The Approximation of a Totally Positive Band Matrix by a Strictly Banded Totally Positive One*

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ABSTRACT

Every nonsingular totally positive m-banded matrix is shown to be the product of m totally positive one-banded matrices and, therefore, the limit of strictly m-banded totally positive matrices. This result is then extended to (bi)infinite m-banded totally positive matrices with linearly independent rows and columns. In the process, such matrices are shown to possess at least one diagonal whose principal sections are all nonzero. As a consequence, such matrices are seen to be approximable by strictly m-banded totally positive ones.

1. INTRODUCTION

In this paper, we prove the result, needed in [1], that a totally positive biinfinite band matrix is the limit of "strictly banded" totally positive matrices of the same band type. But the tool developed for the proof, viz., the factorization of such matrices into "one-banded" totally positive matrices, is of independent and, perhaps, greater interest.

We came to consider such factorizations because of the recent paper by Cavaretta, Dahmen, Micchelli, and Smith [3] in which such a factorization is derived for strictly banded totally positive matrices. But we were unable to adapt their arguments, which involve limits of ratios of entries in a certain

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matrix inverse, to our situation (in which we have neither invertibility nor strict bandedness), and ended up constructing the needed factors by the more familiar device of elimination instead. The factorization is first established for finite matrices and is then extended to biinfinite matrices by a limiting argument. For this, we found it necessary to first prove that a totally positive matrix with linearly independent rows and columns has at least one diagonal with the property that all square finite sections which are principal for that diagonal are nonsingular.

2. BANDEDNESS

The rth diagonal or band of a matrix A is, by definition, the sequence $(A(i,i+r))$. As in [2], we call a matrix A *m*-banded if all nonzero entries of A can be found in at most $m+1$ consecutive bands. Explicitly, the matrix A is m-banded if

$$\text{for some } l, \quad A(i+l,j)\neq 0 \text{ implies } i\leqslant j\leqslant i+m.$$

If both l and $m-l$ are nonnegative, then the $m+1$ nontrivial bands include the "main diagonal" or zeroth band, with l bands to the left of it and $r:=m-l$ bands to the right of it. In this situation, we will at times call such a matrix more explicitly (l,r)-banded.

We call a band matrix strictly banded if the leftmost and the rightmost nontrivial band contain no zero entries. Among banded matrices, the strictly banded ones are particularly important and easier to treat, since they correspond to *nondegenerate* difference operators.

3. TOTAL POSITIVITY

A matrix is said to be totally positive (or TP) in case all its minors are nonnegative.

We use the abbreviation

$$A\begin{bmatrix} i_1,\dots,i_s \\ j_1,\dots,j_t \end{bmatrix} := \left(A(i_\mu,j_\nu)\right)_{\mu=1}^{s}{}_{\nu=1}^{t}$$

for the $s\times t$ matrix which has its (μ,ν)-entry equal to $A(i_\mu,j_\nu)$. Further, if I

and J are index sets, then

$$A_{I,J} := A\begin{bmatrix} I \\ J \end{bmatrix} := A\begin{bmatrix} i_1, \dots, i_s \\ j_1, \dots, j_t \end{bmatrix}$$

with $i_1, \dots, i_s$ and $j_1 \dots j_t$ the elements of I and J, respectively, *in increasing order*. Occasionally, we will use the additional abbreviation

$$A[I] := A\begin{bmatrix} I \\ I \end{bmatrix}.$$

Finally, in case $|I| = |J|$, replacing the square brackets by round brackets gets us from the matrix to its determinant:

$$A\begin{pmatrix} \cdots \\ \cdots \end{pmatrix} := \det A\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}.$$

We will make repeated use of

SYLVESTER'S DETERMINANT IDENTITY (SDI). *If* $A\begin{pmatrix} I \\ J \end{pmatrix} \neq 0$ *and* B *is the matrix obtained from* A *by*

$$B(i,j) := \frac{A\begin{pmatrix} \{i\} \cup I \\ \{j\} \cup J \end{pmatrix}}{A\begin{pmatrix} I \\ J \end{pmatrix}}, \qquad \textit{all} \quad (i,j) \in (\backslash I) \times (\backslash J),$$

then

$$B\begin{pmatrix} I' \\ J' \end{pmatrix} = \frac{A\begin{pmatrix} I' \cup I \\ J' \cup J \end{pmatrix}}{A\begin{pmatrix} I \\ J \end{pmatrix}}.$$

The submatrix $A_{I,J}$ is called the pivot block, since the identity is proved by observing that B is the Schur complement of $A_{I,J}$, i.e., the interesting part of what is left in rows $\backslash I$ and columns $\backslash J$ after rows I have been used to eliminate variables J from the other rows; see, e.g., Gantmacher [5; p. 31] or

Karlin [7; p. 3]. In particular,

COROLLARY 1. $\operatorname{rank} A\begin{bmatrix} I'\cup I \\ J'\cup J \end{bmatrix} = |I| + \operatorname{rank} B\begin{bmatrix} I' \\ J' \end{bmatrix}$ (*for* $I'\cap I = \varnothing = J'\cap J$).

