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Bernstein's comparison theorem and a problem of Braess

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1. Introduction

Let $f, g \in C^{n+1}[a, b]$, and assume that $|g^{(n+1)}(x)| \leq f^{(n+1)}(x)$ for all $x \in [a, b]$. It is a classical result of S. Bernstein [1], commonly referred to as the Bernstein comparison theorem, that an upper bound on the value of the best approximation to g in the uniform (L^∞) norm by polynomials of degree less than or equal to n is the value of the corresponding best approximation to f.

In [3], D. Braess gives a partial generalization of this result to the nonlinear problem of the best approximation of functions, in the uniform norm, by splines of degree n with k variable knots (n and k are fixed). Namely, let $\mathcal{S}_{n,k}$ denote this class of polynomial splines of degree n with k variable (free) knots. Then,

THEOREM 1 (Braess [3]). *Let $f, g \in C^{n+1}[a, b]$ and assume that*

$$0 \leq g^{(n+1)}(x) \leq f^{(n+1)}(x), \qquad x \in [a, b]. \tag{1}$$

Then,

$$\min_{S \in \mathcal{S}_{n,k}} \|g - S\|_\infty \leq \min_{S \in \mathcal{S}_{n,k}} \|f - S\|_\infty \tag{2}$$

The method of proof of this theorem is through the study of monosplines. Unanswered is the question of whether (2) remains valid if the assumption (1) is replaced by

$$|g^{(n+1)}(x)| \leq f^{(n+1)}(x), \qquad x \in [a, b]. \tag{3}$$

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In this note we show how to extend Braess' theorem to the case where (3) replaces (1). In this case (2) is no longer valid. We demonstrate this fact by determining the precise upper bound on the best approximation of the class of functions g satisfying (3), by splines of degree n with k variable knots, i.e., we solve the analogue of Theorem 1 with condition (3). What is especially striking is that we also show that interpolation at $n+k+1$ fixed, distinct points by splines of degree n with k *fixed* knots (the points and knots depend upon f) gives the same upper bound as that which is obtained for the best approximation from $\mathcal{S}_{n,k}$ to the class of functions g satisfying (3). As a simple consequence, we also characterize the best constant in the degree of approximation from $\mathcal{S}_{n,k}$, of functions in the Sobolev space W_∞^{n+1} which satisfy $|g^{(n+1)}|_\infty \leq 1$.

In the proof of the above facts we shall have recourse to another generalization of Bernstein's comparison theorem. Since we were unable to find the result in the literature we have presented, in Section 2, a general version of Bernstein's comparison theorem for best approximations by weak Chebyshev systems in any monotone normed space. However it should be noted that it is only Corollary 2.1 which is subsequently used in Section 3.

2. Bernstein's comparison theorem for weak Chebyshev systems

Given a set of $n+1$ continuous, linearly independent functions $\{u_0, u_1, \ldots, u_n\}$ on $[a, b]$, we say that $\{u_i\}_0^n$ is a weak Chebyshev system of degree n on $[a, b]$ if every linear combination of the $\{u_i\}_{i=0}^n$ has at most n sign changes on $[a, b]$. We say that $\{u_i\}_{i=0}^n$ is a Chebyshev system on $[a, b]$ if no non-trivial linear combination of the $\{u_i\}_{i=0}^n$ has more than n distinct zeros on $[a, b]$, and that they form an extended Chebyshev system on $[a, b]$ if $u_i \in C^{n+1}[a, b]$, $i = 0, 1, \ldots, n$, and no non-trivial linear combination has more than n zeros, counting multiplicites (see e.g., Karlin and Studden [7]). In what follows we shall make use of the orientation of the Chebyshev system. This orientation depends upon the following well-known proposition.

PROPOSITION 2.1. *Let* $u_i \in C[a, b]$, $i = 0, 1, \ldots, n$. *Then* $\{u_i\}_0^n$ *is a Chebyshev system of* $[a, b]$ *iff*

$$U\begin{pmatrix} 0, 1, \ldots, n \\ x_0, x_1, \ldots, x_n \end{pmatrix} = \det(u_i(x_j))_{i,j=0}^n \neq 0 \tag{4}$$

for every choice of $\{x_i\}_{i=0}^n$ *satisfying* $a \leq x_0 < x_1 < \cdots < x_n \leq b$. *The set* $\{u_i\}_{i=0}^n$ *is a weak Chebyshev system on* $[a, b]$ *if the above determinant* (4) *is not identically zero* *and does not change sign for* $(x_0, x_1, \ldots, x_n)$ *satisfying* $a \leq x_0 < x_1 < \cdots < x_n \leq b$. *It is an extended Chebyshev system if the above determinant* (4) *is of one strict sign for* $a \leq x_0 \leq x_1 \leq \cdots \leq x_n \leq b$ *where equalities among the* x_i*'s indicates taking successive derivatives in the columns of the associated matrix.*

The *orientation* which we shall give to our Chebyshev (or weak or extended) systems is to demand that the determinants (4) always be non-negative.

