1}{q}-\frac{1}{p}}\|T\|_q$$

for $1\le q\le 2$, $q\le p\le\infty$. We use this inequality for $p=\infty$ and $q=1$; namely,

$$\|T\|_\infty \le \frac{|K|}{(2\pi)^d}\|T\|_1,$$

and we provide the elementary proof as given in [25]. Let

$$D(x) := \sum_{k\in K}\exp(ik\cdot x)$$

denote the corresponding Dirichlet kernel. Since $T=D*T$ for all $T\in\mathcal{T}_K$, and $\|D\|_2=(|K|/(2\pi)^d)^{1/2}$, then from the inequalities

$$\|T\|_\infty = \|D*T\|_\infty \le \|D\|_2\|T\|_2 = \|D\|_2\|D*T\|_2 \le \|D\|_2^2\|T\|_1 = \frac{|K|}{(2\pi)^d}\|T\|_1,$$

we obtain the desired result. Thus,

$$\alpha^*(\mathcal{T}_K) \ge \frac{(2\pi)^d}{2|K|}.$$

Let

$$\mathcal{T}_m = \bigcup\bigl\{\mathcal{T}_K : |K|\le m\bigr\}.$$

Note that $\mathcal{T}_m$ is a not a linear subspace. Nonetheless, it is a homogeneous subset, and we have

$$\alpha^*(\mathcal{T}_m) \ge \frac{(2\pi)^d}{2m}.$$

That is, if f is a function defined on $[-\pi,\pi)^d$ whose support is of measure at most $(2\pi)^d/(2m)$, then the zero function is a best L^1-approximant from $\mathcal{T}_m$.

What about upper bounds? In general, $\alpha^*(\mathcal{T}_K)$ depends upon arithmetic and combinatorial properties of K, and there are no nontrivial upper estimates for it. In fact, there are known infinite sets K for which $\alpha^*(\mathcal{T}_K) > 0$. Recall that $K \subset \mathbb{Z}^d$ is called a Λ_p set ($p > 1$) if the L^1 and L^p norms are equivalent on $\mathcal{T}_K$; i.e., $A_p < \infty$ for $\mathcal{T}_K$. The constant A_p for $\mathcal{T}_K$ is called the Λ_p constant of K, and we recall that by Proposition 12 we have $\alpha^*(\mathcal{T}_K) \geq \frac{1}{(2A_p)^{p'}}$. We refer the reader to Rudin [31] for an early exposition of this classical notion. We just mention here that (for $d = 1$) if $K = \{n_k\}$ is a lacunary sequence, i.e., if it satisfies $\inf \frac{n_{k+1}}{n_k} > 1$, then it is already proven in Zygmund [37] that K is a Λ_p set for all $p < \infty$. Of course, if $K = \{-n, \dots, 0, \dots, n\}$ then $\mathcal{T}_K = \mathcal{T}_n$, as in Example 1. The analogous result holds whenever K is any set of consecutive integers in $\mathbb{Z}$.

In certain cases, we have upper bounds that asymptotically agree with the lower bounds. For example, let $\mathcal{T}_n^d$ denote the space of real trigonometric polynomials of total degree at most n. That is, $\mathcal{T}_n^d$ is the real subspace generated by $\operatorname{span}\{\exp(ik \cdot x) : |k_1| + \cdots + |k_d| \leq n\}$. Note that the number of such coefficients k is of the order of n^d, and thus,

$$\alpha^*\bigl(\mathcal{T}_n^d\bigr) \geq \frac{C}{n^d}$$

for some constant C. We prove an upper bound of the same order with some other generic constant C:

Proposition 17 *For $\mathcal{T}_n^d$, as above, we have*

$$\alpha^*\bigl(\mathcal{T}_n^d\bigr) \leq \frac{C}{n^d}$$

for some constant C.

Proof By the multivariate Jackson Theorem, see Timan [34, p. 273], for any $f \in L^1(B, \nu)$, we have

$$E_n(f)_{L^1} := \inf_{t \in \mathcal{T}_n^d} \|f - t\|_{L^1} \leq c \sum_{j=1}^{d} \omega_j(f, 1/n)_{L^1},$$

where $\omega_j(f, \cdot)_{L^1}$ denotes the L^1-modulus of continuity with respect to the jth variable, and c is some generic constant.

Let A be any cube in B with edge length a, and denote by χ_A the indicator function of A; i.e., $\chi_A = 1$ on A, and 0 otherwise. Clearly, for any $h > 0$, we have

$$\omega_j(\chi_A, h)_{L^1} \leq 2a^{d-1}h, \quad j = 1, \dots, d.$$

Thus, by Jackson's theorem,

$$E_n(\chi_A)_{L^1} \leq 2cd\frac{a^{d-1}}{n}.$$

Set $a := (\alpha^*(\mathcal{T}_n^d))^{1/d}$. Then $\nu(A) = a^d = \alpha^*(\mathcal{T}_n^d)$. By the definition of $\alpha^*(\mathcal{T}_n^d)$, the zero function is a best L^1-approximant to χ_A from $\mathcal{T}_n^d$; i.e.,

$$\alpha^*\bigl(\mathcal{T}_n^d\bigr) = \nu(A) = \|\chi_A\|_1 = E_n(\chi_A)_{L^1} \leq 2cd\frac{a^{d-1}}{n} = 2cd\frac{\alpha^*(\mathcal{T}_n^d)^{(d-1)/d}}{n},$$

that yields

$$\alpha^*\left(\mathcal{T}_n^d\right) \leq \left(\frac{2cd}{n}\right)^d. \qquad \square$$

Example 3 In this and the next two examples, $B = \mathbb{R}^d$, and we take the usual $L^1(\mathbb{R}^d)$ norm. From Nessel, Wilmes [25], we also have the following result. Let $f \in L^1(\mathbb{R}^d)$, and assume its Fourier transform $\hat{f}$ has compact support. Then $f \in L^\infty(\mathbb{R}^d)$ and

$$\|f\|_\infty \leq \left(\frac{|\operatorname{supp} \hat{f}|}{(2\pi)^d}\right)\|f\|_1,$$

where $|\operatorname{supp} \hat{f}|$ is the Lebesgue measure of the support of $\hat{f}$. The proof of this fact is similar to the proof of the analogous result in the previous example. Thus, if K is any compact set of finite measure, and $\mathcal{S}_K$ denotes the space of functions in $L^1(\mathbb{R}^d)$ whose Fourier transform have their support in K, then

$$\alpha^*(\mathcal{S}_K) \geq \frac{(2\pi)^d}{2|K|}.$$

And if

$$\mathcal{S}_\beta = \bigcup \{\mathcal{S}_K : |K| \leq \beta\},$$

then

$$\alpha^*(\mathcal{S}_\beta) \geq \frac{(2\pi)^d}{2\beta}.$$

Note that $\mathcal{S}_\beta$ is a homogeneous set, but is not a linear subspace. The above states that if f is a function in $L^1(\mathbb{R}^d)$ whose support is of measure at most $(2\pi)^d/(2\beta)$, then the zero function is a best L^1-approximant from $\mathcal{S}_\beta$.