COROLLARY 2. *B is again TP if A is.*

Another result which may be proven by Sylvester's determinant identity (using induction on $|I|$; see Gantmacher and Krein [6; p. 108] or Karlin [7; p. 88]) is

HADAMARD'S INEQUALITY. *If A is TP and* $I = I'\cup I''$ *with* $I'\cap I'' = \varnothing$, *then*

$$A(I) \leqslant A(I')A(I'').$$

4. SHADOWS

In this section, we prove an ancillary result concerning the existence of a diagonal in a TP biinfinite matrix which could serve as the main diagonal in a triangular factorization, i.e., a diagonal all of whose principal sections are nonsingular.

A zero entry in a TP matrix usually "throws a shadow." By this we mean that usually all entries to the left and below it, or else all entries to the right and above it, are also zero. More precisely, call the submatrix $A\begin{bmatrix} i\geqslant i_0 \\ j\leqslant j_0 \end{bmatrix}$ the *left shadow* of the entry $A(i_0, j_0)$ and, correspondingly, call the submatrix $A\begin{bmatrix} i\leqslant i_0 \\ j\geqslant j_0 \end{bmatrix}$ the *right shadow* of $A(i_0, j_0)$. Then the following lemma is known.

LEMMA A. *If A is TP and* $A(i_0, j_0) = 0$, *but neither* $A(\cdot, j_0)$ *nor* $A(i_0, \cdot)$ *is zero, then either the left or the right shadow of* $A(i_0, j_0)$ *is zero.*

Proof. By assumption, $A(i_0, j_1) \neq 0$ for some j_1. If $j_1 < j_0$, then the right shadow of $A(i_0, j_0)$ can be seen to be zero as follows. First, for any $i < i_0$,

$$0 \leqslant A\begin{pmatrix} i, i_0 \\ j_1, j_0 \end{pmatrix} = -A(i_0, j_1)A(i, j_0) \leqslant 0,$$

and $A(i_0, j_1) \neq 0$ implies that $A(i, j_0) = 0$ for all $i < i_0$. Hence there then exists $i_1 > i_0$ for which $A(i_1, j_0) \neq 0$. But now, for any $i \leqslant i_0$ and $j > j_0$,

$$0 \leqslant A\begin{pmatrix} i, i_1 \\ j_0, j \end{pmatrix} = -A(i_1, j_0)A(i, j) \leqslant 0,$$

and $A(i_1, j_0) \neq 0$ implies that $A(i, j) = 0$.

Finally, if instead $j_1 > j_0$, then the left shadow of $A(i_0, j_0)$ is similarly seen to be zero. ∎

As an application for later use, note that a zero in the lower triangular part of an invertible TP matrix necessarily throws a *left* shadow, since all diagonal entries are nonzero, by Hadamard's inequality.

More generally, for any section of A, i.e., any submatrix $A_{I,J}$ of A made up of *consecutive* rows and columns of A, we call the submatrix of A having $A_{I,J}$ as its upper right corner the *left shadow* of $A_{I,J}$, and, correspondingly, we call the submatrix having $A_{I,J}$ as its lower left corner the *right shadow* of $A_{I,J}$. Then we have the following generalization of Lemma A.

PROPOSITION A. *If A is TP and $A_{I,J}$ is a singular section of order n and rank $n-1$, while both $A\begin{bmatrix} I \\ \cdot \end{bmatrix}$ and $A\begin{bmatrix} \cdot \\ J \end{bmatrix}$ are of full rank n, then either the left or the right shadow of $A_{I,J}$ has rank $n-1$.*

REMARK. As in the case $n=1$ discussed earlier, we will describe this last situation by saying that such a section $A_{I,J}$ "throws a (left or right) shadow."

