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Asymptotic Minimum Norm Quadrature Formulae

Allan Pinkus*

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Abstract. The asymptotic limit of minimum norm quadrature formulae for some Hilbert spaces of functions regular and analytic in a domain B is studied, as B expands.

1. Introduction

Let B be a complex domain such that $\mathbb{C}\setminus B$ contains an interior. $D(B)$ is the separable Hilbert space defined by $D(B)=\{f\colon f$ regular analytic in B, $\|f\|_B=(\iint_B |f|^2\,dx\,dy)^{\frac{1}{2}}<\infty\}$. Consider the collection of quadrature formulae (q.f) R expressed by

$$Rf=\int_a^b f(x)\,w(x)\,dx-\sum_{i=1}^{k}\sum_{j=0}^{\mu_i} a_{ij}\,f^{(j)}(\eta_i), \tag{1.1}$$

where

1) $[a, b]$ is a real finite interval in B (assume $a>0$ in all examples),
2) $a\leqq\eta_1<\dots<\eta_k\leqq b$,
3) $a_{ij}\in\mathbb{C}$,
4) $w(x)\in C[a, b]$, $w(x)>0$, $x\in[a, b]$,
5) $\sum_{i=1}^{k}(\mu_i+1)\leqq n$, ($n$ fixed throughout the paper).

Let

$$\|R\|_B=\sup_{f\in D(B)}\frac{|Rf|}{\|f\|_B}.$$

To each $\boldsymbol{a}=(a_{10},\dots,a_{1\mu_1},a_{20},\dots,a_{k\mu_k})$ and $\boldsymbol{\eta}=(\eta_1,\dots,\eta_k)$ satisfying the above conditions, (k, $\{\mu_i\}_1^k$ allowed to vary within the above bounds) there is associated a quadrature formula (thru the series part of (1.1)). Let $K=\inf_{\boldsymbol{a},\boldsymbol{\eta}}\|R_{\boldsymbol{a},\boldsymbol{\eta}}\|_B$. There exists a q.f. in this class which is called the minimum norm q.f. and denoted by R_B^{MN} which attains the above infimum (Theorem 2.1).

Minimum norm q.f. have been studied by Barnhill [1–3], Barnhill and Wixom [4], Richter [12], Valentin [13], Davis [6], and others. We are concerned with the asymptotic limit of the minimum norm q.f. as B grows to all of $\mathbb{C}$. Valentin [13] and Barnhill [3], considered the circle, and ellipse with foci at ± 1, respectively. With the usual parametrization of these domains, they proved that as the circle, and ellipse with foci at ± 1 extend to all of $\mathbb{C}$, the nodes and coefficients of the minimum norm q.f. converge to the nodes and coefficients of the Gaussian q.f. defined with respect to $\{x^i\}_{i=0}^{2n-1}$.

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We show that for any domain B which grows in a radial-like manner to cover all of $\mathbb{C}$, the above result holds (Theorems 3.1 and 3.2). If we consider the L^p norm, $2 \leqq p \leqq \infty$, i.e.,

$$D^p(B) = \left\{f\colon f \text{ regular analytic in } B, \|f\|_{B,p} = \left(\iint_B |f|^p \,dx\,dy\right)^{1/p} < \infty\right\} \quad \text{and}$$

$$\|R\|_{B,p} = \sup_{f\in D^p(B)} \frac{|Rf|}{\|f\|_{B,p}}.$$

then these same conclusions prevail (Theorems 3.3 and 3.4).

Gaussian quadrature formulae may be defined with respect to any Tchebycheff (T) system on $[a, b]$ (see Karlin and Studden [10]). That is, for any T-system $\{u_i(x)\}_{i=0}^{2n-1}$ on $[a, b]$, there exists a unique set of distinct nodes $\{\xi_i\}_{i=1}^n$ in (a, b), and positive coefficients $\{a_i\}_{i=1}^n$ such that

$$\int_a^b u_j(x)\,w(x)\,dx = \sum_{i=1}^n a_i u_j(\xi_i) \qquad j = 0, 1, \ldots, 2n-1.$$

(The uniqueness is with respect to all possible nodes and coefficients such that $\sum_{i=1}^k (\mu_i + 1) \leqq n$.) For an unbounded domain B which does not contain some circle centered at 0, but which does contain the exterior of some other circle, we obtain the result that as B expands towards 0, in a radial-like manner, the minimum norm q.f. converges to the Gaussian q.f. with respect to the T-system $\left\{\frac{1}{x^i}\right\}_{i=2}^{2n+1}$ (Theorem 4.1).

Of some interest is the example of the annulus where the inner radius tends to zero while the outer radius converges to infinity. Here, the result depends upon the ratio of the rates at which the radii tend to zero and infinity. Except for certain degenerate cases, the minimum norm q.f. converges to the Gaussian q.f. defined with respect to $\{x^i\}_{i=k}^{k+2n-1}$, for an appropriate $k = -2n, \ldots, -1$ (Theorem 5.1).

In the above three cases, convergence was to some Gaussian q.f. defined with respect to a subset of the ordinary powers. This need not occur. As illustration, consider the following example.