Example 4 Let $\mathcal{G}_{\sigma_1,\ldots,\sigma_d}$ denote the space of entire functions f defined on $\mathbb{C}^d$ of rectangular exponential type $\sigma_1,\ldots,\sigma_d > 0$. That is, $f \in \mathcal{G}_{\sigma_1,\ldots,\sigma_d}$ if f is entire and for every $\varepsilon > 0$ there exists a constant C_ε such that

$$\left|f(z)\right| \leq C_\varepsilon \exp\left\{\sum_{k=1}^d (\sigma_k + \varepsilon)|z_k|\right\}$$

for all $z \in \mathbb{C}^d$. When $d = 1$, we have that $f \in \mathcal{G}_\sigma$ if

$$f(z) = \sum_{k=0}^\infty a_k z^k,$$

where

$$\limsup_{k\to\infty} \left(k!|a_k|\right)^{1/k} \leq \sigma.$$

From Ibragimov [12]; see also Nessel, Wilmes [25] (and references therein); Nikolskii [27]; Timan [34, p. 234]; and Nikolskii [28, p. 126]; we have the following

Nikolskii-type inequality. Let $f \in L^1(\mathbb{R}^d)$ be the restriction to $\mathbb{R}^d$ of an entire function of rectangular exponential type $\sigma_1, \dots, \sigma_d > 0$. Then f belongs to $L^\infty(\mathbb{R}^d)$, and

$$\|f\|_\infty \le \left(\prod_{k=1}^d \frac{\sigma_k}{\pi}\right)\|f\|_1.$$

This therefore implies that

$$\alpha^*(\mathcal{G}_{\sigma_1,\dots,\sigma_d}) \ge \frac{1}{2}\left(\prod_{k=1}^d \frac{\pi}{\sigma_k}\right).$$

Example 5 From Nessel, Wilmes [25], we also have the following result. Let $\mathcal{H}_\sigma$ denote the space of entire functions defined on $\mathbb{C}^d$ of radial type $\sigma > 0$. That is, $f \in \mathcal{H}_\sigma$ if f is entire and for every $\varepsilon > 0$ there exists a constant C_ε such that

$$\left|f(z)\right| \le C_\varepsilon \exp\left\{(\sigma + \varepsilon)|z|\right\}$$

for all $z \in \mathbb{C}^d$. If $f \in L^1(\mathbb{R}^d)$ is the restriction to $\mathbb{R}^d$ of an entire function of radial type σ, then

$$\|f\|_\infty \le \left(\frac{\sigma^d}{d\Gamma(d/2)2^{d-1}\pi^{d/2}}\right)\|f\|_1.$$

This therefore implies the lower bound

$$\alpha^*(\mathcal{H}_\sigma) \ge \left(\frac{d\Gamma(d/2)2^{d-2}\pi^{d/2}}{\sigma^d}\right).$$

4.2 Algebraic Polynomials, Splines, Müntz Polynomials and More

Example 6 Let $B = [0, 1]$, and Π_n denote the set of algebraic polynomials of degree at most n. In Ho Tho Kau [11]; see also Amir, Ziegler [1]; it is shown that $A_\infty \le (n+1)^2$, implying the lower bound

$$\alpha^*(\Pi_n) \ge \frac{1}{2(n+1)^2}.$$

(The standard Nikolskii-type inequalities as found in Timan [34, p. 236] and DeVore, Lorentz [5, p. 102] are somewhat weaker.) The points that satisfy the Hobby–Rice theorem are known. They are the zeros of the Chebyshev polynomials of the second kind, renormalized to the interval [0, 1]. As such,

$$\min_{0\le i\le n}\{x_{i+1} - x_i\} = x_1 - x_0 = x_{n+1} - x_n = \frac{1 - \cos(\pi/(n+2))}{2} \le \frac{\pi^2}{4(n+2)^2}.$$

Thus, from Corollary 16,

$$\frac{1}{2(n+1)^2} \le \alpha^*(\Pi_n) \le \frac{\pi^2}{4(n+2)^2}.$$

Example 7 Let $B=\mathbb{R}$, and Π_n denote the set of algebraic polynomials of degree at most n. As a special case of Mhaskar [22], we have the following Nikolskii-type inequalities. For $\gamma \geq 2$, let $w_\gamma(x)=e^{-|x|^\gamma}$. Then, for each $P \in \Pi_n$, we have

$$\|w_\gamma P\|_p \leq \left(\gamma^{1/\gamma} n^{1-1/\gamma}\right)^{\frac{1}{q}-\frac{1}{p}} \|w_\gamma P\|_q$$

for every $1 \leq q \leq p \leq \infty$. Taking $p=\infty$ and $q=1$ gives

$$\|w_\gamma P\|_\infty \leq \left(\gamma^{1/\gamma} n^{1-1/\gamma}\right) \|w_\gamma P\|_1.$$

Thus, $A_\infty \leq \gamma^{1/\gamma} n^{1-1/\gamma}$, implying the lower bound

$$\alpha^*(w_\gamma \Pi_n) \geq \frac{1}{2\gamma^{1/\gamma} n^{1-1/\gamma}}.$$

This example was generalized by Nevai, Totik [26] to the case where $0<\gamma<2$. They proved that

$$\|w_\gamma P\|_\infty \leq c \Lambda_n(\gamma) \|w_\gamma P\|_1$$

for some constant c that depends only upon γ, where

$$\Lambda_n(\gamma)=\begin{cases} n^{1-1/\gamma}, & 1<\gamma<2, \\ \ln n, & \gamma=1, \\ 1, & 0<\gamma<1. \end{cases}$$

Thus, we obtain

$$\alpha^*(w_\gamma \Pi_n) \geq \frac{C}{n^{1-1/\gamma}}$$

for $1<\gamma<2$, while for $\gamma=1$,

$$\alpha^*(w_1 \Pi_n) \geq \frac{C}{\ln n},$$

and for $0<\gamma<1$,

$$\alpha^*(w_\gamma \Pi_n) \geq C$$

for some constants $C>0$ that depend only upon γ. (Note that the last of these lower bounds does not tend to 0 as $n \to \infty$.) Nikolskii-type inequalities for other weighted algebraic polynomials on all of $\mathbb{R}$ (with properties similar to those in the next Example 8) may be found in Mthembu [24].

Example 8 Let $B=[-1,1]$, and Π_n denote the set of algebraic polynomials of degree at most n. Lubinsky, Saff [21] consider Nikolskii-type inequalities for algebraic polynomials on B with weights of the form $w:=\exp(-Q)$ where Q satisfies:

(i) Q is even and continuously differentiable in $(-1,1)$, while Q'' is continuous in $(0,1)$;
(ii) $Q' \geq 0$ and $Q'' \geq 0$ in $(0,1)$;
(iii) $\int_0^1 t Q'(t)/\sqrt{1-t^2}\, dt=\infty$;

(iv) the function

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}, \quad x \in (0,1),$$

is increasing in $(0,1)$, $T(0+) > 1$ and $T(x) = O(Q'(x))$, as $x \to 1-$.