Proof. By assumption, we can choose $(i_0, j_0) \in I \times J$ so that $A\begin{pmatrix} I' \\ J' \end{pmatrix} \neq 0$, with $I' := I \setminus \{i_0\}$, $J' := J \setminus \{j_0\}$. The assumptions imply that the Schur complement of $A_{I',J'}$, i.e., the matrix B given by

$$B(r, s) := \frac{A\begin{pmatrix} \{r\} \cup I' \\ \{s\} \cup J' \end{pmatrix}}{A\begin{pmatrix} I' \\ J' \end{pmatrix}}, \qquad \text{all} \quad (r, s) \in (\setminus I') \times (\setminus J'),$$

is again TP (by Corollary 2 of SDI) and vanishes at (i_0, j_0), while (by Corollary 1 of SDI) neither $B(i_0, \cdot)$ nor $B(\cdot, j_0)$ is zero. The lemma therefore implies that either the left or the right shadow of $B(i_0, j_0)$ is zero, and Corollary 1 of SDI then implies that either the left or the right shadow of $A_{I,J}$ has rank $n-1$. ∎

COROLLARY. *If A is an infinite TP matrix, e.g.,* $A \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$, *then all rows and all columns of A are linearly independent if and only if* $A(I)>0$ *for all finite* $I \subseteq \mathbb{N}$.

Proof. If $A(I)=0$ for some finite $I \subseteq \mathbb{N}$, then Hadamard's inequality would imply the existence of some $n \in \mathbb{N}$ for which $A(1,\dots,n)=0$ while $A(1,\dots,n-1)\neq 0$. The proposition then would imply that either the first n rows or else the first n columns of A are linearly dependent. ■

We now state and prove the corresponding result for a biinfinite TP matrix. This is somewhat harder, since it is not clear *a priori* which band is to play the role of main diagonal.

We concentrate on principal sections for a band: A principal section for band r is any submatrix of the form $A_{I,I+r}$, with I an interval. In other words, such a principal section for band r (1) is square, (2) is made up of consecutive rows and columns, and (3) has a piece of band r as its main diagonal. We call such a principal section *minimally singular* if it is singular but contains no smaller principal section for the same band which is also singular. Note that, by Hadamard's inequality, every principal section containing a singular one for the same band is itself singular.

THEOREM A. *Let A be biinfinite TP, and assume that not all minimally singular sections of A throw their shadow in the same direction. Then all rows and all columns of A are linearly independent if and only if all principal sections for some band are nonsingular, i.e.,*

$$\text{there exists } r \text{ such that for all intervals } I, \quad A\binom{I}{I+r}>0.$$

REMARK. This last condition is, if course, equivalent to having $A\binom{I}{I+r}>0$ for all finite index sets I.

Proof. The sufficiency of the condition is obvious. So assume that all rows and all columns of A are linearly independent. If every band r has a singular principal section, then every band has a minimally singular one, and, all rows and all columns being linearly independent, each of these throws a shadow, by Proposition A. By assumption, not all of these shadows go in the same direction, i.e., there exist bands r and s such that some principal section for r throws a shadow to the left while some principal section for s throws a shadow to the right.

We may assume that $r<s$. For, if $r\geqslant s$, then, with A_r the minimally singular r-band section in question, we can pick a minimally singular section A_q for some band q with $q<s$ and in the left shadow of A_r and such that A_r is in the right shadow of A_q. We claim that such A_q must again throw a *left* shadow. For, it throws a shadow by Proposition A, and if this were a right shadow, then both A_r and A_q would have to be of the same rank, hence of the same order, and the union of their shadows would contain a strip of width$\geqslant$order A_r and of rank$=$rank $A_r<$order A_r, thus contradicting the linear independence of rows (or columns) of A.

Further, since every band has a shadow throwing section, we may assume that $s=r+1$. More explicitly now, we assume that, for some r, there is $L\geqslant 0$ and i such that

$$A_r:=A\begin{bmatrix} i,\dots,i+L \\ i+r,\dots,i+r+L \end{bmatrix}$$

is minimally singular with its left shadow of rank L while, for some $R\geqslant 0$ and some k,

$$A_{r+1}:=A\begin{bmatrix} k-R,\dots,k \\ k+r+1-R,\dots,k+r+1 \end{bmatrix}$$

is minimally singular with its right shadow of rank R.

There are three cases, depending on the relative position of these two submatrices.