Let $B(r) = \{z\colon z = x + iy,\ -r < x < \infty,\ -\pi < y < \pi\}$ and

$$E(B(r)) = \Big\{f\colon f \text{ regular analytic in } B(r),$$

$$\|f\|_{B(r)} = \left(\iint_{B(r)} |f|^2 \,dx\,dy\right)^{\frac{1}{2}} < \infty, \quad \text{and} \quad f(z) = \sum_{m=1}^{\infty} a_m e^{-mz} \quad \text{on} \quad B(r)\Big\}.$$

As $r \uparrow \infty$, the minimum norm q.f. converges to the Gaussian q.f. with respect to the T-system $\{e^{-mx}\}_{m=1}^{2n}$ (Theorem 6.1). Results analogous to those obtained for the annulus hold where we consider

$B(r_1, r_2) = \{z\colon z = x + iy,\ -r_2 < x < r_1,\ -\pi < y < \pi\}$ and let r_1 and r_2 tend to infinity (Theorem 6.2).

Although we consider quadrature formulae with variable knots, the methods employed may be applied to other problems. Some of these are outlined in Section 7.

2. Theorems on Existence and Convergence

In what follows, we shall assume, unless otherwise stated, that

$$\|f\|_B = \left[\iint_B |f|^2 \, dx\, dy\right]^{\frac{1}{2}}, \quad \text{and} \quad (f, g)_B = \iint_B f(z)\overline{g(z)}\, dx\, dy.$$

Let Q denote the class of quadrature formulae (q.f.) determined by (1.1) (recall that $\{\mu_i\}_{i=1}^k$ and k are free to vary under the restriction $\sum_{i=1}^k (\mu_i + 1) \leqq n$).

Theorem 2.1. *Under the above conditions and assuming $\mathbb{C}\setminus B$ has an interior, there exists an $R_B^{MN} \in Q$ satisfying $\|R_B^{MN}\|_B = \inf_{R\in Q} \|R\|_B$.*

Proof. The proof of the theorem utilizes methods which are now standard in approximation theory (see Hobby and Rice [8]). We use the fact that because $\|R\|_B = \sup_{f\in D(B)} \frac{|Rf|}{\|f\|_B}$ and since for $\|f\|_B \leqq 1$, $[a, b] \subset B$, $|f^{(l)}(x)|$ is uniformly bounded on $[a, b]$ by a constant depending solely on $d(\partial B, [a, b]) > 0$, it follows that $\|R\|_B < \infty$, for $\{a_{ij}\}_{1\,0}^{k\,\mu_i}$ finite.

Remark 2.1. If B is a simply connected bounded domain, symmetric with respect to the real line, Karlin [14], has shown the existence of an R^* with n distinct simple knots such that

$$\|R^*\|_B = \inf_{R\in Q} \|R\|_B.$$

Remark 2.2. For the subspaces of $D(B)$ to be considered in the following sections, Theorem 2.1 is maintained.

We now prove that if $|R_{B(r)}^{MN}(u_i)| \underset{r\uparrow\infty}{\to} 0$, $i = 0, 1, \ldots, 2n-1$, where $\{u_i(x)\}_{i=0}^{2n-1}$ is an extended complete Tchebycheff (ECT) system on $[a, b]$ (see [10]), then the coefficients and nodes of $R_{B(r)}^{MN}$ converge to the coefficients and nodes of R^G, the Gaussian quadrature formula defined with respect to $\{u_i(x)\}_{i=0}^{2n-1}$ and the weight function $w(x)$.

Before entering into the proof of this fact, it is worth noting that the restriction $w(x) > 0$, $x \in [a, b]$ and $w(x) \in C[a, b]$ is not a necessary condition for the existence of a Gaussian quadrature formula with respect to a Tchebycheff system $\{u_i\}_{i=0}^{2n-1}$ on $[a, b]$. Necessary conditions may be found in Karlin and Studden [10, p. 137]. We will, without loss of generality, drop all references to $w(x)$ and the reader should understand that the statements hold for any $w(x)$ of the above form.

Theorem 2.2. *Assume $B(r)$ satisfies the assumptions of Theorem 2.1 for each $r < \infty$. If $|R_{B(r)}^{MN}(u_i)| \to 0$, $i = 0, 1, \ldots, 2n-1$, where $\{u_i(x)\}_{i=0}^{2n-1}$ is an ECT-system on $[a, b]$, then the coefficients and nodes of the minimum norm (MN) q.f. associated with $B(r)$ converge to the coefficients and nodes of the Gaussian q.f. defined with respect to $\{u_i(x)\}_{i=0}^{2n-1}$.*

Proof. Let

$$R_{B(r)}^{MN}(f) = \int_a^b f(x)\, dx - \sum_{i=1}^{l(r)} \sum_{j=0}^{\mu_i(r)} c_{ij}(r) f^{(j)}(\eta_i(r))$$

where $R^{MN}_{B(r)}$ is a MN q.f. for $B(r)$, and where we assume, without loss of generality, that $\sum_{i=1}^{l(r)} (\mu_i(r)+1) = n$ for all r, $(c_{ij}(r)$ may be zero$)$, and $\mu_i(r) = \mu_i$, $i=1, \ldots, l(r) = l$ (i.e., choose a subsequence of this form). Also assume that

$$\eta_i(r) \underset{r\to\infty}{\to} \xi_j, \quad i=1,\ldots,l; \quad j=1,\ldots,s,$$

where

$$a \leqq \xi_1 < \cdots < \xi_s \leqq b, \quad \nu_j = \sum_{\eta_i(r)\to\xi_j} \mu_i, \quad \left(\sum_{i=1}^{s} (\nu_i+1) = n\right).$$

We show that a contradiction ensues if

1) ν_i are not all zero, i.e. $s \neq n$ or
2) $a = \xi_1$ or $\xi_s = b$.