The constants $a_m := a_m(Q)$, defined by

$$m = \frac{2}{\pi}\int_0^1 \frac{a_m t Q'(a_m t)}{\sqrt{1-t^2}}\,dt,$$

are called the mth *Mhaskar–Rahmanov–Saff numbers*. Lubinsky, Saff [21] proved that for every such weight w and for $P \in \Pi_n$, we have

$$\|wP\|_p \le c\big(nT(a_{2n})^{1/2}\big)^{1/q-1/p}\|wP\|_q$$

for all $0 < q < p \le \infty$ for some universal constant c. Setting $p = \infty$ and $q = 1$, we obtain

$$\alpha^*(w\Pi_n) \ge \frac{C}{nT(a_{2n})^{1/2}}$$

for some constant C.

Example 9 Let $B = [-1, 1]$ and GAP_n denote the set of all generalized nonnegative algebraic polynomials of degree n, i.e., the set of functions

$$P(x) = \lambda \prod_{j=1}^{m} |x - x_j|^{r_j},$$

where $\lambda \in \mathbb{R}$, $r_j > 0$ (not necessarily integers), $x_j \in \mathbb{C}$, and

$$n := \sum_{j=1}^{m} r_j.$$

(Note that the m is arbitrary and n is not necessarily an integer.) GAP_n is not a linear subspace, but it is a homogeneous set. From Borwein, Erdélyi [4, p. 395], we have for $0 < q < p \le \infty$ the Nikolskii-type inequalities

$$\|P\|_p \le \left(\frac{e^2(2+qn)}{2\pi}\right)^{2/q-2/p}\|P\|_q$$

for every $P \in \mathrm{GAP}_n$. Setting $p = \infty$ and $q = 1$ gives

$$\|P\|_\infty \le \left(\frac{e^2(2+n)}{2\pi}\right)^2\|P\|_1.$$

Thus,

$$\alpha^*(\mathrm{GAP}_n) \ge \frac{2\pi^2}{e^4(2+n)^2}.$$

Similar results hold for generalized nonnegative trigonometric polynomials, see Borwein, Erdélyi [4, p. 394], where the asymptotics is of order $1/n$.

Example 10 Let $B=[0,1]$ and $\mathcal{S}_{n,r}$ denote the space of splines of degree n with r simple knots at $\{i/(r+1)\}_{i=1}^r$. That is, $\mathcal{S}_{n,r}$ is the subspace of functions in $C^{n-1}[0,1]$ that, when restricted to each $[(i-1)/(r+1), i/(r+1)]$, $i=1,\dots,r+1$, are algebraic polynomials of degree at most n. We have, for $\mathcal{S}_{n,r}$,

$$\frac{1}{2(r+1)(n+1)^2} \le \alpha^*(\mathcal{S}_{n,r}) \le \frac{1}{n+r+2}.$$

The upper bound is a consequence of Proposition 15, since $\mathcal{S}_{n,r}$ is a WT-system of dimension $n+r+1$. The lower bound follows from the estimate in Example 6: Let $\mathcal{Q}_{n,r}$ denote the space of functions whose restriction to $[(i-1)/(r+1), i/(r+1)]$, $i=1,\dots,r+1$, are algebraic polynomials of degree at most n; i.e., there are no continuity restrictions at the knots $\{i/(r+1)\}_{i=1}^r$. As $\mathcal{S}_{n,r}\subseteq\mathcal{Q}_{n,r}$, we have

$$A_\infty = \sup_{s\in\mathcal{S}_{n,r}} \frac{\|s\|_\infty}{\|s\|_1} \le \sup_{q\in\mathcal{Q}_{n,r}} \frac{\|q\|_\infty}{\|q\|_1}.$$

From Example 6, we have

$$\|P\|_\infty \le (n+1)^2\|P\|_1$$

for every $P\in\Pi_n$ on $[0,1]$. A simple change of variable argument therefore implies that

$$\|q\|_\infty \le (r+1)(n+1)^2\|q\|_1$$

for every $q\in\mathcal{Q}_{n,r}$ which vanishes on r of the $r+1$ intervals $[(i-1)/(r+1), i/(r+1)]$, $i=1,\dots,r+1$, hence for every $q\in\mathcal{Q}_{n,r}$. This gives the lower bound for $\alpha^*(\mathcal{S}_{n,r})$.

When $n=0$, i.e., $\mathcal{S}_{0,r}$ is the space of piecewise constants with knots at $\{i/(r+1)\}_{i=1}^r$, then it is readily verified that $\alpha^*(\mathcal{S}_{0,r})=1/2(r+1)$.

Example 11 Let $B=[0,1]$. We will look at a subclass of Müntz polynomials. Let $0=\lambda_1<\lambda_2<\cdots<\lambda_n$, where $\lambda_{k+1}-\lambda_k\ge 1$ for every k. Set $\Lambda_n=\operatorname{span}\{x^{\lambda_1},\dots,x^{\lambda_n}\}$. Then, see Borwein, Erdélyi [4, p. 298], we have the Nikolskii-type inequalities

$$\|g\|_p \le \left(18\cdot 2^q\sum_{k=1}^n \lambda_k\right)^{1/q-1/p}\|g\|_q,$$

for all $g\in\Lambda_n$, and for any $0<q<p\le\infty$. Setting $q=1$ and $p=\infty$, we obtain

$$\alpha^*(\Lambda_n) \ge \frac{1}{72\sum_{k=1}^n\lambda_k}.$$

Note that as $\lambda_{k+1}-\lambda_k\ge 1$ for every k, it follows that $72\sum_{k=1}^n\lambda_k\ge 36n(n-1)$. (From the Bernstein inequality in Borwein, Erdélyi [4, p. 287] and Proposition 13(ii), we get a similar estimate.) Λ_n is a WT-system on $[0,1]$ for any choice of $0\le\lambda_1<\lambda_2<\cdots<\lambda_n$. As such, Proposition 15 gives

$$\alpha^*(\Lambda_n) \le \frac{1}{n+1},$$

which is undoubtedly not sharp, as it is independent of the values of the λ_k's.

Example 12 Let $B=[0,\infty)$, and $\Gamma_n[0,\infty]=\operatorname{span}\{e^{-\gamma_1 x},\dots,e^{-\gamma_n x}\}$, where the γ_k are distinct positive numbers. Then, see Borwein, Erdélyi [4, p. 281], we have the Nikolskii-type inequalities over $[0,\infty)$ of

$$\|g\|_p \le \left(18\cdot 2^q \sum_{k=1}^n \gamma_k\right)^{1/q-1/p} \|g\|_q,$$

for any $0<q<p\le\infty$ and every $g\in\Gamma_n[0,\infty]$. Set $q=1$ and $p=\infty$ to obtain

$$\alpha^*\big(\Gamma_n[0,\infty]\big) \ge \frac{1}{72\sum_{k=1}^n \gamma_k}.$$

Note that as there is no gap condition on the $\{\gamma_k\}$ (as in the previous Example 11), then for any $c>0$ we can find an infinite number of distinct positive numbers $\{\gamma_k\}$ such that

$$\alpha^*\big(\Gamma_n[0,\infty]\big) \ge c$$

for all n.