Case 1: $k-i\geqslant\max\{L,R\}$.

$$\begin{array}{cccccccccccccc}
 & i+r & & & & i+r+L & & & & & & & \\
 & \downarrow & & & & \downarrow & & & & & & & \\
i\rightarrow & \times & \times & \times & \times & \times & & & & & & & \\
 & & & & & \times & & & & & & & \\
 & & & & & \times & & \times & & & & & \\
 & & & & & & & \times & & & & & \\
 & & & & & & & \times & \times & \times & \times & \times & \times & \leftarrow k \\
 & & & & & & & \uparrow & & & & & \uparrow & \\
 & & & & & & & k+r+1-R & & & & & k+r+1 &
\end{array}$$

Then all columns j in $A\begin{bmatrix} i,\dots,k \\ \cdot \end{bmatrix}$ with $j\leqslant i+r$ are linear combinations of the L columns $i+r+1,\dots,i+r+L$, while all columns j with $j>k+r$ are linear combinations of the R columns $k+r+1-R,\dots,k+r$, and this holds even if, e.g., $L=0$, since then all columns j with $j\leqslant i+r$ are zero themselves. Since

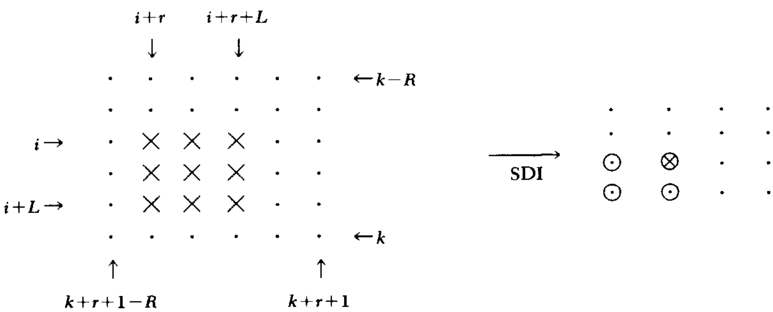
$i+r+1 \leqslant k+r+1-R$ and $i+r+L \leqslant k+r$, all columns are linear combinations of the $k-i$ columns $i+r+1,\ldots,k+r$. We conclude that $A\left[\begin{smallmatrix} i,\ldots,k \\ \cdot \end{smallmatrix}\right]$ has only rank $k-i$, a contradiction to the assumed linear independence of all rows.

Case 2: $i+r+L-(k+r+1-R) \geqslant \max\{L, R\}$. This is treated analogously. It leads to columns $k+r+1-R,\ldots,i+r+L$ being dependent, again a contradiction. This leaves

Case 3: $\max\{k-i, i+L-(k+1-R)\} < \max\{L, R\}$. In this case $i-k < 1+\max\{L, R\}-(L+R) = 1-\min\{L, R\}$, and so

$$\min\{L, R\} \leqslant k-i < \max\{L, R\}.$$

We claim that this contradicts the minimality of the two sections A_r and A_{r+1} chosen. Assume without loss that $L<R$. Since then $L \leqslant k-i < R$, the r-band section A_r lies *inside* the (larger) $(r+1)$-band section A_{r+1}:



By the minimality of A_{r+1}, the L-section

$$A\begin{bmatrix} i,\ldots,i+L-1 \\ i+r+1,\ldots,i+r+L \end{bmatrix}$$

principal for band $r+1$ is nonsingular. In its Schur complement B, the section A_r appears as a zero which, by assumption, throws a left shadow. But this zero appears on the next-to-main diagonal of the submatrix B_{r+1} of B corresponding to A_{r+1}, and this implies that det B_{r+1} is the product of two of its proper complementary minors. But then, by the singularity of A_{r+1}, and hence of B_{r+1}, some proper principal section of A_{r+1} must be singular, contradicting the minimality of A_{r+1}. ∎

The assumption that there ought to be shadows thrown in both directions cannot be omitted, as the following example shows. Let

$$B(i,j):=\begin{cases}1/(i-j)!, & i\geq j,\\ 0, & i<j.\end{cases}$$

Then B is TP, and with $i_1<\cdots<i_s$, $j_1<\cdots<j_s$, we have

$$B\begin{pmatrix} i_1,\ldots,i_s \\ j_1,\ldots,j_s \end{pmatrix}>0 \quad \text{iff} \quad i_t\geq j_t, \quad \text{all } t.$$

From B, construct A by deleting every other row. More explicitly,

$$A(i,j):=B(2i,j), \qquad \text{all } i,j.$$

Then A is again TP and biinfinite, and its rows and columns are linearly independent. But now every band r has singular principal sections; for example, $A(i,i+r)=0$ for all $i<r$.