Let $u(t)$ be the polynomial which

a) has a zero of multiplicity ν_i+1 at ξ_i if ν_i odd or if $\xi_i = a$ or b;
b) has a zero of multiplicity ν_i+2 at ξ_i if ν_i even.

If either 1) or 2) hold, then there exists a polynomial $u(t) = \sum_{i=0}^{2n-1} a_i u_i(t)$ satisfying the above such that $u(t) \geqq 0$, $t \in [a, b]$, and $\sum_{i=0}^{2n-1} a_i^2 = 1$.

Now

$$|R^{MN}_{B(r)}(u(t))| \leqq \sum_{i=0}^{2n-1} |a_i|\, |R^{MN}_{B(r)}(u_i(t))| \underset{r\uparrow\infty}{\to} 0.$$

However

$$|R^{MN}_{B(r)}(u(t))| \underset{r\uparrow\infty}{\to} \left|\int_a^b u(t)\,dt\right| > 0.$$

(This holds since the coefficients $c_{ij}(r)$ must be bounded. A simple argument of the above type proves this fact.) Thus, a contradiction ensues unless $a < \xi_1 < \cdots < \xi_n < b$ and $\nu_i = 0$, $i=1,\ldots,n$. Let c_i be the coefficient associated with each ξ_i, $i=1,\ldots,n$. The quadrature formula with nodes ξ_i and coefficients c_i, $i=1,\ldots,n$ satisfies $R(u_j)=0$, $j=0,1,\ldots,2n-1$. Thus ξ and c are the nodes and coefficients of the Gaussian q.f. associated with the ECT-system $\{u_i(t)\}_0^{2n-1}$.

Since this limit is independent of the subsequence, the theorem is proven. The fact that $R^{MN}_{B(r)}$ is not uniquely defined for each r is not crucial. Q.E.D.

3. Bounded Domains and Radial-Like Limits

Let B be any bounded domain and assume, without loss of generality, that $0 \in B$.

Theorem 3.1. *Let $a(r)$ be a non-decreasing real-valued continuous function of r on $[1, \infty)$ for which $a(1)=1$, and $\lim_{r\to\infty} a(r) = \infty$. Set*

$$B(r) = \left\{z : \frac{z}{a(r)} \in B\right\},$$

and let $R^{MN}_{B(r)}$ be defined as above with

$$\|f\|_{B(r)} = \left[\iint_{B(r)} |f|^2\,dx\,dy\right]^{\frac{1}{2}}.$$

Then the coefficients and nodes of $R^{MN}_{B(r)}$ *converge to the coefficients and nodes of the Gaussian q.f. defined with respect to* $\{x^i\}_{i=0}^{2n-1}$.

Proof. We assume without loss of generality that $[a, b] \subset B(r)$, for all $r \geqq 1$. By Theorem 2.2, we must show that $|R^{MN}_{B(r)}(x^i)| \underset{r\uparrow\infty}{\rightarrow} 0$, $i = 0, 1, \ldots, 2n-1$. Let R^G denote the Gaussian q.f. defined with respect to $\{x^i\}_0^{2n-1}$. Since $0 \in B$, there exists a circle of radius s, center 0, which we denote by $U(s)$, such that $U(s) \subseteqq B$. Let $U(sa(r)) = \left\{z: \frac{|z|}{sa(r)} \leqq 1\right\}$. Thus $U(sa(r)) \subseteqq B(r)$. Without loss of generality, let $s = 1$. In the proof of the theorem, we use the following lemma.

Lemma 3.1. *Under the above assumptions,*

$$[a(r)]^{2n} \|R^{MN}_{B(r)}\|_{B(r)} \underset{r\uparrow\infty}{\rightarrow} 0.$$

Proof.

$$\left\{\sqrt{\frac{m+1}{\pi}}\,\frac{z^m}{(a(r))^{m+1}}\right\}_0^\infty$$

is an orthonormal complete sequence in $D(U(a(r)))$. Thus

$$\|R^G\|^2_{U(a(r))} = \sum_{m=0}^{\infty} \left|R^G\left(\sqrt{\frac{m+1}{\pi}}\,\frac{z^m}{(a(r))^{m+1}}\right)\right|^2$$
$$= \sum_{m=2n}^{\infty} \left|R^G\left(\sqrt{\frac{m+1}{\pi}}\,\frac{z^m}{(a(r))^{m+1}}\right)\right|^2$$

since $R^G(z^i) = 0$, $i = 0, 1, \ldots, 2n-1$. It follows that

$$[a(r)]^{4n} \|R^G\|^2_{U(a(r))} = [a(r)]^{4n} \sum_{m=2n}^{\infty} \left|R^G\left(\sqrt{\frac{m+1}{\pi}}\,\frac{z^m}{(a(r))^{m+1}}\right)\right|^2 \underset{r\uparrow\infty}{\rightarrow} 0$$

since $\|R^G\|_{U(a(r_0))} < \infty$, and

$$\frac{[a(r)]^{4n}}{[a(r)]^{2m+2}} \downarrow 0$$

as $r \uparrow \infty$ for $m \geqq 2n$. By definition

$$\|R^G\|_{U(a(r))} = \sup_{f \in D(U(a(r)))} \frac{|R^G f|}{\|f\|_{U(a(r))}}.$$

Since $D(B(r)) \subset D(U(a(r)))$, and for $f \in D(B(r))$, $\|f\|_{U(a(r))} \leqq \|f\|_{B(r)}$, it follows that

$$\|R^G\|_{U(a(r))} \geqq \sup_{f \in D(B(r))} \frac{|R^G f|}{\|f\|^{B(r)}}$$
$$= \|R^G\|_{B(r)}$$
$$\geqq \|R^{MN}_{B(r)}\|_{B(r)}.$$

Thus $[a(r)]^{2n} \|R^{MN}_{B(r)}\|_{B(r)} \underset{r\uparrow\infty}{\rightarrow} 0$. Q.E.D.