Example 13 Let $B=[a,b]$ be any finite interval, and $\Gamma_n[a,b]=\operatorname{span}\{e^{-\gamma_1 x},\dots,e^{-\gamma_n x}\}$, where the γ_k are distinct real numbers. From Erdélyi [9], we have the Nikolskii-type inequalities over $[a,b]$ of

$$\|g\|_p \le c\left(n^2+\sum_{k=1}^n |\gamma_k|\right)^{1/q-1/p} \|g\|_q,$$

for any $0<q<p\le\infty$ and every $g\in\Gamma_n[a,b]$. The constant c depends upon p, q, a and b. Set $q=1$ and $p=\infty$ to obtain

$$\alpha^*\big(\Gamma_n[a,b]\big) \ge \frac{C}{n^2+\sum_{k=1}^n |\gamma_k|}$$

for some constant C depending on a and b.

Example 14 For a convex body K in $\mathbb{R}^d$, we denote by ω_K its width, i.e., the minimal distance between two parallel supporting hyperplanes of K. A set B is said to be *noncuspidal* if there exists a constant $c_B>0$ such that each point of B is contained in some convex subset $K\subseteq B$ whose width is larger than c_B.

Let B be a compact noncuspidal subset of $\mathbb{R}^d$, and let ν be the d-dimensional Lebesgue measure on B. Let Π_n^d denote the space of algebraic polynomials of total degree at most n; that is,

$$\Pi_n^d := \left\{\sum_{|k|\le n} a_k x^k : a_k\in\mathbb{R},\ k\in\mathbb{Z}_+^d,\ x\in\mathbb{R}^d\right\},$$

where for $x=(x_1,\dots,x_d)\in\mathbb{R}^d$ and $k=(k_1,\dots,k_d)\in\mathbb{Z}_+^d$, we set $x^k=x_1^{k_1}\cdots x_d^{k_d}$ and $|k|=k_1+\dots+k_d$. From Kroó, Schmidt [19, p. 426], we have that

$$r_B\big(\Pi_n^d;\delta\big) \le \exp\big(c' n\delta^{1/(2d)}\big).$$

(If B is convex, then we can set $c' = 6$.) Choosing $\delta = (c'n)^{-2d}$,

$$r_B\big(\Pi_n^d; \delta\big) \le e,$$

and by Proposition 14 and the choice of δ,

$$\alpha^*\big(\Pi_n^d\big) \ge \sup_{\{\delta: 0<\delta<1\}} \frac{\delta \nu(B)}{2 r_B(\Pi_n^d; \delta)} \ge \frac{c}{n^{2d}}$$

for some constant $c > 0$.

It is known, see Wilhelmsen [35], that the Bernstein–Markov factor $b(\Pi_n^d)$ is bounded above by $4n^2/c_B$. Thus, if B is convex, then Proposition 13(ii) also gives the lower bound

$$\alpha^*\big(\Pi_n^d\big) \ge \frac{c}{n^{2d}}.$$

This includes the case $d = 1$ considered in Example 6, but there we have an explicit constant c.

Remark Kroó, Saff, Yattselev [20] studied the Remez factors of homogeneous polynomials H_n^d in d variables of degree n, $d \ge 2$, on star-like surfaces, namely on images $\mathcal{S}_r$ of S^{d-1} under maps of the form $u \to r(u)u$, where $r : S^{d-1} \to \mathbb{R}_+$ is an even Lip α function. Under these assumptions, the surface area is well defined, and they obtained tight estimates with respect to this measure. In particular, when $\alpha = 1$ (for example when the interior of $\mathcal{S}_r$ is convex), they obtained that

$$r_{\mathcal{S}_r}\big(H_n^d; \delta\big) \le \exp\bigg(cn\delta^{1/(d-1)} \ln \frac{1}{\delta}\bigg).$$

Thus,

$$\alpha^*\big(H_n^d\big) \ge \left(\frac{c}{n \ln n}\right)^{d-1}.$$

If r is smooth, the ln terms can be eliminated in both formulae.

Example 15 As above, let $B \subset \mathbb{R}^d$ be a compact set, Π_n^d denote the space of algebraic polynomials of total degree at most n, and ν be the usual d-dimensional Lebesgue measure on B. For each $x \in B$, let $R_B(x)$ denote the radius of the largest ball contained in B such that x is on the surface of this ball. Set

$$R(B) := \inf_{x \in B} R_B(x).$$

We say that the compact $B \subset \mathbb{R}^d$ is *smooth* if $R(B) > 0$. This condition essentially requires that B have C^2 boundary. Under these assumptions on B, it is proven, in Kroó, Schmidt [19], that

$$r_B\big(\Pi_n^d; \delta\big) \le \exp\big(c' n \delta^{1/(d+1)}\big).$$

Choosing $\delta = (c'n)^{-(d+1)}$ gives $r_B(\Pi_n^d; \delta) \le e$. Thus, under these assumptions on B, we have

$$\alpha^*\big(\Pi_n^d\big) \ge \frac{c}{n^{d+1}}$$

for some constant c. Compare this with Example 14.

Remark The notion of C^2-smoothness used in the above example is based on inscribing Euclidean balls into a domain. If we use instead l_p-balls, with $1 \le p \le 2$, then we are led to the more general notion of C^p-smoothness. It is shown, in Kroó [17], that for C^p-domains the Remez factor can be bounded by

$$r_B\big(\Pi_n^d;\delta\big) \le \exp\big(cn\delta^{\frac{p}{2d+2p-2}}\big),$$

hence,

$$\alpha^*\big(\Pi_n^d\big) \ge \frac{c}{n^{\frac{2d+2p-2}{p}}}.$$

Note that when $p=1$ (e.g., when B is convex), this leads to the lower bound of Example 14, while for $p=2$ (C^2-boundary), the lower estimate of Example 15 follows.

In Examples 14 and 15, we gave two different lower bounds for $\alpha^*(\Pi_n^d)$ dependent upon the geometry of $B \subset \mathbb{R}^d$. We will here prove upper bounds that, up to powers of $\ln n$, are of the same orders, and also depend upon the geometry of $B \subset \mathbb{R}^d$. The geometric conditions on B are similar, but different, from those in Examples 14 and 15, and we therefore consider them as distinct examples.

Example 16 As previously, assume that B is a compact subset of $\mathbb{R}^d$, Π_n^d is the space of algebraic polynomials of total degree at most n, and ν is the usual d-dimensional Lebesgue measure on B. We say that B *has a vertex at* $a \in \partial B$ if there exist convex polytopes D_1 and D_2 such that a is a vertex for both D_1 and D_2, and $D_1 \subseteq B \subseteq D_2$.