The following known proposition implies that it suffices, in what follows, to prove our theorem only for extended Chebyshev systems (see [7]).

PROPOSITION 2.2. *Let* $\{u_i\}_{i=0}^n$ *be a weak Chebyshev system on* $[a, b]$. *There exists a family of extended Chebyshev systems* $\{u_i^\varepsilon\}_{i=0}^n$, $\varepsilon > 0$, *with the property that*

$$\lim_{\varepsilon \to 0+} u_i^\varepsilon(x) = u_i(x), \qquad i = 0, 1, \ldots, n$$

uniformly for $x \in [a, b]$.

We shall deal with the problem of approximating continuous functions by weak Chebyshev systems with respect to any norm $\|\cdot\|$ on $C[a, b]$ which satisfies the following two properties.

I) If $|g(x)| \leq |f(x)|$ for all $x \in [a, b]$, then $\|g\| \leq \|f\|$.

II) If $f \in C[a, b]$ and $\{u_i\}_0^n$ is a weak Chebyshev system on [a, b], then there exists a best approximation $\sum_{i=0}^n a_i^* u_i$ to f, i.e.,

$$\min_{a_0, a_1, \ldots, a_n} \left\| f - \sum_{i=0}^n a_i u_i \right\| = \left\| f - \sum_{i=0}^n a_i^* u_i \right\|,$$

for which $f - \sum_{i=0}^n a_i^* u_i$ has at least $n+1$ zeros in $[a, b]$.

It is clear that any weighted L^p norm, $1 \leq p \leq \infty$, has the above properties. Any norm which satisfies property I) is known as a monotone norm. Not every monotone norm satisfies property II). However monotone norms have been studied by Kimchi and Richter-Dyn [8], [9] and they show that any monotone norm satisfies property II) if $\{u_i\}_{i=0}^n$ is an extended Chebyshev system and if we count zeros of multiplicity up to 2. Since for any monotone norm and for any $f \in C[a, b]$, $\|f\| \leq C\|f\|_\infty$, where C may be taken as the monotone norm of the function identically one on $[a, b]$, it follows that the extended Chebyshev system $\{u_i^\varepsilon\}_{i=0}^n$ of Proposition 2.2 tend to the weak Chebyshev system $\{u_i\}_{i=0}^n$ in any monotone norm. This fact permits us, in the proof of the following theorem, to consider only extended Chebyshev systems.

Let S_h denote a best approximation to $h \in C[a, b]$ from span $\{u_0, u_1, \ldots, u_n\} = [u_0, u_1, \ldots, u_n]$.

THEOREM 2. *Assume that $\{u_i\}_{i=0}^n$ is a weak Chebyshev system on $[a, b]$, and that $\|\cdot\|$ is any monotone norm thereon. Assume that $f \pm g$ are both in the convexity cone of $\{u_0, u_1, \ldots, u_n\}$, i.e., $\{u_0, u_1, \ldots, u_n, f \pm g\}$ are also weak Chebyshev systems on $[a, b]$. Then*

$$\|g - S_g\| \leq \|f - S_f\|.$$

Proof. As indicated above, we may assume that $\{u_i\}_{i=0}^n$ and $\{u_0, u_1, \ldots, u_n, f \pm g\}$ are extended Chebyshev systems on $[a, b]$. By the result of Kimchi and Richter-Dyn [8], we have that $Z(f - S_f) \geq n+1$, where Z counts zeros up to multiplicity at most two.

Choose any $n+1$ zeros of $(f - S_f)(x)$, $\{x_i\}_{i=1}^{n+1}$, $a \leq x_1 \leq \cdots \leq x_{n+1} \leq b$, where each zero is listed to its multiplicity. Let $\tilde{S}_g(x) = \sum_{i=0}^n b_i u_i(x)$ denote the unique function in $[u_0, u_1, \ldots, u_n]$ which interpolates to g at the points $\{x_i\}_{i=1}^{n+1}$. Note that $f - S_f$ and $g - \tilde{S}_g$ have $n+1$ common zeros. Since $f \pm g$ are strictly in the convexity cone of $\{u_i\}_0^n$, i.e., $\{u_0, u_1, \ldots, u_n, f \pm g\}$ are extended Chebyshev systems, it follows that $f \pm g - (S_f \pm \tilde{S}_g)$ have no additional zeros in $[a, b]$.

