Reprinted from:
APPROXIMATION THEORY, II

ACADEMIC PRESS, INC.
New York San Francisco London

THE EXACT ASYMPTOTIC VALUE FOR THE N-WIDTH OF SMOOTH FUNCTIONS IN L^∞

Charles A. Micchelli and Allan Pinkus

In this paper we answer a question raised by Chui and Smith and obtain the exact asymptotic value for the N-width of the set $D_r = \{f: ||Lf||_\infty \le 1,\ f \in W^r_\infty[0,1]\}$ where L is an r-th order totally disconjugate differential operator and $||\cdot||_\infty$ = sup norm on [0,1].

1. Introduction

Let $W^r_\infty[0,1] = \{f: f^{(r-1)}$ absolutely continuous on [0,1], $f^{(r)} \in L^\infty[0,1]\}$, $\lambda_j \in C^{r-j}[0,1]$, $j=1,\ldots,r$, and $(Lf)(x) = \prod_{j=1}^{r} (\frac{d}{dx} + \lambda_j(x))f(x)$. The N-width of the set

(1) $D_r = \{f: f \in W^r_\infty[0,1],\ ||Lf||_\infty \le 1\}$

(relative to $C[0,1]$) is defined by

(2) $d_N(D_r) = \inf_{X_N} \sup_{f \in D_r} \inf_{g \in X_N} ||f-g||_\infty,$

where the infimum is taken over all N-dimensional linear subspaces X_N of $C[0,1]$.

The purpose of this paper is to prove the following theorem which answers a question raised in Chui and Smith [1].

Let

$$e_r = \frac{2}{\pi^{r+1}} \sum_{-\infty}^{+\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^{r+1}} .$$

THEOREM. For any $r \geq 1$,

$$\lim_{N\to\infty} N^r d_N(D_r) = e_r$$

The proof of the above theorem relies on several results of [3] which we summarize below.

2. Proof of theorem

Let $K(x,y)$ be the Green's function for the initial value problem

$$(Lf)(x) = h(x)$$
$$f^{(i)}(0) = 0,\ i=0,1,\ldots,r-1.$$

Also, define $k_0(x),\ldots,k_{r-1}(x)$ as the unique set of functions in the null space of L satisfying the conditions

$$k_i^{(j)}(0) = \delta_{ij},\ i,j=0,1,\ldots,r-1.$$

The Green's function $K(x,y)$ has the property that

$$K(x,y) = \begin{cases} 0 & , x < y \\ H(x,y), & x > y \end{cases}$$

where for each fixed y, $H(x,y)$ is in the null space of L, as a function of x.

Thus D_r has the equivalent representation

$$(3)\quad D_r = \{\sum_{j=0}^{r-1} a_j k_j(x) + \int_0^1 K(x,y)h(y)dy:\ ||h||_\infty \leq 1,\ (a_0,\ldots,a_{r-1}) \in R^r\}.$$

S. Karlin proves in [2] that for every integer $s \geq 0$ there exists a function

(4) $$P_s(x) = \sum_{j=0}^{r-1} b_j k_j(x) + \sum_{j=0}^{s} (-1)^j \int_{\xi_j}^{\xi_{j+1}} K(x,y)dy$$

$$0 = \xi_0 < \xi_1 <\ldots< \xi_s < \xi_{s+1} = 1$$

which equioscillates $r+s+1$ times, that is,

$$P_s(\tau_i) = (-1)^{i+1}||P_s||_\infty, \quad i = 1,\ldots,r+s+1,$$

for some points $0 \le \tau_1 < \tau_2 <\ldots< \tau_{r+s+1} \le 1$ (see also [3]). We will denote by Q_s the class of all functions which may be expressed as

$$P(x) = \sum_{j=0}^{r-1} a_j k_j(x) + \sum_{j=0}^{\ell} (-1)^j \int_{\eta_j}^{\eta_{j+1}} K(x,y)dy$$

for some constants $(a_0,\ldots,a_{r-1}) \in R^r$ and points $0 = \eta_0 < \eta_1 <\ldots< \eta_\ell < \eta_{\ell+1} = 1$, $\ell \le s$. Then P_s has the following properties:

(5) $$||P_s||_\infty \le ||P||_\infty, \quad P \in Q_s,$$

and

(6) $$\min_{1\le j\le r+s+1} |f(x_j)| \le ||P_s||_\infty,$$

where f is any function in D_r such that for some points $0 \le x_1 <\ldots< x_{r+s+1} \le 1$, $f(x_i)f(x_{i+1}) \le 0$, $i=1,\ldots,r+s$.

The importance of the function P_s rests on the equation

(7) $$d_N(D_r) = ||P_{N-r}||_\infty, \quad N \ge r,$$

which, along with (5) and (6), was proven in [3].

We are now prepared to prove the theorem.

