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# ON SOME ZERO-INCREASING OPERATORS

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**Abstract.** We characterize the set of linear operators of the form

$$Tx^n = x^n + \sum_{k=0}^{n-1} b_{n,k} x^k, \qquad n = 0, 1, \dots,$$

which satisfy $Z(Tp) \geqq Z(p)$ for every polynomial $p$, where $Z$ counts the number of zeros on all of $\mathbf{R}$. We also consider the analogous question on $[0, \infty)$.

## 1. Introduction

Mathematicians have for many centuries studied problems connected with the number and location of the roots of a polynomial under linear transformations. A classic example thereof is Rolle's Theorem (from 1691). Further examples abound.

The Hermite–Poulain Theorem (Hermite [4], Poulain [16]) is the following (see also Obreschkoff [12, p. 4], Pólya and Szegő [15, Part V, No. 63]).

Hermite–Poulain Theorem A. *Assume*

$$g(x) = \sum_{k=0}^{m} b_k x^k$$

*is a polynomial with only real zeros. If $p$ is any real polynomial, then the number of non-real zeros of*

$$h(x) = \sum_{k=0}^{m} b_k p^{(k)}(x)$$

*is at most the number of non-real zeros of $p$. (If $b_0 \neq 0$, then we say that the number of real zeros of $h$ is at least the number of real zeros of $p$.)*

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There are various proofs of this theorem. The essential idea is to note that

$$h(x) = \big(g(D)p\big)(x)$$

where $D = d/dx$. As $g$ has only real zeros, $g(D)$ applied to $p$ is a composition of the operators $(D - \alpha I)p$, with $\alpha \in \mathbf{R}$. It thus suffices to verify that this operator has the desired property. For $\alpha = 0$ we apply Rolle's Theorem (here $b_0 = 0$). If $\alpha \neq 0$, then it is easily checked that between every two real zeros of $p$ there is a zero of $(D - \alpha I)p$. Thus $(D - \alpha I)p$ has at least the number of real zeros of $p$ less 1. However as $\alpha \neq 0$, the degree of the polynomials $(D - \alpha I)p$ and $p$ are the same. Thus the parity of the number of real zeros of $(D - \alpha I)p$ and $p$ is the same, and $(D - \alpha I)p$ must have at least as many real zeros as $p$.

A specialization of this theorem is the following.

HERMITE–POULAIN THEOREM B. *Let $g$ be as above and assume $g$ has all positive zeros. Then the number of non-negative zeros of $h$ is at least the number of non-negative zeros of $p$.*

In fact if $0 \leqq x_1 \leqq \cdots \leqq x_k$ are the non-negative zeros of $p$, listed to their multiplicity, then there exist zeros of $h$, $0 \leqq y_1 \leqq \cdots \leqq y_k$, listed to their multiplicity, satisfying $x_i \leqq y_i$, $i = 1, \ldots, k$. This theorem is a consequence of the previous theorem, and the additional fact that for $\alpha > 0$, $(D - \alpha I)p$ has the same sign as $p$ near and to the right of $x_k$, but the opposite sign to $p$ at infinity. Thus $(D - \alpha I)p$ has an additional zero in $(x_k, \infty)$.

We introduce the notation $Z_I(p)$ to denote the number of zeros (counting multiplicity) of a polynomial $p$ on the interval $I$. For convenience we will drop the $I$ when counting zeros on all of $\mathbf{R}$. In addition, we let $\Pi$ denote the space of real-valued polynomials.

Another famous theorem about zero-counting is the following due to Laguerre [11] (see also Obreschkoff [12, p. 6]).

LAGUERRE'S THEOREM. *Assume $g$ is a polynomial with only real zeros, all of which lie outside the interval $[0, n]$. Then for any real polynomial of degree at most $n$,*

$$p(x) = \sum_{k=0}^{n} a_k x^k$$

*we have*

$$Z\left(\sum_{k=0}^{n} a_k g(k) x^k\right) \geqq Z\left(\sum_{k=0}^{n} a_k x^k\right).$$