In fact we may write

$$[(f \pm g) - (S_f \pm \tilde{S}_g)](x) = \frac{U\begin{pmatrix} 0, 1, \ldots, n, f \pm g \\ x_1, x_2, \ldots, x_{n+1}, x \end{pmatrix}}{U\begin{pmatrix} 0, 1, \ldots, n \\ x_1, x_2, \ldots, x_{n+1} \end{pmatrix}},$$

where

$$U\begin{pmatrix} 0, 1, \ldots, n, f \pm g \\ x_1, x_2, \ldots, x_{n+1}, x \end{pmatrix} = \begin{vmatrix} u_0(x_1) \cdots u_0(x_{n+1}) & u_0(x) \\ \vdots \qquad\quad \vdots & \vdots \\ u_n(x_1) \cdots u_n(x_{n+1}) & u_n(x) \\ (f \pm g)(x_1) \cdots (f \pm g)(x_{n+1}) & (f \pm g)(x) \end{vmatrix}.$$

Due to the orientation of the Chebyshev systems,

$$[(f - S_f)(x) \pm (g - \tilde{S}_g)(x)](-1)^{i+n+1} \geq 0, \qquad x_i \leq x \leq x_{i+1}$$

$i = 0, 1, \ldots, n+1$, where $x_0 = a$, $x_{n+2} = b$.

Thus $|(f-S_f)(x)| \geq |(g-\tilde{S}_g)(x)|$ for all $x \in [a, b]$ and, since $\|\cdot\|$ is a monotone norm, $\|f-S_f\| \geq \|g-\tilde{S}_g\|$. Because $\tilde{S}_g$ is one possible, but not necessarily a best, approximant it immediately follows that $\|f-S_f\| \geq \|g-S_g\|$.

COROLLARY 2.1. *Let $\mathcal{S}_{n,k}(\boldsymbol{\xi})$ denote the class of splines of degree n with k fixed knots given by $\boldsymbol{\xi}=(\xi_1, \ldots, \xi_k)$, $a<\xi_1<\cdots<\xi_k<b$. Let f be any function for which $f^{(n+1)}(x)(-1)^{i+k} \geq 0$ a.e. on $[\xi_i, \xi_{i+1}]$, $i=0, 1, \ldots, k$, where $\xi_0=a$, $\xi_{k+1}=b$. Let g satisfy*

$$|g^{(n+1)}(x)| \leq |f^{(n+1)}(x)| \quad a.e.\ in \quad [a, b].$$

Then the error in the best approximation of g from $\mathcal{S}_{n,k}(\boldsymbol{\xi})$ in any monotone norm is less than or equal to the error in the best approximation to f from $\mathcal{S}_{n,k}(\boldsymbol{\xi})$.

Proof. The corollary rests on the fact that any f as above lies in the convexity cone of the weak Chebyshev system $\{1, x, \ldots, x^n, (x-\xi_1)_+^n, \ldots, (x-\xi_k)_+^n\}$ which is a basis for $\mathcal{S}_{n,k}(\boldsymbol{\xi})$, see [10].

3. A problem of Braess

Let $\mathcal{S}_{n,k}$ denote the class of polynomial splines of degree n with k variable (free) knots. As stated in the introduction, Braess [3] proved that if we denote by S_f and S_g the best approximations to $f, g \in C[a, b]$ from $\mathcal{S}_{n,k}$ in the uniform norm, then under the assumption that $0 \leq g^{(n+1)}(x) \leq f^{(n+1)}(x)$ a.e. in $[a, b]$, it follows that $\|g-S_g\|_\infty \leq \|f-S_f\|_\infty$.