Proposition 18 *If B, as above, has a vertex, then there exists a constant c, dependent upon B and d but independent of n, such that*

$$\alpha^*\big(\Pi_n^d\big) \le c\left(\frac{\ln n}{n}\right)^{2d}.$$

Proof We assume, without loss of generality, that $a=(-1,0,\dots,0)\in\mathbb{R}^d$ is a vertex of B, and D_1, D_2 are convex polytopes such that a is a vertex for both D_1 and D_2, and $D_1 \subseteq B \subseteq D_2$. We also assume, without loss of generality, that

$$D_2 \subset \big\{x=(x_1,\dots,x_d) : |x_1| \le 1\big\},$$

and if $x \in D_2\backslash\{a\}$, then $x_1 > -1$.

It therefore follows that if

$$B_h = \{x : x \in B, -1 \le x_1 \le -1+h\},$$

then, for all h sufficiently small,

$$c_2 h^d \le \nu(B_h) \le c_1 h^d.$$

Now, there exists a univariate polynomial P of degree n such that $|P(t)| \le 1$ for all $t \in [-1+h, 1]$, while $|P(t)| \ge \exp(c_3 n\sqrt{h})$ for $t \in [-1, -1+h/2]$. (This P can be taken to be the standard Chebyshev polynomial transformed to the interval $[-1+h, 1]$, see Borwein, Erdelyi [4, p. 30].)

Thus, for P and B_h, as above,

$$\int_{B\setminus B_h} \left|P(x_1)\right| d\nu(x) \le \nu(B),$$

while

$$\int_{B_h} \left|P(x_1)\right| d\nu(x) \ge \int_{B_{h/2}} \left|P(x_1)\right| d\nu(x) \ge c_4 h^d \exp\bigl(c_3 n\sqrt{h}\bigr).$$

Setting $h = c_5^2(\ln n/n)^2$, we obtain

$$c_4 h^d \exp\bigl(c_3 n\sqrt{h}\bigr) = c_4 c_5^{2d} \left(\frac{\ln n}{n}\right)^{2d} n^{c_3 c_5}.$$

Thus, for c_5 sufficiently large (but independent of n),

$$c_4 c_5^{2d} \left(\frac{\ln n}{n}\right)^{2d} n^{c_3 c_5} > \nu(B),$$

and therefore

$$\int_{B_h} \left|P(x_1)\right| d\nu(x) > \int_{B\setminus B_h} \left|P(x_1)\right| d\nu(x).$$

This implies that

$$\alpha^*\bigl(\Pi_n^d\bigr) \le \nu(B_h) \le c_1 h^d = c\left(\frac{\ln n}{n}\right)^{2d}. \qquad \square$$

Does a similar upper bound hold for $\alpha^*(\Pi_n^d)$ for all B? It cannot, as is evident from Example 14. In fact, again up to a $(\ln n)^{d+1}$ factor, the asymptotics given in Example 15 are optimal if we assume that B has a C^2 boundary.

Example 17 As previously, we assume that B is a compact subset of $\mathbb{R}^d$, Π_n^d is the space of algebraic polynomials of total degree at most n, and ν is the usual d-dimensional Lebesgue measure on B.

Proposition 19 *If B, as above, has C^2 boundary, then there exists a constant c, dependent upon B and d but independent of n, such that*

$$\alpha^*\bigl(\Pi_n^d\bigr) \le c\left(\frac{\ln n}{n}\right)^{d+1}.$$

Proof The proof is very similar to that of Proposition 18, except that here we use the fact that as the boundary of B is C^2, then there exists a point $a \in \partial B$, and balls B_1, B_2, such that $a \in \partial B_1, \partial B_2$ and $B_1 \subseteq B \subseteq B_2$. To see this, let B_2 be the smallest ball containing B. Then the boundaries of B and B_2 must have nonempty intersection. Let a be in this intersection. By the C^2 smoothness, there exists a ball $B_1 \subseteq B$ with a being on the boundary of B_1.

As above, without loss of generality, let us assume that $a = (-1, 0, \dots, 0) \in \mathbb{R}^d$, and

$$B \subset \bigl\{x = (x_1, \dots, x_d) : |x_1| \le 1\bigr\}.$$

Set $B_h = \{x : x \in B, -1 \leq x_1 \leq -1 + h\}$. As $B_1 \subseteq B \subseteq B_2$, it follows that

$$c_2 h^{(d+1)/2} \leq \nu(B_h) \leq c_1 h^{(d+1)/2}.$$

We now follow the proof of Proposition 18, essentially verbatim. □

Remark In the proof of Proposition 19, we only used the property that there exists a point $a \in \partial B$ and balls B_1, B_2 such that $a \in \partial B_1, \partial B_2$ and $B_1 \subseteq B \subseteq B_2$. This can, of course, hold without the boundary of B being C^2.

Remark It would be interesting to know whether the $\ln n$ terms in Propositions 18 and 19 are necessary. Note that in the trigonometric case, this term does not appear, see Proposition 17.

Example 18 Let $B = [-1, 1]^d$, and Π_n^d be the space of algebraic polynomials of total degree at most n. Ditzian, Tikhonov [6] consider Nikolskii-type inequalities for this space with Jacobi weights w on the cube B. That is, let $w := w_{\alpha,\beta}(x) = \prod_{i=1}^d w_{\alpha_i,\beta_i}(x_i)$, where $w_{\alpha_i,\beta_i}(x_i) = (1 - x_i)^{\alpha_i}(1 + x_i)^{\beta_i}$, $\alpha_i > -1$, $\beta_i > -1$, $\alpha_i + \beta_i > -1$. Then, for all $P \in \Pi_n^d$ and $0 < q < p \leq \infty$, we have

$$\|wP\|_p \leq cn^{\gamma(1/q-1/p)}\|wP\|_q,$$

where c is some constant and $\gamma = \sum_{i=1}^d \max(2 + 2\max\{\alpha_i, \beta_i\}, 1)$. Set $q = 1$ and $p = \infty$ to obtain

$$\alpha^*\big(w\Pi_n^d\big) \geq \frac{C}{n^\gamma}$$

for some other constant C. If $w = 1$, i.e., $\alpha_i = \beta_i = 0$ for all i, we obtain $\alpha^*(\Pi_n^d) \geq (C/n^{2d})$, as also follows from Example 14.

5 Dimension Independent Exact and Lower Bounds

In this section, we present three examples, or rather three families of examples, where $\alpha^*(M)$ is either exactly computed or bounded below by a constant independent of the dimension of M. The common feature of these examples, which makes it relatively easy to do the computations, is that M will have the property that all $m \in M$ with a fixed L^1 norm have the same distribution. Thus, $\alpha^*(M)$ can be computed by considering any $m \in M$, $m \neq 0$.

Finding the optimal $\alpha^*(M)$ for a one-dimensional subspace $M = \text{span}\{m\}$ is intimately connected with the topic of decreasing rearrangements of functions. What immediately follows is mainly taken from Bennett, Sharpley [2], but can also be found in many other sources. We assume, as previously, that ν is a nonatomic measure.