This theorem is a consequence of the fact that if

$$g(x) = b \prod_{j=1}^{m} (x - \alpha_j),$$

then

$$\sum_{k=0}^{n} a_k g(k) x^k = b \prod_{j=1}^{m} (xD - \alpha_j I) p(x).$$

As these linear operators commute, Laguerre's Theorem now follows from a carefully checked Rolle's Theorem argument which implies that

$$Z\big((xD - \alpha I)p\big) \geqq Z(p)$$

for $\alpha \notin [0, n]$. (Consider the sign of $(xD - \alpha I)p$ at the zeros of $p$. For $\alpha < 0$, the sign of $(xD - \alpha I)p$ at 0 is of opposite sign to the sign of $(xD - \alpha I)p$ evaluated at the least positive and/or greatest negative zeros of $p$. For $\alpha > n$, the polynomials $p$ and $(xD - \alpha I)p$ are of opposite sign at $\infty$ and $-\infty$.)

In addition to this general result, various particular and limiting cases of this result may be found in the literature. For example, from this theorem of Laguerre we obtain the inequalities

$$Z\left(\sum_{k=0}^{n} a_k \binom{n}{k} x^k\right) \geqq Z\left(\sum_{k=0}^{n} a_k x^k\right)$$

and

$$Z\left(\sum_{k=0}^{n} a_k q^{k^2} x^k\right) \geqq Z\left(\sum_{k=0}^{n} a_k x^k\right)$$

for $|q| \leqq 1$.

To explain some further extensions of these results and what will be proven in this paper, we first recall what we may obtain as limiting cases of the polynomials $g$ in the above-mentioned theorems.

Based on work of Laguerre [10] and Pólya [13], see e.g., Obreschkoff [12, pp. 31–42], Karlin [9, Ch. 7, §2], Widder [18, Ch. 7], the following is known. (More exact details may be found in the above references.)

Let $\mathcal{A}_1$ denote the class of entire functions of the form

$$\phi_1(z) = C z^m e^{-c^2 z^2 + az} \prod_{k=1}^{\infty} (1 + \alpha_k z) e^{-\alpha_k z}$$

where $C, c, a, \alpha_k \in \mathbf{R}$, $m$ is a nonnegative integer, and $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. Then $\mathcal{A}_1$ is the analytic extension of those functions defined on $\mathbf{R}$ which may be obtained as the uniform limit (on $[-A, A]$, every $A > 0$) of polynomials having only real zeros.

With regard to the two exponential functions in the above, note that

$$e^{ax} = \lim_{m \to \infty} \left(1 + \frac{ax}{m}\right)^m$$

and

$$e^{-c^2x^2} = \lim_{m\to\infty} \left(1 - \frac{cx}{\sqrt{m}}\right)^m \left(1 + \frac{cx}{\sqrt{m}}\right)^m .$$

Let $\mathcal{A}_2$ denote the class of entire functions of the form

$$\phi_2(z) = Cz^m e^{-az} \prod_{k=1}^{\infty} (1 - \alpha_k z)$$

where $C \in \mathbf{R}$, $a \geqq 0$, $m$ is a nonnegative integer, $\alpha_k \geqq 0$ and $\sum_{k=1}^{\infty} \alpha_k < \infty$. Then $\mathcal{A}_2$ is the analytic extension of the functions defined on $\mathbf{R}$ which can be obtained as the uniform limit (on $[-A, A]$, every $A > 0$) of polynomials having only real positive zeros. (Of course there is the analogous result for limits of polynomials having only real negative zeros.)

Finally we also have, based on $\mathcal{A}_1$ and $\mathcal{A}_2$, the following class (see Laguerre's Theorem). We let $\mathcal{A}_3$ denote the class of entire functions of the form

$$\phi_3(z) = Cz^m e^{-c^2z^2+az} \prod_{k=1}^{\infty} (1 + \alpha_k z) e^{-\alpha_k z}$$

where $C, c, a \in \mathbf{R}$, $m$ is a nonnegative integer, $\alpha_k \geqq 0$ and $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. The class $\mathcal{A}_3$ is the analytic extension of those functions defined on $\mathbf{R}$ which can be obtained as the uniform limit (on $[-A, A]$, every $A > 0$) of polynomials having only real zeros, all of which are outside $[0, n]$ for any $n \in \mathbf{N}$. (This latter condition is to be understood within the context of Laguerre's Theorem. That is, $\phi_3$ is the uniform limit, as above, of polynomials $g_m$, where the $g_m$ have only real roots and where for each $n \in \mathbf{N}$ there exists an $M(n)$ such that for all $m > M(n)$ the roots of $g_m$ lie outside $[0, n]$.)