5. FACTORIZATION OF A FINITE BAND MATRIX

THEOREM B. *A TP nonsingular (l,r)-banded $n\times n$ matrix A can be factored as*

$$A=L^{(l)}\cdots L^{(1)}DU^{(1)}\cdots U^{(r)}$$

with $L^{(k)}$ unit-diagonal $(1,0)$-banded TP, all k, D diagonal TP, and $U^{(k)}$ unit-diagonal $(0,1)$-banded TP, all k.

Proof. We obtain the factorization by the standard device of elimination. In the typical step, we have a nonsingular TP (l,r)-banded matrix A with zeros already in band $-l$ in columns $1,\ldots,k-1$:

$$\begin{array}{c|ccccccccc}
 & & & k & & & & & l+k+r & \\
 & & & \downarrow & & & & & \downarrow & \\
 & \times & \times & \times & \times & & & & & \\
 & \times & \times & \times & \times & \times & & & & \\
 & 0 & \times & \times & \times & \times & \times & & & \\
 & & 0 & \times & \times & \times & \times & \times & & \\
l+k\rightarrow & & & \otimes & \times & \times & \times & \times & \times & \\
 & & & & \times & \times & \times & \times & \times & \times
\end{array}$$

From it, we obtain the matrix B by subtracting c times row $l+k-1$ from row $l+k$. Thus B differs from A only in row $l+k$, and there only in entries $k, k+1, \cdots, l+k+r-1$, because of the zeros in the other entries in row $l+k-1$. In particular, B is again (l, r)-banded, with zeros in band $-l$ in columns $1, \ldots, k-1$.

We now choose c so that also $B(l+k, k)=0$. If $A(l+k, k)=0$, then the choice $c=0$ will do. Otherwise $A(l+k, k)\neq 0$ and then necessarily also $A(l+k-1, k)\neq 0$ [since $A(l+k-1, k)=0$ would imply that $A(l+k-1, j)=0$ for $j>k$, hence $A(l+k-1, \cdot)=0$, therefore A could not have full row rank]. But then the positive number

$$c=A(l+k, k)/A(l+k-1, k)$$

does the job.

Note that $B=CA$, with C the identity matrix except for a $-c$ in position $(l+k, l+k-1)$. Since the action of C is undone by *adding* c times row $l+k-1$ to row $l+k$, it follows that

$$A=EB$$

with E the identity matrix except for a c in position $(l+k, l+k-1)$.

In order to carry out these steps repeatedly, we need to know

LEMMA B. *B is again TP.*

Proof. Since B differs from A only in row $l+k$, we only need to consider minors of B which involve row $l+k$. Among these, we only need to consider those minors which do not involve row $l+k-1$, since the others retain their (nonnegative) value in going from A to B. Thus we must show that

$$B\binom{I}{J}\geqslant 0$$

whenever I and J are index sets of like cardinality and I contains $l+k$, but not $l+k-1$. Let

$$I':=\{i\in I: i<l+k\}, \qquad I'':=\{i\in I: i>l+k\}.$$

Then

$$B\binom{I}{J}=A\binom{I}{J}-cA\binom{I',l+k-1,I''}{J}$$

and there is nothing to prove unless, as we now assume,

$$A\binom{I',l+k-1,I''}{J}>0,$$

which implies, by Hadamard's inequality, that every principal minor of the corresponding submatrix is strictly positive. We must then show that

$$\frac{A\binom{I}{J}}{A\binom{I',l+k-1,I''}{J}}\geqslant\frac{A\binom{l+k}{k}}{A\binom{l+k-1}{k}}\qquad(=c).$$

For this, let, correspondingly,

$$J=J'\cup\{j\}\cup J''$$

with J' the $|I'|$ smallest, and J'' the $|I''|$ largest, elements of J.