In order to ease our exposition, we introduce the following notation:

$$W_\infty^{n+1}=\{f : f^{(n)} \text{ abs. cont., } \|f^{(n+1)}\|_\infty<\infty\}$$
$$W(h)=\{f : f \in W_\infty^{n+1}, |f^{(n+1)}(x)| \leq h(x) \text{ a.e.}\}$$

where h is some non-negative function in L^∞. Also

$$\mathcal{S}_{n,k}=\left\{S(x)=\sum_{i=0}^{n} a_i x^i+\sum_{i=1}^{l} \sum_{j=0}^{\mu_i-1} b_{ij}(x-\xi_i)_+^{n-j},\right.$$
$$\left. a_i, b_{ij} \in R,\ a<\xi_1<\cdots<\xi_l<b,\ \sum_{i=1}^{l} \mu_i \leq k\right\}.$$

The idea of the proof of Theorem 3 is to construct an $f^* \in W(h)$ which has the zero function as a best approximation from $\mathcal{S}_{n,k}$, and for which $\|g-S_g\|_\infty \leq \|f^*\|_\infty$

for all $g \in W(h)$. This generalizes Braess' result. We shall in fact prove more, namely that $\|g-\tilde{S}_g\|_\infty \leq \|f^*\|_\infty$, where $\tilde{S}_g$ is an interpolation operator to g at $n+k+1$ fixed points by splines of degree n with k fixed knots.

The construction of $f^*(x)$ is based on the following proposition.

PROPOSITION 3.1. *Let $W(h)$ be as given above. There exists a function $f^* \in W(h)$ with the following properties:*

1) There exist knots $\xi^*=(\xi_1^*, \ldots, \xi_k^*)$, $\xi_0^*=a<\xi_1^*<\cdots<\xi_k^*<b=\xi_{k+1}^*$ such that $(f^*)^{(n+1)}(x)=\varepsilon(-1)^i h(x)$, a.e. for $x \in (\xi_i^*, \xi_{i+1}^*)$, $i=0,1,\ldots,k$, where $\varepsilon=\pm 1$ fixed.

2) $f^*(x)$ equioscillates at $n+k+2$ points, i.e., there exist $\{x_i^*\}_{i=1}^{n+k+2}$, $a \leq x_1^* < \cdots < x_{n+k+2}^* \leq b$, for which

$$f^*(x_i^*)=\delta(-1)^i\|f^*\|_\infty, \qquad i=1,\ldots,n+k+2$$

where $\delta=\pm 1$, fixed. (In fact $\delta=\varepsilon(-1)^n$.)

This theorem represents a slight generalization of the existence of equioscillating perfect splines due to Karlin [6]. (See also Cavaretta [4]). We here present a proof of the proposition for completeness and because it is short and direct.

Proof. Let $\Xi_k=\{\mathbf{z} : \mathbf{z}=(z_1,\ldots,z_{k+1}), \sum_{i=1}^{k+1}|z_i|=b-a\}$. Let

$$t_i^{(\mathbf{z})}=\sum_{j=1}^{i}|z_j|+a, \; i=1,\ldots,k+1.$$

Thus

$$t_0^{(\mathbf{z})}=a \leq t_1^{(\mathbf{z})} \leq \cdots \leq t_k^{(\mathbf{z})} \leq t_{k+1}^{(\mathbf{z})}=b.$$

Set

$$F_{\mathbf{z}}(x)=\sum_{i=1}^{k+1}(\operatorname{sgn} z_i)\int_{t_{i-1}^{(\mathbf{z})}}^{t_i^{(\mathbf{z})}}(x-y)_+^n h(y)\,dy.$$

Note that $F_{\mathbf{z}}(x)$ is a function in $W(h)$ for which $F_{\mathbf{z}}^{(n+1)}(x)=(\operatorname{sgn} z_i)h(x)$, a.e. for $x \in (t_{i-1}^{(\mathbf{z})}, t_i^{(\mathbf{z})})$. Now, $\{x^i\}_{i=0}^{n+k}$ is a Chebyshev system on $[a, b]$ and as such there exists, for each $F_{\mathbf{z}}(x)$, a unique best L^∞-approximation from polynomials of degree $n+k$. Let $\sum_{i=0}^{n+k} a_i(\mathbf{z})x^i$ denote this best L^∞-approximation to $F_{\mathbf{z}}(x)$. Since $\{x^i\}_{i=0}^{n+k}$ is a Chebyshev system, either $F_{\mathbf{z}}(x)-\sum_{i=0}^{n+k} a_i(\mathbf{z})x^i$ exhibits at lest $n+k+2$ points of equioscillation or $F_z(x)$ is a polynomial of degree $n+k$. Furthermore by the uniqueness of the best approximation, the coefficients $\{a_i(\mathbf{z})\}_{i=0}^{n+k}$ are continuous functions of $\mathbf{z}$, and since $F_{-\mathbf{z}}(x)=-F_{\mathbf{z}}(x)$, they are also odd, i.e., $a_i(-\mathbf{z})=-a_i(\mathbf{z})$, $i=0,1,\ldots,n+k$.