The functions of class $\mathcal{A}_1$, $\mathcal{A}_2$ (and $\mathcal{A}_3$) are sometimes called Pólya–Laguerre functions (see Karlin [9, Ch. 7]).

Returning to our previous results, Pólya and Schur [14] partially generalized the result of Laguerre's Theorem in the following way.

PÓLYA–SCHUR THEOREM. *The following are equivalent for a sequence* $\{\gamma_k\}_{k=0}^{\infty}$.

1) *If* $\sum_{k=0}^{n} a_k x^k$ *has all real zeros, then* $\sum_{k=0}^{n} a_k \gamma_k x^k$ *has all real zeros, all* $n$.

2) $\sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k$ *has all real zeros of one fixed sign, all* $n$.

3) *The function*

$$\phi(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$$

*is an entire function and either* $\phi(z)$ *or* $\phi(-z)$ *is in* $\mathcal{A}_2$.

In other words, Pólya and Schur considered the "diagonal" linear operators given by

$$Tx^n = \gamma_n x^n, \qquad n = 0, 1, \dots, \tag{1.1}$$

and asked for conditions which exactly characterize when such operators have the property that if $p$ has all real zeros, then $Tp$ has all real zeros.

Iserles, Nørsett and Saff found, in a series of papers (see Iserles, Nørsett and Saff [7] and references therein), numerous examples of linear operators $T$ (taking polynomials of degree $n$ to polynomials of degree $n$, all $n$) with the property that if $p$ has all its zeros in some given real interval $I$ then $Tp$ has all its zeros in some interval $J$. Their motivation and techniques were connected with orthogonal polynomials.

Based on the theorem of Pólya and Schur, it is natural to ask for conditions on a linear operator $T$ of the form (1.1) such that $Z(Tp) \geqq Z(p)$ for every polynomial $p$. This non-trivial question was considered by Craven and Csordas in a series of papers, see e.g. Craven and Csordas [3], Bakan, Craven, Csordas and Golub [2], Bakan, Craven and Csordas [1], and references therein. A direct generalization of Laguerre's Theorem implies that if $g \in \mathcal{A}_3$, and

$$Tx^n = g(n)x^n, \qquad n = 0, 1, \dots,$$

then $Z(Tp) \geqq Z(p)$ for every polynomial $p$, see Karlin [9, p. 382]. In addition we can replace these $g(n)$ by $(-1)^n g(n)$ and the same inequality holds. It is not known if, for $g(0) \neq 0$, this is the full family of such operators.

In this paper we consider a different class of linear operators and totally characterize all such operators satisfying $Z(Tp) \geqq Z(p)$ for every polynomial $p$. Our class of linear operators is given by

$$Tx^n = x^n + \sum_{k=0}^{n-1} b_{n,k} x^k. \tag{1.2}$$

That is, we will consider operators for which the degree of the polynomials $p$ and $Tp$ is the same, but which explicitly do not include operators of the form (1.1), i.e., they are unit lower triangular. The main theorem of this paper is the following result.

THEOREM 1. *Let $T$ be as in* (1.2). *Then*

$$Z(Tp) \geqq Z(p) \tag{1.3}$$

*for every polynomial $p$ if and only if $T = F(D)$ where $F \in \mathcal{A}_1$ with $C = 1$ and $m = 0$. That is,*

$$F(x)=e^{-c^2x^2+ax}\prod_{k=1}^{\infty}(1+\alpha_k x)e^{-\alpha_k x} \tag{1.4}$$

*with* $\alpha_k, a, c \in \mathbf{R}$ *and* $\sum_{k=1}^{\infty}\alpha_k^2<\infty$.

That $F$ of this form satisfies (1.3) is a consequence of the Hermite–Poulain Theorem A. It is the converse result which needs proving. In Section 2 we prove and explain this theorem. In Section 3 we consider the analogous question when we count zeros on $[0,\infty)$. Here the situation is less evident.

## 2. Proof of Theorem 1

We divide the proof of Theorem 1 into a series of steps. We start with the following.