Proof of Theorem 3.1 (Continued). $1, z, \ldots, z^{2n-1} \in D(B)$. Choose b_k such that $\|b_k z^k\|_B = 1$. It is easily seen that

$$\frac{b_k z^k}{[a(r)]^{k+1}} \in D(B(r)), \quad \text{and} \quad \left\|\frac{b_k z^k}{[a(r)]^{k+1}}\right\|_{B(r)} = 1.$$

Since

$$[a(r)]^{2n} \| R^{MN}_{B(r)} \|_{B(r)} \underset{r\uparrow\infty}{\to} 0,$$

we have

$$[a(r)]^{2n} \left| R^{MN}_{B(r)} \left(\frac{b_k z^k}{[a(r)]^{k+1}} \right) \right| \underset{r\uparrow\infty}{\to} 0.$$

It follows that

$$|R^{MN}_{B(r)}(z^k)| = |R^{MN}_{B(r)}(x^k)| \underset{r\uparrow\infty}{\to} 0,$$

for $k = 0, 1, \ldots, 2n-1$. Q.E.D.

In Theorem 3.1, $B(r)$ grew in a radial manner. We generalize this condition in the following theorem.

Theorem 3.2. *Let $a(r)$ and $U(s)$ be defined as above. Let $t > s > 0$ be such that $U(s) \subseteqq B \subseteqq U(t)$. Define $\tilde{B}(r)$ to be any domain such that $U(s\,a(r)) \subseteqq \tilde{B}(r) \subseteqq U(t\,a(r))$, for all $r \geqq 1$. Then the results of Theorem 3.1 hold.*

Proof. The proof is totally analogous to that of Theorem 3.1.

Remark 3.1. The method of proof used in Theorems 3.1 and 3.2 is a generalization of that used by both Valentin [13] and Barnhill [3]. The cases which they each considered are subsumed by Theorem 3.2.

Corollary 3.1. *Let $\tilde{D}(B)$ be any closed Hilbert subspace of $D(B)$ containing the functions $1, z, \ldots, z^{2n-1}$, and assume otherwise that the conditions of Theorem 3.1 hold. Define*

$$\|R\|_B = \sup_{f \in \tilde{D}(B)} \frac{|R(f)|}{\|f\|_B}.$$

etc .... *Then the results of Theorem 3.1 and Theorem 3.2 hold.*

Proof. The same as for Theorems 3.1 and 3.2.

In the foregoing we have considered the problem on the Hilbert space $L^2(B)$. We now extend the above results to $L^p(B)$, $2 \leqq p \leqq \infty$.

Let $B(r)$ be as in Theorem 3.1, and let

$$D^p(B(r)) = \left\{ f\colon f \text{ regular analytic in } B(r), \text{ and } \|f\|_{B(r),p} = \left[\iint_{B(r)} |f|^p \, dx\, dy \right]^{1/p} < \infty \right\}$$

for $2 \leqq p < \infty$. As usual, define

$$\|R\|_{B(r),p} = \sup_{f \in D^p(B(r))} \frac{|Rf|}{\|f\|_{B(r),p}},$$

and specify $R^{MN}_{B(r),p}$ in the analogous manner.

Theorem 3.3. *Under the above conditions, the results of Theorem 3.1 hold for the quadrature formulae $R^{MN}_{B(r),p}$ $2 \leqq p < \infty$.*

Proof. Let $m(B) = \iint_B dx\, dy < \infty$. Then,

$$\begin{aligned}\|f\|_{B(r),2} &= \left(\iint_{B(r)} |f|^2 \, dx\, dy \right)^{\frac{1}{2}} \leqq \left(\iint_{B(r)} |f|^p \, dx\, dy \right)^{1/p} \left(\iint_{B(r)} dx\, dy \right)^{\frac{p-2}{2p}} \\ &= \|f\|_{B(r),p} \left[\sqrt{m(B)}\, a(r) \right]^{1-2/p},\end{aligned}$$

by Holder's inequality. Thus, $D^p(B(r)) \subseteq D^2(B(r))$, and if $\|f\|_{B(r),p} \leqq 1$, then $\|f\|_{B(r),2} \leqq [\sqrt{m(B)}\, a(r)]^{1-2/p}$. Therefore,

$$\begin{aligned}
\|R^G\|_{B(r),2} &= \sup_{f \in D^2(B(r))} \frac{|R^G f|}{\|f\|_{B(r),2}} \\
&\geqq \sup_{f \in D^p(B(r))} \frac{|R^G f|}{\|f\|_{B(r),p}[\sqrt{m(B)}\, a(r)]^{1-2/p}} \\
&= \frac{\|R^G\|_{B(r),p}}{[\sqrt{m(B)}\, a(r)]^{1-2/p}} \\
&\geqq \frac{\|R^{MN}_{B(r),p}\|_{B(r),p}}{[\sqrt{m(B)}\, a(r)]^{1-2/p}}.
\end{aligned}$$

From Lemma 3.1, $[a(r)]^{2n}\|R^G\|_{B(r),2} \underset{r\uparrow\infty}{\to} 0$. Choose b_k such that $\|b_k z^k\|_{B,p} = 1$. Since

$$\left\|\left(\frac{1}{a(r)}\right)^{2/p} \frac{b_k z^k}{(a(r))^k}\right\|_{B(r),p} = 1,$$

$$[a(r)]^{2n-1+2/p}\left|\left(\frac{1}{a(r)}\right)^{2/p} R^{MN}_{B(r),p}\left(\frac{b_k z^k}{(a(r))^k}\right)\right| \underset{r\uparrow\infty}{\to} 0.$$

The result follows. Q.E.D.