We claim that

$$\frac{A\binom{I}{J}}{A\binom{I',l+k-1,I''}{J}}\geqslant\frac{A\binom{l+k,I''}{j,J''}}{A\binom{l+k-1,I''}{j,J''}}.$$

This inequality follows by $|I'|$-fold application of the inequality

$$\frac{C\binom{1,\cdot,s-1,s+1,\cdot,t+1}{1,\cdot\quad\cdot\quad\cdot\quad\cdot,\ t}}{C\binom{1,\cdot,s,s+2,\cdot,t+1}{1,\cdot\quad\cdot\quad\cdot\quad\cdot,\ t}}\geqslant\frac{C\binom{2,\cdot,s-1,s+1,\cdot,t+1}{2,\cdot\quad\cdot\quad\cdot\quad\cdot,\ t}}{C\binom{2,\cdot,s,s+2,\cdot,t+1}{2,\cdot\quad\cdot\quad\cdot\quad\cdot,\ t}}\qquad(*)$$

valid for any TP matrix C, because of the identity

$$C\begin{pmatrix} 1, & \cdot, & s-1, & s+1, & \cdot, & t+1 \\ 1, & \cdot & \cdot & \cdot & \cdot, & t \end{pmatrix} C\begin{pmatrix} 2, & \cdot, & s, & s+2, & \cdot, & t+1 \\ 2, & \cdot & \cdot & \cdot & \cdot, & t \end{pmatrix}$$

$$-C\begin{pmatrix} 1, & \cdot, & s, & s+2, & \cdot, & t+1 \\ 1, & \cdot & \cdot & \cdot & \cdot, & t \end{pmatrix} C\begin{pmatrix} 2, & \cdot, & s-1, & s+1, & \cdot, & t+1 \\ 2, & \cdot & \cdot & \cdot & \cdot, & t \end{pmatrix}$$

$$= C\begin{pmatrix} 2,\dots,t+1 \\ 1,\dots,t \end{pmatrix} C\begin{pmatrix} 1, & \cdot, & s-1, & s+2, & \cdot, & t+1 \\ 2, & \cdot & \cdot & \cdot & \cdot, & t \end{pmatrix},$$

valid for such matrices. This identity is proved, e.g., in Karlin [7; p. 8]. It may also be proven by SDI applied to the $(t+1)\times(t+1)$ matrix obtained by adjoining to the first $t+1$ rows and t columns of C the additional column $(1,0,\dots,0)^T$, and taking

$$C\begin{bmatrix} 1, & \cdot, & s-1, & s+2, & \cdot, & t+1 \\ 2, & \cdot & \cdot & \cdot & \cdot, & t \end{bmatrix}$$

as the pivot block.

Unfortunately, the corresponding argument involving dropping of the *last* few rows and columns reverses the sign in (*) and so provides the irrelevant inequality

$$\frac{A\begin{pmatrix} l+k, I'' \\ j, J'' \end{pmatrix}}{A\begin{pmatrix} l+k-1, I'' \\ j, J'' \end{pmatrix}} \leqslant \frac{A\begin{pmatrix} l+k \\ j \end{pmatrix}}{A\begin{pmatrix} l+k-1 \\ j \end{pmatrix}}.$$

Instead, we observe next that

$$\frac{A\begin{pmatrix} l+k, I'' \\ j, J'' \end{pmatrix}}{A\begin{pmatrix} l+k-1, I'' \\ j, J'' \end{pmatrix}} \geqslant \frac{A\begin{pmatrix} l+k, I'' \\ k, J'' \end{pmatrix}}{A\begin{pmatrix} l+k-1, I'' \\ k, J'' \end{pmatrix}}.$$

This follows from the fact that

$$\frac{A\begin{pmatrix} l+k, I'' \\ j, J'' \end{pmatrix}}{A\begin{pmatrix} l+k-1, I'' \\ j, J'' \end{pmatrix}} = \frac{C\begin{pmatrix} l+k \\ j \end{pmatrix}}{C\begin{pmatrix} l+k-1 \\ j \end{pmatrix}}$$

with the matrix C given by

$$C(\mu,\nu) := A\begin{pmatrix} \mu, I'' \\ \nu, J'' \end{pmatrix}, \qquad \text{all } \mu, \nu,$$

hence TP (by SDI), and therefore the ratio is monotone nondecreasing in j for j to the left of J'', while the strict positivity of

$$A\begin{pmatrix} I', l+k-1, I'' \\ J \end{pmatrix}$$

implies, via Hadamard's inequality, that

$$A\begin{pmatrix} l+k-1 \\ j \end{pmatrix} > 0,$$

and so $k \leqslant j$ (recall that

$$A\begin{pmatrix} l+k-1 \\ \nu \end{pmatrix} = 0 \qquad \text{for} \quad \nu < k).$$

This leaves us, finally, with the task of showing that

$$\frac{A\begin{pmatrix} l+k, I'' \\ k, J'' \end{pmatrix}}{A\begin{pmatrix} l+k-1, I'' \\ k, J'' \end{pmatrix}} \geqslant \frac{A\begin{pmatrix} l+k \\ k \end{pmatrix}}{A\begin{pmatrix} l+k-1 \\ k \end{pmatrix}}.$$