PROPOSITION 1. *Let* $T$ *be as above satisfying* $Z(Tp) \geqq Z(p)$ *for all* $p \in \Pi$. *Then there exist real* $\{c_k\}_{k=1}^{\infty}$ *such that*

$$T=I+\frac{c_1}{1!}D+\frac{c_2}{2!}D^2+\cdots. \tag{2.1}$$

*Furthermore for every* $n$

$$P_n(x)=\sum_{k=0}^{n}\binom{n}{k}c_kx^k$$

*has only real zeros* ($c_0=1$).

PROOF. If $T$ has the form (2.1), then

$$Tx^n=\sum_{k=0}^{n}\binom{n}{k}c_kx^{n-k}$$

has $n$ real zeros (since $x^n$ has $n$ real zeros). Substitute $1/x$ for $x$ and multiply by $x^n$. This implies that $P_n$ has only real zeros.

It remains to prove that $T$ is of the form (2.1). For each $n$

$$Tx^n=\sum_{j=0}^{n}b_{n,j}x^j,$$

with $b_{n,n}=1$. We will prove, by induction on $k$, that

$$b_{n,n-k}=\binom{n}{k}c_k. \tag{2.2}$$

We start with the case $k = 1$. For each $a \in \mathbf{R}$, $Z\big((x+a)^n\big) = n$ and thus

$$Z\big(T(x+a)^n\big) = n.$$

By Rolle's Theorem it therefore follows that $\big[T(x+a)^n\big]^{(n-2)}$ has 2 real zeros. A formal calculation shows that

$$\big[T(x+a)^n\big]^{(n-2)} = (n-2)!\bigg[\frac{n(n-1)}{2}(x+a)^2 + (n-1)b_{n,n-1}(x+a)$$

$$+ a\big(nb_{n-1,n-2} - (n-1)b_{n,n-1}\big) + b_{n,n-2}\bigg].$$

This in turn implies that

$$\frac{n(n-1)}{2}y^2 + (n-1)b_{n,n-1}y + a(nb_{n-1,n-2} - (n-1)b_{n,n-1}) + b_{n,n-2}$$

has 2 real zeros for every $a \in \mathbf{R}$. A necessary condition for this to hold is that the coefficient of $a$ in the above vanish, i.e.,

$$nb_{n-1,n-2} - (n-1)b_{n,n-1} = 0.$$

Setting $b_{1,0} = c_1$ it is then easily seen that for all $n = 1, 2, \ldots$, $b_{n,n-1} = nc_1$.

We now assume, by induction, that

$$b_{n,n-j} = \binom{n}{j}c_j, \qquad j = 1, \ldots, k-1,$$

for all $n \geqq j$. Note that this implies that

$$Tx^n = \bigg(\sum_{j=0}^{k-1}\frac{c_j}{j!}D^j\bigg)x^n + \sum_{j=0}^{n-k}b_{n,j}x^j,$$

where $c_0 = 1$ and $D^0 = I$. We wish to prove (2.2) for all $n \geqq k$.

As previously we consider $T(x+a)^n$. Using the induction a calculation shows that

$$T(x+a)^n = \sum_{j=0}^{k-1}\binom{n}{j}c_j(x+a)^{n-j} + b_{n,n-k}x^{n-k} + b_{n,n-k-1}x^{n-k-1}$$

$$+ nab_{n-1,n-k-1}x^{n-k-1} + \text{lower order terms}.$$

Taking the $(n-k-1)$st derivative gives a polynomial of degree $k+1$ with $k+1$ real roots. It has the exact form

$$\frac{n!}{(k+1)!}(x+a)^{k+1}+\binom{n}{1}c_1\frac{(n-1)!}{k!}(x+a)^k+\cdots$$
$$+\binom{n}{k-1}c_{k-1}\frac{(n-k+1)!}{2!}(x+a)^2+b_{n,n-k}(n-k)!(x+a)$$
$$+a(n-k-1)!\big[nb_{n-1,n-k-1}-(n-k)b_{n,n-k}\big]+b_{n,n-k-1}(n-k-1)!.$$

Let $y=x+a$. In order that this polynomial have $k+1$ real zeros for all $a\in\mathbf{R}$ it is necessary that

$$nb_{n-1,n-k-1}-(n-k)b_{n,n-k}=0.$$

This is true for each $n=k,k+1,\ldots$. Set $b_{k,0}=c_k$. Then (2.2) holds for all $n$. □

The following is well-known, see e.g., Obreschkoff [12, pp. 31–42], Karlin [9, Ch. 7, §2], Widder [18, Ch. 7].