For $p = \infty$, set

$$\|f\|_{B(r),\infty} = \sup_{z \in B(r)} |f(z)|,$$

while $D^\infty(B(r))$, $\|R\|_{B(r),\infty}$, and $R^{MN}_{B(r),\infty}$ are defined analogously.

Theorem 3.4. *The results of Theorem 3.1 hold for* $p = \infty$.

Proof. The proof is a repetition of Theorem 3.3, where we note that

$$\|f\|_{B(r),2} \leqq \|f\|_{B(r),\infty} [\sqrt{m(B)}\, a(r)]. \quad \text{Q.E.D.}$$

Note that Theorem 3.2 extends to cover the cases considered in Theorems 3.3 and 3.4.

4. Unbounded Domains and Radial—Like Limits

In the last section we considered bounded domains with a chosen fixed point, which was used as the centre, and radial-like growth. In this section we briefly consider the reciprocal problem.

Let B be any unbounded domain which contains the exterior of some circle and such that $d(O, \partial B) > 0$. We shall let B expand towards 0, and prove that

$$R^{MN}_{B(r)}\left(\frac{1}{x^k}\right) \underset{r\uparrow\infty}{\to} 0 \qquad \text{for } k = 2, 3, \ldots, 2n+1.$$

Note that $\left\{\frac{1}{x^k}\right\}^2_{k=2n+1}$ is an ECT-system on $[a, b]$.

Theorem 4.1. *Let* $a(r)$ *be a non-increasing real-valued continuous function of* r *on* $[1, \infty]$, *for which* $a(1) = 1$, *and* $\lim_{r\to\infty} a(r) = 0$. *If*

$$B(r) = \left\{z : \frac{z}{a(r)} \in B\right\},$$

and $R^{MN}_{B(r)}$ *is defined as usual with*

$$\|f\|_{B(r)} = \left[\iint_{B(r)} |f|^2 dx\, dy\right]^{\frac{1}{2}},$$

then the coefficients and nodes of $R^{MN}_{B(r)}$ *converge to the coefficients and nodes of the Gaussian q.f. defined with respect to* $\left\{\frac{1}{x^k}\right\}_{k=2}^{2n+1}$.

Proof. We assume without loss of generality that $[a, b] \in B(r)$, for all $r \geqq 1$. By Theorem 2.1, it is sufficient to show that $R^{MN}_{B(r)}\left(\frac{1}{x^k}\right) \underset{r\uparrow\infty}{\rightarrow} 0, k = 2, \ldots, 2n+1$.

The proof is similar to that of Theorem 3.1 where we choose a $V(s) \subseteqq B$, where $V(s) = \{z : |z| > s\}$. Note that $\left\{\frac{\sqrt{m-1}}{\sqrt{\pi}} \frac{s^{m-1}}{z^m}\right\}_{m=2}^{\infty}$ is an orthonormal complete sequence in $D(V(s))$. The proof then follows in an analogous manner to Theorem 3.1. Q.E.D.

These results do not extend to $L^p(B)$, $2 < p \leqq \infty$. However, an analogue of Theorem 3.2 does hold.

5. The Annulus

In this section, we consider the following interesting example. Let

$$U(r, R) = \{z : r < |z| < R\},$$

and let $D(U(r, R))$ be analogously defined.

We shall consider the problem of convergence of the MN q.f. as $r \downarrow 0$ and $R \uparrow \infty$. We state the following well-known proposition.

Proposition 5.1. $\{\phi_m(z)\}_{m=-\infty}^{\infty}$ *is an orthonormal complete sequence in* $D(U(r, R))$, *where*

$$\phi_m(z) = \sqrt{\frac{m+1}{\pi}} \frac{z^m}{\sqrt{R^{2m+2} - r^{2m+2}}}; \quad m = -\infty, \ldots, \infty; \quad m \neq -1$$

and

$$\phi_{-1}(z) = \frac{1}{\sqrt{2\pi}} \frac{1}{z} \frac{1}{\sqrt{\log R/r}}.$$

Let $r = \frac{1}{R^\lambda}$, $0 < \lambda < \infty$, and consider the series $\{2m+2\}_{m=0}^{2n-2}$, and $\{\lambda(2m-2)\}_{m=2}^{2n}$. Combine and order the two series, and choose the smallest $2n-1$ terms. Assume that the $2n-1^{\text{st}}$ term is strictly less than the $2n^{\text{th}}$ term.

Let $m_1 \geqq -1$ be the largest integer such that $2m_1 + 2$ is in the set, where $m_1 = -1$ indicates that none of the terms of the first series lie in the set. Similarly, let $m_2 \geqq 1$ be the largest integer such that $\lambda(2m_2 - 2)$ lies in this set, where $m_2 = 1$ indicates that none of the terms of the second series lie in the set. Thus, $m_1 + m_2 = 2n - 1$.