Let $m \in L^1(B, \nu)$. The *distribution function* μ_m of the function $|m|$ is defined on $[0, \infty)$ by

$$\mu_m(\lambda) := \nu\big\{x : \big|m(x)\big| > \lambda\big\}, \quad \lambda \geq 0.$$

μ_m is nonnegative, nonincreasing, and right-continuous on $[0, \infty)$. The *decreasing rearrangement* of m is defined by

$$m^*(t) := \inf\{\lambda : \mu_m(\lambda) \le t\}, \quad t \ge 0,$$

where it is to be understood that the infimum of the empty set is defined as ∞. Note that we have

$$m^*(t) = \sup\{\lambda : \mu_m(\lambda) > t\}, \quad t \ge 0.$$

Thus, m^* may also be regarded as a distribution function (of μ_m) and, as such, is also nonnegative, nonincreasing, and right-continuous on $[0, \infty)$. An important property of m^* is that $|m|$ and m^* are *equimeasurable*, i.e., have the same distribution function, the former with respect to ν and the latter with respect to Lebesgue measure.

An additional important property of m^* is that

$$\int_B \bigl|m(x)\bigr|^p \, d\nu(x) = \int_0^\infty m^*(t)^p \, dt = p \int_0^\infty \lambda^{p-1} \mu_m(\lambda) \, d\lambda$$

for all $p \in (0, \infty)$ (and the integrals are infinite together). Also $\|m\|_\infty = \|m^*\|_\infty$. Our interest is in the case $p = 1$, where we have

$$\int_B \bigl|m(x)\bigr| \, d\nu(x) = \int_0^\infty m^*(t) \, dt = \int_0^\infty \mu_m(\lambda) \, d\lambda.$$

As ν is nonatomic, it follows that

$$|||m|||_\alpha = \sup_{\nu(N) \le \alpha} \int_N |m| \, d\nu = \int_0^\alpha m^*(t) \, dt.$$

Thus,

$$\frac{|||m|||_\alpha}{\|m\|_1} \le \frac{1}{2}$$

if and only if

$$\frac{\int_0^\alpha m^*(t) \, dt}{\int_0^\infty m^*(t) \, dt} \le \frac{1}{2}.$$

Example 19 (Symmetric p-Stable Random Variables) A random variable m on a probability space (B, Σ, ν) is called a *symmetric p-stable random variable* if there is a constant $c > 0$ such that its characteristic function $\varphi_m(t) = \mathbb{E} e^{itm}$ is given by $\varphi(t) = e^{-c|t|^p}$. The p-stable laws were introduced and studied by Paul Lévy in the 1920s, and they play an important role in probability theory. For a proof of the following classical theorem, see parts (i) and (iii) of Benyamini, Lindenstrauss [3, Appendix D, Theorem D.8]. (And also see there references to further basic facts on symmetric p-stable random variables.)

Theorem 20

(i) *For each* $0 < p \le 2$, *there is a symmetric* p-*stable random variable with characteristic function* $\varphi(t) = e^{-|t|^p}$.

(ii) *If* $p < 2$ *and* m *is a symmetric* p*-stable random variable, then* $\|m\|_r = (\mathbb{E}|m|^r)^{1/r}$ *is finite if and only if* $r < p$. (*When* $p = 2$, *we obtain Gaussian random variables, which will be discussed in detail in the next example. In this case,* $\mathbb{E}|m|^r < \infty$ *for every* $r < \infty$.)

A standard fact in measure theory is that when X is any random variable and (B, Σ, ν) is nonatomic, then it carries a random variable with the same distribution as X. More generally, it carries a sequence of independent random variables $\{X_j\}$ with the same distribution as X.

Recall also that when X and Y are independent random variables with characteristic functions φ_X, φ_Y, respectively, then the characteristic function of $aX + bY$ is given by

$$\varphi_{aX+bY}(t) = \mathbb{E}e^{it(aX+bY)} = \mathbb{E}e^{itaX}\mathbb{E}e^{itbY} = \varphi_X(at)\varphi_Y(bt).$$

Now fix $p \leq 2$, and let $\{m_j\}$ be a sequence (finite or infinite) of independent random variables with the same characteristic function $e^{-|t|^p}$. It follows that if $m = \sum a_j m_j$, then

$$\varphi_m(t) = \prod e^{-|t|^p|a_j|^p} = e^{-|t|^p \sum |a_j|^p}.$$

Thus, m is also p-stable and has the same distribution as $(\sum |a_j|^p)^{1/p} m_1$.

By (ii), $\{m_j\} \subset L^1(B, \nu)$, and we let M be the closed subspace they span in $L^1(B, \nu)$. By the above computations, every $m = \sum a_j m_j$ satisfies

$$\|m\|_1 = \Bigl(\sum |a_j|^p\Bigr)^{1/p} \|m_1\|_1,$$

and $M = \{\sum a_j m_j : \sum |a_j|^p < \infty\}$.

Now fix any r with $1 < r < p$; then similarly, every $m = \sum a_j m_j \in M$ satisfies

$$\|m\|_r = \Bigl(\sum |a_j|^p\Bigr)^{1/p} \|m_1\|_r.$$

Thus, $\frac{\|m\|_r}{\|m\|_1}$ is the constant $\frac{\|m_1\|_r}{\|m_1\|_1}$ for all $0 \neq m \in M$, and therefore

$$A_r = \sup \frac{\|m\|_r}{\|m\|_1} = \frac{\|m_1\|_r}{\|m_1\|_1} < \infty.$$

From Proposition 12, we obtain $\alpha^*(M) \geq (\frac{\|m_1\|_1}{2\|m_1\|_r})^{1/r'}$.

Example 20 (Gaussian Random Variables) Let (B, Σ, ν) be a nonatomic probability space, and let $\{m_j\}$ be a sequence (finite or infinite) of independent standard Gaussian random variables on (B, Σ, ν); i.e., each m_j has $N(0, 1)$ distribution. Let $M = \{\sum a_j m_j : \sum |a_j|^2 < \infty\}$ be the closed linear span in $L^1(B, \nu)$ of the m_j's. Rather then just obtaining a lower bound, as above, we shall here compute $\alpha^*(M)$ explicitly to obtain:

$$\alpha^*(M) = \widetilde{\alpha} \approx 0.239\ldots$$

As in Example 19, all $m \in M$ with the same L^1 norm have the same distribution, and we may therefore assume that M is actually one dimensional, spanned by an m which is a standard Gaussian random variable on (B, Σ, ν). Thus,

$$\nu\{x : m(x) < \lambda\} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\lambda} e^{-s^2/2}\, ds,$$

for all $\lambda \in \mathbb{R}$. Using previous notation, the distribution function of each $|m|$ is given by

$$\mu_m(\lambda) := \nu\{x : |m(x)| > \lambda\} = \frac{2}{(2\pi)^{1/2}} \int_{\lambda}^{\infty} e^{-s^2/2}\, ds$$

for all $\lambda \ge 0$, and m^* is given by

$$m^*(t) = \begin{cases} \infty, & t = 0, \\ \lambda, & \frac{2}{(2\pi)^{1/2}} \int_{\lambda}^{\infty} e^{-s^2/2}\, ds = t \text{ if } t \in (0, 1), \\ 0, & t \ge 1. \end{cases}$$