PROPOSITION 2. *Let* $\{c_k\}_{k=0}^{\infty}$ *be real numbers and assume*

$$P_n(x)=\sum_{k=0}^{n}\binom{n}{k}c_kx^k$$

*has only real zeros* ($c_0=1$). *Then the function*

$$F(x)=\sum_{k=0}^{\infty}\frac{c_k}{k!}x^k$$

*is an entire function of the form* (1.4).

In fact $P_n(\cdot/n)$ converges uniformly to $F$ on $[-A,A]$, any $A>0$.

The remaining claim of Theorem 1 is a consequence of this next proposition.

PROPOSITION 3. *Assume that the function*

$$F(x)=\sum_{k=0}^{\infty}\frac{c_k}{k!}x^k$$

*is an entire function of the form* (1.4). *Then* $Z\big(F(D)p\big)\geqq Z(p)$ *for every* $p\in\Pi$.

We first recall the direct proof of this result and then explain the result in more detail.

PROOF. Let $F$ be as above and $g_n$ be any sequence of polynomials with all real zeros which converges uniformly to $F$ on $[-A, A]$, any $A > 0$. One could, for example, take $g_n = P_n(\cdot/n)$ with $P_n$ as above. As $F(0) = 1$, and thus $g_n(0) \neq 0$ for all $n$ sufficiently large, it follows from the Hermite–Poulain Theorem A that $Z\big(g_n(D)p\big) \geqq Z(p)$ for every $p \in \Pi$. From Hurwitz's Theorem concerning convergent sequences of analytic functions (see e.g., Hille [5, p. 205]) it follows that $Z\big(F(D)p\big) \geqq Z(p)$ for every $p \in \Pi$. $\square$

Propositions 1, 2 and 3 together prove Theorem 1.

To understand more fully what $F(D)$ does to each $p$, let us consider each of the factors $e^{-c^2D^2}$, $e^{aD}$ and $I + \alpha D$ applied to $p$. The factor $I + \alpha D$ is what appears in the Hermite–Poulain Theorem A. We will not discuss it again.

Assume $g(x) = e^{ax}$. Then the operator $e^{aD}$ applied to $p$ is just the shift operator. That is,

$$g(D)x^n = e^{aD}x^n = \sum_{k=0}^{\infty} \frac{a^k D^k}{k!} x^n = \sum_{k=0}^{n} \binom{n}{k} a^k x^{n-k} = (x+a)^n.$$

Thus for every $p$, $\big(g(D)p\big)(x) = p(x+a)$. Obviously we have $Z\big(g(D)p\big) = Z(p)$.

Assume $g(x) = e^{-c^2x^2}$. Before explaining the operator $e^{-c^2D^2}$, we note that

$$e^{-c^2D^2}x^n = \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!k!} (-1)^k x^{n-2k} c^{2k}.$$

The $n$th degree Hermite polynomial (with leading coefficient $2^n$) is given by

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!k!} (-1)^k (2x)^{n-2k}.$$

Thus

$$e^{-c^2D^2}x^n = c^n H_n\left(\frac{x}{2c}\right).$$

The inequality $Z\big(g(D)p\big) \geqq Z(p)$ thus may be seen to be equivalent to

$$Z\left(\sum_{k=0}^{n} a_k H_k(x)\right) \geqq Z\left(\sum_{k=0}^{n} a_k x^k\right)$$

for all real $\{a_k\}_{k=0}^n$ and all $n$. This inequality is a special case of what was proven above. We present another proof. It is lengthier, but worth noting. It is taken from Iserles and Saff [8]. They were only interested in one particular consequence, namely if $\sum_{k=0}^{n} a_k x^k$ has only real zeros, then $\sum_{k=0}^{n} a_k H_k(x)$ has only real zeros. But of course, more is true.