Set $G(\mathbf{z})=(a_{n+1}(\mathbf{z}),\ldots,a_{n+k}(\mathbf{z}))$. G is an odd, continuous map of Ξ_k into R^k and hence by the Borsuk–Antipodensatz [2], there exists a point $\mathbf{z}^*$ for which $G(\mathbf{z}^*)=\mathbf{0}$, i.e., $a_{n+i}(\mathbf{z}^*)=0$, $i=1,\ldots,k$.

Set $f^*(x)=F_{\mathbf{z}^*}(x)-\sum_{i=0}^{n} a_i(\mathbf{z}^*)x^i$. The function $f^*(x)$ is not identically zero and thus satisfies the conditions of the theorem if we can show that $z_i^*z_{i+1}^*<0$, $i=1,\ldots,k$. Since $f^*(x)$ has at least $n+k+2$ points of equioscillation, it also has at least $n+k+1$ sign changes. A Rolle's theorem argument easily shows that $f^{*(n+1)}(x)=F_{\mathbf{z}^*}^{(n+1)}(x)$ must have at least k sign changes. This is possible only if $z_i^*z_{i+1}^*<0$, $i=1,\ldots,k$. The proposition is proved.

PROPOSITION 3.2. *Let $f^*(x)$ be as above. Then the zero function is a best approximation to $f^*(x)$ from $\mathcal{S}_{n,k}$ in the L^∞-norm.*

Proof. That the zero function is a best L^∞-approximation to f^* from $\mathcal{S}_{n,k}$ follows from the fact that f^* equioscillates at $n+k+2$ points. Indeed, assume that $\tilde{S}\in\mathcal{S}_{n,k}$ is a better approximant. Thus $\|f^*-\tilde{S}\|_\infty<\|f^*\|_\infty$. Since f^* equioscillates at $n+k+2$ points, $\tilde{S}=f^*-(f^*-\tilde{S})$ has at least $n+k+1$ sign changes on $[0,1]$. However it is well-known that any spline in $\mathcal{S}_{n,k}$ has at most $n+k$ sign changes on any finite interval. This contradiction proves the proposition.

THEOREM 3. *Let $f^*\in W(h)$ and $\mathcal{S}_{n,k}$ be as previously defined. If $g\in W(h)$, then*

$$\min_{S\in\mathcal{S}_{n,k}}\|g-S\|_\infty\leq\min_{S\in\mathcal{S}_{n,k}}\|f^*-S\|_\infty=\|f^*\|_\infty \tag{5}$$

Proof. On the basis of Propositions 3.1 and 3.2, it remains to prove that we can find an approximant (not necessarily the best) $\hat{S}_g$ to g from $\mathcal{S}_{n,k}$ for which $\|g-\hat{S}_g\|_\infty\leq\|f^*\|_\infty$. Let $\boldsymbol{\xi}^*=(\xi_1^*,\ldots,\xi_k^*)$ denote the k knots of $f^*(x)$ as defined in Proposition 3.1. Apply Corollary 2.1 to the class $\mathcal{S}_{n,k}(\boldsymbol{\xi}^*)$. Let $\hat{S}_g$ be any best approximant to g from $\mathcal{S}_{n,k}(\boldsymbol{\xi}^*)$. From Corollary 2.1 we have

$$\|g-\hat{S}_g\|_\infty\leq\|f^*-\hat{S}_{f^*}\|_\infty=\|f^*\|_\infty,$$

since $|g^{(n+1)}(x)|\leq|f^{*(n+1)}(x)|$ a.e., and $f^{*(n+1)}(x)$ changes sign at the $\{\xi_i^*\}_{i=1}^k$ and only there. The theorem is proved.