In addition,

$$\|m\|_1 = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} |s| e^{-s^2/2}\, ds = \sqrt{\frac{2}{\pi}}.$$

We therefore want to calculate

$$\alpha^*(M) = \sup\left\{\alpha : \sup_{\nu(N) \le \alpha} \int_N |m|\, d\nu \le \frac{1}{2}\sqrt{\frac{2}{\pi}}\right\}.$$

The interior supremum is clearly attained on the set

$$N = \{x : |m(x)| > \beta\},$$

where $\beta > 0$ is defined by

$$\int_{\{x:|m(x)|>\beta\}} |m(x)|\, d\nu(x) = 2\int_{\{x:m(x)>\beta\}} m(x)\, d\nu(x) = \frac{1}{2}\sqrt{\frac{2}{\pi}}.$$

Now

$$\int_{\{x:m(x)>\beta\}} m(x)\, d\nu(x) = \frac{1}{(2\pi)^{1/2}} \int_{\beta}^{\infty} s e^{-s^2/2}\, ds = \frac{1}{(2\pi)^{1/2}} e^{-\beta^2/2},$$

whence $\beta = \sqrt{2 \ln 2}$. The value $\alpha^*(M)$ is therefore given by

$$\alpha^*(M) = \nu(N) = \nu\{x : |m(x)| > \sqrt{2 \ln 2}\} = 2\bigl(1 - \Phi\bigl(\sqrt{2 \ln 2}\bigr)\bigr),$$

where $\Phi(t) = \nu\{x : m(x) \le t\}$. Using tables, we get $\alpha^*(M) := \widetilde{\alpha} \approx 0.239\ldots$.

In fact, we conjecture the following:

Conjecture *For every infinite dimensional subspace M of $L^1(B, \nu)$, with finite $\nu(B)$, we have $\alpha^*(M) \le \widetilde{\alpha}\, \nu(B)$.*

What is the largest value of $\alpha^*(M_n)$ as we vary over all M_n of dimension n? We do not know the answer to this question. Let us assume that $\nu(B) < \infty$. Then among all subspaces M_1 of dimension 1, the largest $\alpha^*(M_1)$ is $(1/2)\,\nu(B)$, and it is attained if M_1 is spanned by a function $\widetilde{m}$ such that $|\widetilde{m}|$ is a constant function. Indeed $m^*(t)=0$ for all $t\ge\nu(B)$, and if $|||m|||_\alpha\le(1/2)\|m\|_1$, then $\alpha\le(1/2)\,\nu(B)$ with equality if and only if $|m|$ is a constant function. What can be said when M_n is of dimension $n>1$? Example 20 shows that

$$\sup\bigl\{\alpha^*(M_n):\dim M_n=n\bigr\}\ge\widetilde{\alpha}\,\nu(B).$$

In fact, strict inequality holds in the above, as is verified in this next example:

Example 21 (Linear Functions on the Sphere) Let $\|\cdot\|_2$ denote the Euclidean norm on $\mathbb{R}^n$, and let $S^{n-1}=\{x:\|x\|_2=1\}$ denote the unit sphere. For $n>1$, let M_n denote the n-dimensional linear space of functions $\{\langle x,a\rangle\}$ restricted to S^{n-1}. That is, the elements of M_n are the linear functions $m_a(\cdot)=\langle\cdot,a\rangle$ for $a\in\mathbb{R}^n$.

We consider $L^1(S^{n-1},\nu_n)$ equipped with the normalized Lebesgue measure ν_n. The rotation invariance of ν_n implies that if $\|a_1\|_2=\|a_2\|_2$, then m_{a_1} and m_{a_2} have the same distribution function. Hence, in particular, they have the same norm in $L^1(S^{n-1},\nu_n)$ and the same α-norms. Thus, in order to compute $\alpha^*(M_n)$, it suffices to compute what happens with $m=m_{e_1}$, where $e_1=(1,0,\dots,0)$; i.e.,

$$\alpha^*(M_n)=\nu_n\bigl\{x:\bigl|\langle x,e_1\rangle\bigr|>\beta_n\bigr\},$$

where $\beta_n>0$ is defined by the equation

$$\int_{|\langle x,e_1\rangle|>\beta_n}\bigl|\langle x,e_1\rangle\bigr|\,d\nu_n=\frac{1}{2}\int_{S^{n-1}}\bigl|\langle x,e_1\rangle\bigr|\,d\nu_n.$$

The surface area of S^{n-1} is given by

$$I_n=\frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

If θ is the angle between a point $x\in S^{n-1}$ and the hyperplane spanned by $e_2,\dots,e_n$, then we have

$$I_n=I_{n-1}\int_{-\pi/2}^{\pi/2}\cos^{n-2}\theta\,d\theta,$$

for $n=2,3,\dots$, and thus,

$$\begin{aligned}\int_{S^{n-1}}\bigl|\langle x,e_1\rangle\bigr|\,d\nu_n&=\frac{2I_{n-1}}{I_n}\int_0^{\pi/2}\sin\theta\cos^{n-2}\theta\,d\theta=\frac{-2I_{n-1}}{(n-1)I_n}\cos^{n-1}\theta\Big|_0^{\pi/2}\\&=\frac{2I_{n-1}}{(n-1)I_n},\end{aligned}$$

while

$$\int_{|\langle x,e_1\rangle|>\beta_n}\bigl|\langle x,e_1\rangle\bigr|\,d\nu_n=\frac{2I_{n-1}}{I_n}\int_{\beta_n}^{\pi/2}\sin\theta\cos^{n-2}\theta\,d\theta=\frac{2I_{n-1}}{(n-1)I_n}\cos^{n-1}\beta_n.$$

Thus, β_n is explicitly given by

$$\cos^{n-1}\beta_n=\frac{1}{2}.$$

We also have the following asymptotics for β_n. From Taylor's theorem, $\cos x = 1 - \frac{x^2}{2} + O(x^4)$ and $(1/2)^x = 1 + x \ln \frac{1}{2} + O(x^2)$, and therefore,

$$1 - \frac{\beta_n^2}{2} + O\big(\beta_n^4\big) = \cos \beta_n = \left(\frac{1}{2}\right)^{\frac{1}{n-1}} = 1 + \frac{1}{n-1} \ln \frac{1}{2} + O\big(n^{-2}\big).$$

Solving, we obtain

$$\beta_n = \sqrt{\frac{2 \ln 2}{n-1}} + O\big(n^{-1}\big).$$

We can precisely compute β_n and $\alpha^*(M_n)$ in the cases $n = 2$ and $n = 3$. For $n = 2$, we have $\beta_2 = \pi/3$ and $\alpha^*(M_2) = 1/3$, while for $n = 3$, we have $\beta_3 = \pi/4$ and $\alpha^*(M_3) = (\sqrt{2} - 1)/\sqrt{2} \approx 0.293$.