The proof is based on the well-known extended total positivity (ETP) property (on $\mathbf{R}\times\mathbf{R}$) of $e^{-(x-t)^2}$ (or equivalently of $e^{xt}$), see Karlin [9]. We first show that

$$e^{2xt-t^2}=\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!}.$$

This may be found, for example, in Rainville [17].

$$\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]}\frac{n!}{(n-2k)!k!}(-1)^k(2x)^{n-2k}\right)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{(n-2k)!k!}(2x)^{n-2k}t^n=\sum_{k=0}^{\infty}\sum_{n=2k}^{\infty}\frac{(-1)^k}{(n-2k)!k!}(2x)^{n-2k}t^n$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{k!}\sum_{n=2k}^{\infty}\frac{(2xt)^{n-2k}}{(n-2k)!}=e^{-t^2}e^{2xt}.$$

We also make use of the well-known orthogonality conditions

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_n(x)H_m(x)e^{-x^2}\,dx=\delta_{nm}2^n n!.$$

Finally we note that

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_n(x)e^{-(x-t)^2}\,dx=(2t)^n.$$

This is a consequence of

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_n(x)e^{-(x-t)^2}\,dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_n(x)e^{2xt-t^2}e^{-x^2}\,dx$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_n(x)\left(\sum_{m=0}^{\infty}\frac{H_m(x)t^m}{m!}\right)e^{-x^2}\,dx$$

$$=\sum_{m=0}^{\infty}\frac{t^m}{m!}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_n(x)H_m(x)e^{-x^2}\,dx=\frac{t^n}{n!}2^n n!=(2t)^n.$$

With these preliminaries, we can now prove the result. As mentioned, the ETP property of $e^{-(x-t)^2}$ is well-known. As such for any polynomial $p$

$$Z\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} p(x)e^{-(x-t)^2}\,dx\right) \leqq Z(p).$$

In particular, setting $p(x)=\sum_{k=0}^{n} a_k H_k(x)$ we get

$$Z\left(\sum_{k=0}^{n} a_k (2t)^k\right) \leqq Z\left(\sum_{k=0}^{n} a_k H_k(x)\right).$$

Since $Z\left(\sum_{k=0}^{n} a_k (2t)^k\right) = Z\left(\sum_{k=0}^{n} a_k x^k\right)$ this implies

$$Z\left(\sum_{k=0}^{n} a_k x^k\right) \leqq Z\left(\sum_{k=0}^{n} a_k H_k(x)\right).$$

Assume $T$ has the above form, i.e., $Z(Tp) \geqq Z(p)$ for every polynomial $p$, and $Tx^n = x^n + \cdots$. What can we say about the inverse of $T$?

There are two cases to consider depending on the exact form of the above $F$. If $F(x)=e^{ax}$, then $(Tp)(x)=p(x+a)$ and obviously the inverse operator is simply a shift by $-a$.

Assume $F$ is not of this form. Then it follows from Karlin [9, Ch. 7], that $1/F$ is the Laplace transform of a Pólya frequency (PF) density. That is,

$$\frac{1}{F(x)} = \int_{-\infty}^{\infty} e^{-tx}\Lambda(t)\,dt$$

where $\Lambda(t-s)$ is totally positive (TP), and

$$\int_{-\infty}^{\infty}\Lambda(t)\,dt = 1.$$

Furthermore if $p$ and $q$ are polynomials and $F(D)p=q$ then

$$p(x) = \int_{-\infty}^{\infty}\Lambda(x-t)q(t)\,dt.$$

In other words, this integral operator is the inverse of $T$. We therefore have the following.

THEOREM 2. *Assume $S$ is a linear operator on $\Pi$ of the specific form*

$$Sx^n = x^n + \sum_{j=0}^{n-1} a_{n,j} x^j,$$

*which satisfies $Z(Sq) \leqq Z(q)$ for every polynomial $q$. Then either* (i) *or* (ii) *hold.*

(i) *$(Sq)(x) = q(x-a)$ for some $a \in \mathbf{R}$.*

(ii) *There is a PF density $\Lambda$ such that*

$$(Sq)(x) = \int_{-\infty}^{\infty} \Lambda(x-t) q(t)\, dt.$$

REMARK. As previously, let

$$Tx^n = \sum_{k=0}^{n} \binom{n}{k} c_{n-k} x^k.$$

The operator $S$ can also be written in this form (calculate or see Karlin [9, p. 344]). It is

$$Sx^n = \sum_{j=0}^{n} \binom{n}{j} \mu_{n-j} (-1)^{n-j} x^j,$$

where

$$\mu_j = \int_{-\infty}^{\infty} \Lambda(t) t^j\, dt.$$

(Since $\Lambda$ is non-negative, all $\mu_{2j}$ are strictly positive.) The relations between the $c_k$ and $\mu_j$ are given by $c_0 = \mu_0 = 1$ and

$$\sum_{j=0}^{m} \binom{m}{j} c_{m-j} \mu_j (-1)^j = 0$$

for all $m \geqq 1$.