Let R, $R^{MN}_{U(R^{-\lambda}, R)}$ be defined in the usual manner and assume $[a, b] \subset U(R_0^{-\lambda}, R_0)$, R_0 finite, $a > 0$. Then,

Theorem 5.1. *Under the above assumption, the coefficients and nodes of the MN q.f.* $R^{MN}_{U(R^{-\lambda}, R)}$ *converge, as* $R \to \infty$, *to the coefficients and nodes of the Gaussian q.f. associated with the ECT-system* $\{x^k\}_{k=-m_2}^{m_1}$ *on* $[a, b]$.

Remark 5.1. Note that $k=-1$ is always in the set. The condition

$$\max\{2m_1+2,\ \lambda(2m_2-2)\}<\min\{2(m_1+1)+2,\ \lambda(2(m_2+1)-2)\}$$

is necessary in the theorem. The cases with equality are, in effect, the boundary points, and we make no definite statement on the convergence at these values of λ.

Proof. The analysis follows the pattern of Theorem 3.1. Let R_λ^G indicate the Gaussian q.f. with respect to the functions given in the statement of the theorem. Let l be any number such that

$$\max\{2m_1+2,\ \lambda(2m_2-2)\}<l<\min\{2(m_1+1)+2,\ \lambda(2(m_2+1)-2)\}.$$

Lemma 5.1. *Under the above assumptions,*

$$R^l\|R^{MN}_{U(R^{-\lambda},R)}\|^2_{U(R^{-\lambda},R)}\to 0 \quad as \quad R\uparrow\infty.$$

Proof. Since

$$\|R_\lambda^G\|_{U(R^{-\lambda},R)}\geqq\|R^{MN}_{U(R^{-\lambda},R)}\|_{U(R^{-\lambda},R)}$$

for all $R>0$, it is sufficient to prove that $R^l\|R_\lambda^G\|^2_{U(R^{-\lambda},R)}\to 0$ as $R\uparrow\infty$.

$$\begin{aligned}R^l\|R_\lambda^G\|^2_{U(R^{-\lambda},R)}&=R^l\sum_{m=-\infty}^{\infty}|R_\lambda^G(\phi_m(z))|^2\\&=R^l\left[\sum_{m=m_1+1}^{\infty}|R_\lambda^G(\phi_m(z))|^2+\sum_{m=-m_2-1}^{-\infty}|R_\lambda^G(\phi_m(z))|^2\right]\end{aligned}$$

by the definition of R_λ^G. For given R_0, $\|R_\lambda^G\|_{U(R_0^{-\lambda},R_0)}<\infty$. Now,

$$\begin{aligned}R^l|R_\lambda^G(\phi_m(z))|^2&=R^l\frac{|m+1|}{\pi}\left|R_\lambda^G\left(\frac{z^m}{(R^{2m+2}-R^{-\lambda(2m+2)})^{\frac12}}\right)\right|^2\\&=\frac{R^l|R_0^{2m+2}-R_0^{-\lambda(2m+2)}|}{|R^{2m+2}-R^{-\lambda(2m+2)}|}\ \frac{|m+1|}{\pi}\left|R_\lambda^G\left(\frac{z^m}{(R_0^{2m+2}-R_0^{-\lambda(2m+2)})^{\frac12}}\right)\right|^2,\end{aligned}$$

and for $m=m_1+1,\ m_1+2,\ldots,\ \infty$, and $m=-m_2-1,\ -m_2-2,\ldots,\ -\infty$,

$$\frac{R^l|R_0^{2m+2}-R_0^{-\lambda(2m+2)}|}{|R^{2m+2}-R^{-\lambda(2m+2)}|}\to 0 \quad \text{as } R\uparrow\infty,$$

R_0 fixed. The lemma follows. Q.E.D.

Proof of Theorem 5.1 (continued). By the above, $R^l\|R^{MN}_{U(R^{-\lambda},R)}\|^2_{U(R^{-\lambda},R)}\to 0$ as $R\uparrow\infty$. Hence, $R^l|R^{MN}_{U(R^{-\lambda},R)}(\phi_m(z))|^2\to 0$ as $R\uparrow\infty$, $m=-\infty,\ldots,\infty$.

Thus, for $m=0,1,\ldots,m_1$,

$$R^l\left|R^{MN}_{U(R^{-\lambda},R)}\left(\frac{\sqrt{m+1}}{\sqrt{\pi}}\ \frac{z^m}{(R^{2m+2}-R^{-\lambda(2m+2)})^{\frac12}}\right)\right|^2\to 0$$

as $R\uparrow\infty$. Since $l>2m_1+2$, this implies that

$$|R^{MN}_{n,U(R^{-\lambda},R)}(z^m)|\to 0 \quad \text{as } R\uparrow\infty,$$

$m=0,1,\ldots,m_1$. This same method applies to $m=1,2,\ldots,-m_2$.

Since $\{x^k\}_{k=-m_2}^{m_1}$ is an ECT-system on $[a,b]$, $a>0$, the result is obtained via application of Theorem 2.2. Q.E.D.

The above results extend to the L^p norms as was the case in Section 3. We mention various corollaries, the proofs of which are analogous to the proof of the above theorem.