But that is now obvious since $A(i,k)=0$ for $i>l+k$, hence

$$\frac{A\begin{pmatrix} l+k, I'' \\ k, J'' \end{pmatrix}}{A\begin{pmatrix} l+k-1, I'' \\ k, J'' \end{pmatrix}} = \frac{A\begin{pmatrix} l+k \\ k \end{pmatrix} A\begin{pmatrix} I'' \\ J'' \end{pmatrix}}{A\begin{pmatrix} l+k-1 \\ k \end{pmatrix} A\begin{pmatrix} I'' \\ J'' \end{pmatrix}} = \frac{A\begin{pmatrix} l+k \\ k \end{pmatrix}}{A\begin{pmatrix} l+k-1 \\ k \end{pmatrix}}. \qquad \blacksquare$$

We conclude that a nonsingular (l, r)-banded TP matrix A can be factored as

$$A = E^{(1)} \cdot \cdots \cdot E^{(n-l)} B$$

with B again TP but only $(l-1, r)$-banded, while for each k, $E^{(k)}$ is the identity matrix except for some nonnegative c_k in position $(l+k, l+k-1)$. But then

$$L^{(l)} := E^{(1)} \cdot \cdots \cdot E^{(n-l)}$$

is a $(1,0)$-banded matrix with unit diagonal and the nonnegative number c_k in position $(l+k, l+k-1)$, $k = 1, \ldots, n-l$, and zero everywhere else. Consequently, $L^{(l)}$ is $(1,0)$-banded and TP.

We conclude that a nonsingular (l, r)-banded TP matrix A can be factored as

$$A = L^{(l)} \cdot \cdots \cdot L^{(1)} B$$

with B a $(0, r)$-banded TP matrix and each $L^{(k)}$ a TP $(1,0)$-banded matrix with unit diagonal. Applying this last statement to B^T and transposing the result finishes the proof of Theorem B. ■

Lemma B establishes the following proposition of independent interest.

PROPOSITION B. *Let $A \in \mathbb{R}^{n \times m}$ be a TP matrix of full row rank of the partitioned form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

with the first column of A_{22} zero except for possibly nonzero entries in rows 1 and 2. Then $A = LB$, with $L \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ again TP and of the corresponding form

$$L = \begin{bmatrix} 1 & 0 \\ 0 & L_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_{11} & A_{12} \\ 0 & B_{22} \end{bmatrix},$$

with L_{22} the identity except for a possibly nonzero entry in column 1 in row 2, and the first column of B_{22} zero except for a possibly nonzero entry in row 1.

REMARK. A. Whitney's result [9] corresponds to the special case $A_{11} = \varnothing$. Repeated application of Proposition B recovers Cryer's result [4] on factoring a nonsingular TP matrix into lower and upper triangular TP matrices.

6. FACTORIZATION OF A (BI)INFINITE BAND MATRIX

THEOREM C. *A TP (bi)infinite m-banded matrix A whose rows and columns are linearly independent can be factored as*

$$A = R^{(1)} \cdots R^{(m)} D,$$

with each $R^{(k)}$ a TP one-banded matrix with maximum entry in each column equal to 1, *and D a TP diagonal matrix with* $0 < D(j,j) \leq \max_i A(i,j)$, *all j.*

Proof. If A is biinfinite, then we know from Theorem A that all principal sections for some band of A are nonsingular. Assume without loss of generality that the zeroth band is such a distinguished band and that A is, more explicitly, (l,r)-banded. Then we know that

$$A_n := A[-n,\dots,n]$$

is nonsingular. If A is only infinite, $A \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ say, then we know from the Corollary to Proposition A that, for all n,

$$A_n := A[1,\dots,n]$$

is nonsingular.

In either case, Theorem B assures us that A_n has a factorization

$$A_n = L_n^{(l)} \cdots L_n^{(1)} D_n U_n^{(1)} \cdots U_n^{(r)},$$

with $L_n^{(k)}$ unit-diagonal (1,0)-banded TP, $U_n^{(k)}$ unit-diagonal (0,1)-banded TP, and D_n diagonal TP. We intend to let n go to infinity and therefore must consider the possibility that these factors may not be bounded independently of n. There is no such difficulty in case A is *strictly* banded, the case treated earlier in [3], since in that case the finite factors can even be seen to converge *monotonely*. But, without strict bandedness, we must deal with the possible unboundedness of the finite factors.