It may in fact be shown by a Rolle's theorem argument that the function $f^*(x)$

has exactly $n+k+1$ zeros which we denote by $\{y_i^*\}_{i=1}^{n+k+1}$, $a<y_1^*<\cdots<y_{n+k+1}^*<b$. One may also prove, see Karlin [6], that

$$y_i^*<\xi_i^*<y_{i+n+1}^*,\quad i=1,\ldots,k. \tag{6}$$

The inequalities (6) guarantee the existence of a unique interpolant to any $g\in C[a,b]$ at the point $\{y_i^*\}_{i=1}^{n+k+1}$ from $\mathcal{S}_{n,k}(\boldsymbol{\xi}^*)$. Thus from the proofs of Theorems 2 and 3 we have

THEOREM 4. *Let f^* and $W(h)$ be as previously defined. Let $\tilde{S}_g$ denote the unique interpolant to g at the points $\{y_i^*\}_{i=1}^{n+k}$ from $\mathcal{S}_{n,k}(\boldsymbol{\xi}^*)$. Then for every $g\in W(h)$*

$$\|g-\tilde{S}_g\|_\infty\le\|f^*\|_\infty.$$

As a result of Theorem 3, we also have

COROLLARY 3.1. *Let $g\in W_\infty^{n+1}$, and let S_g denote a best approximation to g from $\mathcal{S}_{n,k}$ in the L^∞-norm. Then,*

$$\|g-S_g\|_\infty\le C^*\,\|g^{(n+1)}\|_\infty \tag{7}$$

where C^* is the norm of the equioscillating perfect spline of degree $n+1$ with k knots, whose $(n+1)$ derivative is equal to one in absolute value. Furthermore, the inequality (7) is exact.

A perfect spline of degree $n+1$ with k knots is a function of the form

$$P(x)=\sum_{i=0}^{n}a_ix^i+c[x^{n+1}-2(x-\xi_1)_+^{n+1}+2(x-\xi_2)_+^{n+1}\cdots+(-1)^k2(x-\xi_k)_+^{n+1}].$$

where

$$\xi_1<\cdots<\xi_k.$$

Remark 3.1. The above analysis applies in many situations other than splines. For example, let $K(x,y)$ denote an extended totally positive kernel on $[a,b]\times[c,d]$ (see Karlin [5]). Set

$$\mathcal{K}_n=\left\{\sum_{i=1}^{l}\sum_{j=0}^{\mu_i-1}a_{ij}\frac{\partial^jK(x,\xi_i)}{\partial y^j}:c\le\xi_1<\cdots<\xi_l\le d,\ a_{ij}\in R,\ \sum_{i=1}^{l}\mu_i\le n\right\}.$$

Each $f\in C[a,b]$ possesses a best approximation S_f from $\mathcal{K}_n$ in the uniform norm.

Let

$$K(h^*)=\left\{g(x)=\int_c^d K(x,y)h(y)\,dy: |h(y)|\leq h^*(y)\right\}$$

where h^* is a given non-negative L^∞-function.

Then the results of this section apply. That is to say, we can in a totally analogous manner, prove the existence of an $f^*\in K(h^*)$ which is of the form

$$f^*(x)=\sum_{i=0}^{n}(-1)^i\int_{\xi_i^*}^{\xi_{i+1}^*} K(x,y)h^*(y)\,dy,$$

where $\xi_0^*=c<\xi_1^*<\cdots<\xi_n^*<\xi_{n+1}^*=d$, and which equioscillates at $n+1$ points in $[a,b]$. A best approximation to $f^*(x)$ from $\mathcal{K}_n$ is the zero function, and for every $g\in K(h^*)$

$$\|g-S_g\|_\infty\leq\|f^*\|_\infty.$$

Again interpolation at the zeros of $f^*(x)$ by a function of the form $\sum_{i=1}^n a_iK(x,\xi_i^*)$ is a sufficiently good approximant.

Remark 3.2. Braess obtained an exact upper bound on the error function for g satisfying $0\leq g^{(n+1)}(x)\leq h(x)$. We did likewise for g satisfying $-h(x)\leq g^{(n+1)}(x)\leq h(x)$. Surprisingly neither method seems to carry over to the class of functions g satisfying $h_2(x)\leq g^{(n+1)}(x)\leq h_1(x)$ for general $h_1, h_2\in L^\infty$.

Remark 3.3. Since the results of Section 2 are valid for any monotone norm and thus for any L^p-norm, it is natural to ask whether the theorems of Section 3 may be extended to any L^p-norm for $1\leq p<\infty$. Each best approximant to $f\in C[a,b]\setminus\mathcal{S}_{n,k}$ from $\mathcal{S}_{n,k}$ in the L^p-norm, $1\leq p<\infty$, must have k active knots (and hence is never the zero function). It follows from this fact that approximation by splines with variable knots in L^p for $1\leq p<\infty$ is strictly better than approximation by splines with fixed knots over the class $W(h)$. Thus the methods used above are restricted to problems of L^∞-approximation.

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