PROOF. For every integer N, let

$$G_N(x) = \sum_{j=0}^{N-1} (-1)^j \int_{\frac{j}{N}}^{\frac{j+1}{N}} K(x,y)dy.$$

We claim that there exists a v_N in the null space of L such that for the function $H_N = G_N + v_N$

(8) $$\lim_{N\to\infty} N^r ||H_N||_\infty = e_r$$

and there exist N point $0 \le x_1^N < \ldots < x_N^N \le 1$ such that

(9) $$\lim_{N\to\infty} \max_{1\le i\le N} \left| N^r H_N(x_i^N) - (-1)^{i+1} e_r \right| = 0.$$

These facts, together with (5), (6), and (7) imply

$$\min_{1\le i\le N+1} |H_{N+1}(x_i^{N+1})| \le d_N(D_r) \le ||H_{N-r+1}||_\infty .$$

Thus we conclude the validity of the theorem.

Let us then prove (8) and (9). Recall that the r-th Euler polynomial is defined by the relation

$$E_r(x+1) + E_r(x) = \frac{2x^r}{r!}, \quad x \in R.$$

Here we have normalized E_r so that $E_r^{(r)}(x) = 1$. We perform the usual surgery on E_r and define $\bar{E}_r$ by

(10) $$\begin{aligned} \bar{E}_r(x) &= E_r(x), \quad x \in [0,1] \\ \bar{E}_r(x+1) &= -\bar{E}_r(x), \quad x \in R. \end{aligned}$$

It is evident that we may express G_N as

$$G_N(x) = \frac{1}{N^r} \int_0^1 \left(\frac{d^r}{dy^r} \bar{E}_r(Ny)\right) K(x,y)dy.$$

Our next step is to integrate the above expression by parts. For any $f \in W_\infty^r[0,1]$, there exists constants $c_0, c_1, \ldots, c_{r-1}$ such that

$$\int_0^1 f^{(r)}(y)K(x,y)\,dy = f(x) - \sum_{j=0}^{r-1} c_j k_j(x) + \int_0^x J(x,y)f(y)\,dy$$

where

$$J(x,y) = (-1)^r \frac{\partial^r}{\partial y^r} K(x,y), \quad y < x.$$

Applying this identity to $G_N(x)$ it follows that there exists v_N in the null space of L such that

$$G_N(x) = \frac{1}{N^r}\overline{E}_r(Nx) - v_N(x) + \frac{1}{N^r}\int_0^x J(x,y)\overline{E}_r(Ny)\,dy.$$

We define $H_N = G_N + v_N$. From (10) we note that (8) and (9) will follow provided that

(11) $\quad ||E_r||_\infty = e_r$

and

(12) $\quad \lim\limits_{N\to\infty} \max\limits_{0\le x\le 1} \left|\int_0^x J(x,y)\overline{E}_r(Ny)\,dy\right| = 0.$

The expression (11) for the L^∞-norm of E_r is well-known. It is easily deduced from the Fourier series expansion of $e^{i\pi x}E_r(x)$ and the fact that

$$||E_r||_\infty = \begin{cases} |E_r(\frac{1}{2})|, & r \text{ even} \\ |E_r(0)|, & r \text{ odd} \end{cases}$$

Thus it remains to verify (12).

Let M be an integer. Divide $[0,1]$ into M equal pieces, $I_i = [\frac{i}{M}, \frac{i+1}{M}]$, $i = 0,1,\ldots,M-1$. Let $g_i(y)$ be the characteristic function of the interval I_i and define $S_M(x,y) = \sum_{i=0}^{M-1} J(x,\frac{i}{M})g_i(y)$. Then $\lim\limits_{M\to\infty} ||J-S_M||_\infty = 0$, where $||\cdot||_\infty$ denotes the L^∞-norm on $[0,1] \times [0,1]$. Now, for any

$x \in [0,1]$

$$\left|\int_0^x J(x,y)\overline{E}_r(Ny)\,dy\right|$$
$$\le ||J-S_M||_\infty \int_0^1 |\overline{E}_r(Ny)|\,dy + M||J||_\infty \max_{0\le i\le M+1} \left|\int_{I_i} \overline{E}_r(Ny)\,dy\right|.$$

However, $\left|\int_{I_i} \overline{E}_r(Ny)\,dy\right| \le \frac{1}{N}\int_0^1 |E_r(y)|\,dy$ for all $i=0,1,\ldots,M-1$.

Thus

$$\lim_{N\to\infty} \max_{0\le x\le 1} \left|\int_0^x J(x,y)\overline{E}_r(Ny)\,dy\right| \le ||J-S_M||_\infty \int_0^1 |E_r(y)|\,dy.$$

Letting $M\to\infty$ we obtain (12) and thus the proof is complete.

REMARK. We conjecture that the above theorem remains valid for any r-th order differential operator

$$L = D^r + \sum_{j=0}^{r-1} a_j D^j.$$

Our proof, however, requires the upper and lower bounds given by (5) and (6) which were proven in [3] only for differential operators L allowing a global factorization into real linear factors.

References

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Charles A. Micchelli
Mathematical Sciences Department
IBM T. J. Watson Research Center
Yorktown Heights, N.Y. 10598

Allan Pinkus
Mathematics Research Center
University of Wisconsin
Madison, Wisconsin 53706