Corollary 5.1. *Let $\tilde{D}(U(R^{-\lambda}, R))$ be any closed subspace of $D(U(R^{-\lambda}, R))$ containing the functions $\{z^k\}$, $k = -m_2, \ldots, 0, \ldots, m_1$. Then the results of Theorem 5.1 hold.*

Corollary 5.2. *Let $C^1(U(R^{-\lambda}, R)) = \{f : f$ regular analytic in $U(R^{-\lambda}, R)$, $\|f\|_{U(R^{-\lambda}, R)} < \infty$, and $f(z) = \sum_{m=i_1}^{i_k} a_m z^m + \sum_{m=-j_1}^{-j_l} k_m z_m\}$, where $\{i_s\}_{s=1}^k$ is an increasing sequence of non-negative integers (k may be infinite), and $\{j_s\}_{s=1}^l$ is an increasing sequence of positive integers (l may be infinite).*

Then the M.N. q.f. on $C^1(U(R^{-\lambda}, R))$ tends as $R \uparrow \infty$, to the Gaussian q.f., with respect to the ECT-system

$$\{x^k\}_{k=i_1}^{i_{m_1}} \cup \{x^k\}_{k=-j_1}^{-j_{m_2}},$$

where $\{i_1+1, \ldots, i_{m_1}+1, \lambda(j_1-1), \ldots, \lambda(j_{m_2}-1)\}$ are the $2n$ smallest terms in $\{i_s+1\}_{s=1}^k \cup \{\lambda(j_s-1)\}_{s=1}^l$ and the $2n$-th term is strictly less than the $2n+1$-st term in the sequence.

6. Hilbert Subspaces and Bounded and Unbounded Rectangles

The examples have so far dealt with the functions $\{x^k\}_{k=-\infty}^{\infty}$. In the following, this is not the case.

Let $B = \{z : z = x + iy, 0 < x < \infty, -\pi < y < \pi\}$ and $B(r) = \{z : z = x + iy, -r < x < \infty, -\pi < y < \pi\}$.

Let

$$E(B(r)) = \Big\{ f : f \text{ regular analytic in } B(r), \Big(\iint_{B(r)} |f|^2 \, dx \, dy \Big)^{\frac{1}{2}} = \|f\|_{B(r)} < \infty, \text{ and } f(z) = \sum_{m=1}^{\infty} a_m e^{-mz} \text{ on } B(r) \Big\}.$$

Then,

Theorem 6.1. *Under the above assumptions, the M.N. q.f. on $E(B(r))$ tends, as $r \uparrow \infty$ to the Gaussian q.f. with respect to the ECT-system $\{e^{-mx}\}_{m=1}^{2n}$, where $[a, b] \in (0, \infty)$.*

Proof. Use the fact that $\left\{ \sqrt{\frac{m}{\pi}}\, e^{-mr} e^{-mz} \right\}_{m=1}^{\infty}$ is an orthonormal complete sequence in $E(B(r))$ and the methods of Theorem 3.1. Q.E.D.

A similar result holds for $\tilde{B}(r) = \{z : z = x + iy, -\infty < x < r, -\pi < y < \pi\}$ and $E(\tilde{B}(r))$ defined analogously to $E(B(r))$.

The situation corresponding to the annulus is the following. Let

$$B(r_1, r_2) = \{z : z = x + iy, -r_2 < x < r_1, -\pi < y < \pi\}$$

where $r_1, r_2 > 0$, and $E(B(r_1, r_2))$ is defined analogously to $E(B(r))$.

Let $r_2=\lambda r$, $r_1=r$. Choose the smallest $2n-1$ terms in the set $\{m\}_{m=1}^{2n-1}\cup\{\lambda m\}_{m=1}^{2n-1}$ and assume that the $2n-1$-st term is strictly less than the $2n$-th term. Assume the $2n-1$ terms contain m_1 and λm_2, but not m_1+1, $\lambda(m_2+1)$.

Theorem 6.2. *Under the above assumptions, the M.N. q.f. on $E(B(r,\lambda r))$ tends, as $r\uparrow\infty$ to the Gaussian q.f. associated with the ECT-system $\{e^{mx}\}_{m=-m_2}^{m_1}$ on $[a,b]$ where $-\infty<a<b<\infty$.*

Proof. Corresponding to past proofs.

We now list other examples, which are stated without proof. In these examples assume, for sake of convenience, that $[a,b]\subset(0,1)$.

I. Let $B(r)=\{z\colon |z|<r,\ z\neq -x,\ x\geqq 0\}$.

Since this is not a Caratheodory domain (see Markushevich [11]), it is known that $\{z^k\}_{k=0}^\infty$ is not a basis for $D(B(r))$.

a) Consider the subspace of $D(B(r))$, $C^1(B(r))$ spanned by $\{z^k\}_{k=0}^\infty$. Then the limit, as $r\uparrow\infty$, is the Gaussian q.f. associated with $\{x^k\}_{k=0}^{2n-1}$.

b) Consider the subspace of $D(B(r))$, $C^2(B(r))$ spanned by $\{\log z,\ \{z^k\}_{k=0}^\infty\}$. Then the limit, as $r\uparrow\infty$, is the Gaussian q.f. associated with $\{\log x,\ \{x^k\}_{k=0}^{2n-2}\}$.

c) Consider the subspace of $D(B(r))$, $C^3(B(r))$ spanned by $\{z^{k-1/2}\}_{k=0}^\infty$. Then the limit, as $r\uparrow\infty$, is the Gaussian q.f. associated with $\{x^{k-1/2}\}_{k=0}^{2n-1}$.

d) Consider the subspace of $D(B(r))$, $C^4(B(r))$ spanned by

$$\{\{z^k\}_{k=0}^\infty\cup\{z^{k-1/2}\}_{k=0}^N\}.$$

Then the limit, as $r\uparrow\infty$, is the Gaussian quadrature formula associated with $\{\{x^k\}_{k=0}^l\cup\{x^{k-1/2}\}_{k=0}^m\}$, where $l=m=n-1$ if $N\geqq n-1$, and if $N<n-1$, $m=N$ and $l=2n-2-N$. The analogous result holds when considering the subspace spanned by $\{\{z^k\}_{k=0}^N\cup\{z^{k-1/2}\}_{k=0}^\infty\}$.