In the next result, we prove that $\{\beta_n\}$ is a monotone decreasing sequence tending to zero, while the $\{\alpha^*(M_n)\}$ monotonically decrease to $\widetilde{\alpha}$, where $\widetilde{\alpha}$ is the value from the Gaussian space (see the previous Example 20).

Theorem 21 *Let β_n and $\alpha^*(M_n)$ be as above. Then*

(i) $\{\beta_n\}$ *is a monotone decreasing sequence tending to zero.*
(ii) $\{\alpha^*(M_n)\}$ *is a monotone decreasing sequence.*
(iii) $\lim_{n\to\infty} \alpha^*(M_n) = \widetilde{\alpha}$.

Proof (i) The montonicity of the $\{\beta_n\}$ follows from the fact that since $\cos^{n-1} \beta_n = \cos^n \beta_{n+1} = \frac{1}{2}$, then $\cos^n \beta_n < \cos^n \beta_{n+1}$. As $\beta_n, \beta_{n+1} \in (0, \pi/2)$, we have $\beta_n > \beta_{n+1}$.

(ii) We have that

$$\alpha^*(M_n) = \frac{2 \int_{\beta_n}^{\pi/2} \cos^{n-2} \theta \, d\theta}{2 \int_0^{\pi/2} \cos^{n-2} \theta \, d\theta},$$

while

$$\cos^{n-1} \beta_n = \frac{1}{2}.$$

Substitute $t = \cos^{n-1} \theta$ to obtain $dt = -(n-1) \cos^{n-2} \theta \sin \theta \, d\theta$. Since $\sin \theta = \sqrt{1 - \cos^2 \theta}$, we obtain

$$-\frac{1}{n-1} \frac{dt}{\sqrt{1 - t^{2/(n-1)}}} = \cos^{n-2} \theta \, d\theta.$$

Thus,

$$\int_{\beta_n}^{\pi/2} \cos^{n-2} \theta \, d\theta = \frac{1}{n-1} \int_0^{1/2} \frac{dt}{\sqrt{1 - t^{2/(n-1)}}},$$

while

$$\int_0^{\pi/2} \cos^{n-2} \theta \, d\theta = \frac{1}{n-1} \int_0^1 \frac{dt}{\sqrt{1 - t^{2/(n-1)}}}.$$

We therefore wish to prove that

$$\frac{\int_0^{1/2} \frac{dt}{\sqrt{1-t^{2/(n-1)}}}}{\int_0^1 \frac{dt}{\sqrt{1-t^{2/(n-1)}}}} > \frac{\int_0^{1/2} \frac{dt}{\sqrt{1-t^{2/n}}}}{\int_0^1 \frac{dt}{\sqrt{1-t^{2/n}}}}.$$

We claim that

$$\frac{\int_0^{c} \frac{dt}{\sqrt{1-t^{2/(n-1)}}}}{\int_0^1 \frac{dt}{\sqrt{1-t^{2/(n-1)}}}} > \frac{\int_0^{c} \frac{dt}{\sqrt{1-t^{2/n}}}}{\int_0^1 \frac{dt}{\sqrt{1-t^{2/n}}}}$$

for every $c \in (0, 1)$; i.e.,

$$\int_0^c \frac{dt}{\sqrt{1-t^{2/(n-1)}}} > A \int_0^c \frac{dt}{\sqrt{1-t^{2/n}}},$$

where the positive constant A is such that equality holds for $c = 1$.

To prove this, it suffices to prove that

$$\frac{\sqrt{1-t^{2/n}}}{\sqrt{1-t^{2/(n-1)}}}$$

is decreasing on $(0, 1)$; i.e.,

$$\frac{1-t^{2/n}}{1-t^{2/(n-1)}}$$

is decreasing on $(0, 1)$.

Set $s = t^{2/n(n-1)}$. Thus, $t^{2/n} = s^{n-1}$ and $t^{2/(n-1)} = s^n$, and we wish to show that

$$\frac{1-s^{n-1}}{1-s^n}$$

is decreasing on $(0, 1)$. Differentiating, this is then equivalent to

$$-(n-1)s^{n-2}\big(1-s^n\big) + \big(1-s^{n-1}\big)ns^{n-1} < 0,$$

which can be rewritten as

$$s < \frac{n-1}{n} + \frac{1}{n}s^n,$$

which, in turn, is easily proven.

(iii) To show the desired convergence, write

$$\begin{aligned}\alpha^*(M_n) &= \frac{\int_{\beta_n}^{\pi/2} \cos^{n-2}\theta\, d\theta}{\int_0^{\pi/2} \cos^{n-2}\theta\, d\theta} \\ &= \frac{\int_{\beta_n\sqrt{n-2}}^{\pi\sqrt{n-2}/2} \cos^{n-2}(t/\sqrt{n-2})\, dt}{\int_0^{\pi\sqrt{n-2}/2} \cos^{n-2}(t/\sqrt{n-2})\, dt} = \frac{\int_{\beta_n\sqrt{n-2}}^{\infty} f_n(t)\, dt}{\int_0^{\infty} f_n(t)\, dt},\end{aligned}$$

where $t = (\sqrt{n-2})\,\theta$ and where $f_n(t) = \cos^{n-2}(t/\sqrt{n-2})$ for $0 \le t \le \pi\sqrt{n-2}/2$ and 0 for $t > \pi\sqrt{n-2}/2$.

From the asymptotics for β_n, we have

$$\lim_{n\to\infty}\beta_n\sqrt{n-2}=\lim_{n\to\infty}\left(\sqrt{\frac{2\ln 2}{n-1}}+O\big(n^{-1}\big)\right)\sqrt{n-2}=\sqrt{2\ln 2}.$$

We also note that $0\le f_n(t)\le e^{-t^2/2}$ (because $\cos x\le e^{-x^2/2}$ for $x\in[0,\pi/2]$), and that

$$0\le\left(1-\frac{1}{2}\big(t/\sqrt{n-2}\big)^2\right)^{n-2}\le f_n(t)$$

when $0\le t/\sqrt{n-2}\le\sqrt{2}$ (because $0\le 1-x^2/2\le\cos x$ for $x\in[0,\sqrt{2}]$).

It follows from these inequalities that $f_n(t)\to e^{-t^2/2}$ pointwise, and since $e^{-t^2/2}$ is integrable and $0\le f_n(t)\le e^{-t^2/2}$, Lebesgue's dominated convergence theorem gives

$$\lim_{n\to\infty}\alpha^*(M_n)=\frac{\int_{\sqrt{2\ln 2}}^{\infty}e^{-t^2/2}\,dt}{\int_0^{\infty}e^{-t^2/2}\,dt}=\widetilde{\alpha}.\qquad\square$$

Remark The above is an example of the known fact (usually attributed to Maxwell) that for a fixed k (here we have $k=1$), the projections of the uniform measures on $\sqrt{n-1}\,S^{n-1}\subset\mathbb{R}^n$ on $\mathbb{R}^k$ converge, as $n\to\infty$, to the standard Gaussian measure on $\mathbb{R}^k$.

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