For this, define one-banded matrices $S^{(-l)},\dots,S^{(r)}$ as follows. Starting with $M^{(-l)}:=1$, define $S^{(-l)},\dots,S^{(-1)}$ successively by

$$S^{(-k)}:=M^{(-k)}L_n^{(k)}\left(M^{(-k+1)}\right)^{-1},$$

with $M^{(-k+1)}$ the diagonal matrix having $\max_i(M^{(-k)}L_n^{(k)})(i,j)$ in its jth diagonal position. This number cannot be zero (by induction on k), since $M^{(-l)}=1$ and $L_n^{(k)}$ is unit-diagonal. It follows that each $S^{(-k)}$ is a $(1,0)$-banded TP matrix with maximum entry 1 in each column, and

$$L_n^{(l)}\cdots L_n^{(1)}=S^{(-l)}\cdots S^{(-1)}M^{(0)}.$$

Now continue the process, starting with $M^{(1)}:=M^{(0)}D_n$, getting successively $S^{(1)},\dots,S^{(r)}$ by

$$S^{(k)}:=M^{(k)}U_n^{(k)}\left(M^{(k+1)}\right)^{-1},$$

with $M^{(k+1)}$ the diagonal matrix having $\max_i(M^{(k)}U_n^{(k)})(i,j)$ in its jth position.

We arrive at the factorization

$$A_n=S^{(-l)}\cdots S^{(-1)}S^{(1)}\cdots S^{(r)}M^{(r)}=:R_n^{(1)}\cdots R_n^{(m)}E_n,$$

with each $R_n^{(k)}$ one-banded TP and maximum entry 1 in each column, and E_n a diagonal TP matrix. We claim that

$$0<E_n(j,j)\leqslant\max_i A(i,j),\qquad \text{all } j.$$

We know that

$$A(i,j)=\sum_{j_1}\cdots\sum_{j_m}R_n^{(1)}(i,j_1)R_n^{(2)}(j_1,j_2)\cdots R_n^{(m)}(j_{m-1},j_m)E_n(j_m,j)$$

with all summands nonnegative. Further, for at least one choice of i, one of the summands is just $E_n(j,j)$, since, starting with $j_m=j$, we can pick $j_{m-1},j_{m-2},\dots,j_0=:i$ in sequence so that $R_n^{(k)}(j_{k-1},j_k)=1$. But then $A(j_0,j)\geqslant E_n(j,j)$.

We can now let n go to infinity through a subsequence of $\mathbb{N}$ in such a way that each of the matrices $R_n^{(k)}$ converges entrywise to some (bi)infinite matrix

$R^{(k)}$, necessarily one-banded TP with maximum entry 1 in each column, and E_n likewise converges to some diagonal matrix D satisfying $0 \leqslant D(j,j) \leqslant \max_i A(i,j)$, all j, while

$$A = R^{(1)} \cdots R^{(m)} D.$$

But then $0 < D(j,j)$, all j, since otherwise $A(\cdot, j) = 0$, contradicting the linear independence of the columns of A. ∎

COROLLARY. *Let A be a (bi)infinite TP m-banded matrix whose rows and columns are linearly independent. Then A is the limit of strictly m-banded (bi)infinite TP matrices, and this limit is uniform (i.e., in norm) if A is bounded.*

Proof. Replace each zero entry in the two interesting bands of $R^{(k)}$ above by $\varepsilon > 0$ to obtain the strictly one-banded TP matrix $R_\varepsilon^{(k)}$, all k. Then

$$A_\varepsilon := R_\varepsilon^{(1)} \cdots R_\varepsilon^{(m)} D$$

is strictly m-banded TP (as a product of strictly banded TP matrices) and converges entrywise to A as $\varepsilon \to 0$. Since the entries of $R^{(k)}$ are bounded by 1 while those of D are bounded by $\|A\|_\infty$, this convergence is obviously uniform in case $\|A\|_\infty < \infty$. ∎

REMARK. The assumption that the rows and columns of A are linearly independent is not bothersome in the intended use of this corollary in [1], since there A is even boundedly invertible. But it would be nice to know whether this assumption is necessary. We note that Metelmann [8] has obtained strictly one-banded TP factorizations for finite strictly (l, r)-banded TP matrices, and that Cryer [4] has obtained one-banded TP factorizations for arbitrary finite TP matrices. But the procedure given by Cryer may produce more than m 1-banded factors unless the matrix is *strictly* m-banded.

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