II. Let $B(r,l)=\{z\colon |z|<r,\ z=se^{i\theta}$, and $-\frac{\pi}{l}<\theta<\frac{\pi}{l}$, l a positive integer, $l>1\}$, $D(B(r,l))$ is spanned by $\{z^k\}_{k=0}^\infty$ since $B(r,l)$ is a Caratheodory domain. The sets of functions $\{z^{ml+k}\}_{m=0}^\infty$, $k=0,1,\ldots,l-1$ are each orthogonal on $B(r,l)$, i.e., $(z^{ml+k}, z^{nl+k})=0$, $m\neq n$, for fixed $k=0,1,\ldots,l-1$.

a) Let $C^1(B^k(r,l))$ be the subspace of $D(B(r,l))$ spanned by $\{z^{ml+k}\}_{m=0}^\infty$, k fixed. Then the limit, as $r\uparrow\infty$, is the Gaussian q.f. associated with $\{x^{ml+k}\}_{m=0}^{2n-1}$.

b) Let $C^2(B(r,l);j_0,\ldots,j_{l-1})$ be the subspace of $D(B(r,l))$ spanned by $\bigcup_{k=0}^{l-1}\{z^{ml+k}\}_{m=0}^{j_k}$, where at most are of the j_k is infinite, and $j_k=-1$ indicates that the corresponding set is empty. Then the limit, as $r\uparrow\infty$, is the Gaussian q.f. associated with the lowest $2n$ powers in $\bigcup_{k=0}^{l-1}\{x^{ml+k}\}_{m=0}^{j_k}$.

The results of I and II also extend in the various directions as indicated by the corollaries of Section 5.

7. Related Questions

In this section we indicate other areas in which the previous analysis is applicable.

I. *Minimum norm Quadrature Formulae with Fixed knots*

Let

$$Rf = \int_a^b f(x)w(x)\,dx - \sum_{i=1}^{k}\sum_{j=0}^{\mu_i} a_{ij} f^{(j)}(\eta_i), \tag{7.1}$$

where $c \leqq \eta_1 < \cdots < \eta_k \leqq d$, η_i, μ_i fixed, $i = 1, \ldots, k$, $[a, b] \subseteq [c, d]$ (c, d finite), and $\sum_{i=1}^{k}(\mu_i + 1) = n$. Associated with each quadrature formula of this form there exist unique $\{a_{ij}\}_{1\,0}^{k\,\mu_i}$ such that $Ru_i = 0$, $i = 0, 1, \ldots, l$, where $\{u_i\}_{j=0}^{l}$ is an ECT-system on $[c, d]$. Depending on the choice of $\{\eta_i\}_{i=1}^{k}$ and $\{\mu_i\}_{i=1}^{k}$,

$$n - 1 \leqq l \leqq \sum_{i=1}^{k}(\mu_i + 1) + k - E - 1,$$

where E counts the η_i for which μ_i is odd, or $\eta_i \notin (a, b)$.

We consider the class of quadrature formulae obtained by allowing the coefficients $\{a_{ij}\}_{i=1}^{k}{}_{j=0}^{\mu_i}$ to vary. Analogues to Theorems 2.1 and 2.2 are easily proven, and with this follows the extension to the results of Sections 3–6.

II. *Minimum Norm Quadrature Formulae with Variable Knots of Fixed Degrees*

Consider the class of q.f. of the form (7.1) where the μ_i are fixed $i = 1, \ldots, k$, and each μ_i is even, but the η_i are permitted to vary within $[c, d]$. Then there exist unique

$$\{\eta_i\}_{i=1}^{k}, \quad a < \eta_1 < \cdots < \eta_k < b, \quad \text{and} \quad \{a_{ij}\}_{i=1}^{k}{}_{j=0}^{\mu_i}$$

such that $Ru_i = 0$, $i = 0, \ldots, \sum_{i=1}^{k}(\mu_i + 1) + k - 1 = p$, where $\{u_i\}_{i=0}^{p}$ is an ECT-system on $[c, d]$ (see Karlin and Pinkus [9]). All the above theorems hold with the obvious changes.

III. *Interpolation*

For the case of the ellipse with foci at ± 1, the reader is refered to Barnhill and Wixom [5]. The problem is as follows. Consider $z_0, z_1, \ldots, z_k \in B$, where $z_i \neq z_j$, $i, j = 0, 1, \ldots, k$. Let $R_n f$ be the linear functional defined by

$$R_n f = f(z_0) - \sum_{i=1}^{k}\sum_{j=0}^{\mu_i} a_{ij} f^{(j)}(z_i),$$

where $\{z_i\}_{i=0}^{k}$, $\{\mu_i\}_{i=1}^{k}$ are fixed, and $\sum_{i=1}^{k}(\mu_i + 1) = n$. We consider the class of linear functionals obtained by allowing the $\{a_{ij}\}$ to vary. The results then follow those of Sections 2–6.

In general, any class of remainder formulae

$$Rf = Lf - \sum_{i=1}^{n} a_i L_i(f),$$

where $L_i f = f(z_i)$, $z_i \in B$, Lf is a bounded linear functional on $D(B)$, and the a_i are free to vary, will satisfy the analysis of Sections 2–6 (see Golomb and Weinberger [7]).

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Dr. Allan Pinkus
Math. Sciences Dept. IBM
Box 218
Yorktown Heights N.Y. 10598/U.S.A.