We end this section by asking how one can characterize the linear operators of the form

$$Tx^n = \sum_{k=0}^{n} b_{n,k} x^k$$

(here $b_{n,n}$ is not necessarily equal to 1 for all $n$) which satisfy $Z(Tp) \geqq Z(p)$ for every $p \in \Pi$. Can all such operators be factored into products of operators of the form given in Theorem 1 and those "diagonal" operators which have this same zero-increasing property? Unfortunately these two sets of operators do not commute. As such, any factorization will not be simple.

## 3. The interval $[0,\infty)$

It follows from the Hermite–Poulain Theorem B that if $g$ is a polynomial with all positive zeros then

$$Z_{[0,\infty)}\bigl(g(D)p\bigr) \geqq Z_{[0,\infty)}(p)$$

for every $p \in \Pi$. The correct closure of the set of all polynomials with all positive zeros (normalized to be 1 at $z=0$) is a subset of $\mathcal{A}_2$. It is the set of entire functions of the form

$$F(z) = e^{-az} \prod_{k=1}^{\infty} (1-\alpha_k z) \tag{3.1}$$

where $a \geqq 0$, $\alpha_k \geqq 0$, and $\sum_{k=1}^{\infty} \alpha_k < \infty$. Thus for each such $F$

$$Z_{[0,\infty)}(F(D)p) \geqq Z_{[0,\infty)}(p)$$

for every $p \in \Pi$.

It is natural to conjecture, paralleling Theorem 1, that any linear operator $T$ of the form

$$Tx^n = x^n + \sum_{k=0}^{n-1} b_{n,k} x^k$$

satisfying

$$Z_{[0,\infty)}(Tp) \geqq Z_{[0,\infty)}(p) \tag{3.2}$$

for every $p \in \Pi$ is of the form $T = F(D)$ for some $F$ as in (3.1).

We do prove an analogue of Theorem 1. However we shall show that the above conjecture is not valid.

THEOREM 3. *Let $T$ be a linear operator on $\Pi$ of the form*

$$Tx^n = x^n + \sum_{k=0}^{n-1} b_{n,k} x^k.$$

*The following properties are equivalent:*

(i) *$T = F(D)$ where $F$ is of the form* (3.1).

(ii) $Z_{[c,\infty)}(Tp) \geqq Z_{[c,\infty)}(p)$ *for every* $p \in \Pi$ *and for all* $c \geqq 0$.
(iii) $Z_{[c,\infty)}(Tp) \geqq Z_{[c,\infty)}(p)$ *for every* $p \in \Pi$ *and for all* $c \in \mathbf{R}$.
(iv) $Z(Tp) \geqq Z(p)$ *and* $Z_{[0,\infty)}(Tp) \geqq Z_{[0,\infty)}(p)$ *for every* $p \in \Pi$.

PROOF. (i) $\Rightarrow$ (iii). Using the same idea as in the proof of the Hermite–Poulain Theorem it is easily verified that if $p$ has $k$ zeros greater than or equal to $c$, then $(I - \alpha D)p$, $\alpha \geqq 0$, has at least $k$ zeros greater than or equal to $c$. Similarly $(e^{-aD}p)(x) = p(x-a)$, $a \geqq 0$, shifts zeros to the right by $a$ units. Thus (iii) holds.

(iii) $\Rightarrow$ (iv). For a given $p$ we show $Z(Tp) \geqq Z(p)$ by taking $c \in \mathbf{R}$ less than the least real zero of $p$. Taking $c = 0$ completes the proof.

(iv) $\Rightarrow$ (i). Since $Z(Tp) \geqq Z(p)$ for all $p \in \Pi$, we have from Theorem 1 that $T = F(D)$ for some $F$ of the form (1.4). Since $Z_{[0,\infty)}(Tx^n) = n$ this implies that

$$P_n(x) = \sum_{k=0}^{\infty} \binom{n}{k} c_k x^k$$

has only positive zeros, $P_n(0) = 1$. As $P_n(\cdot/n)$ converges to $F$, this implies that $F$ is of the form (3.1).

(iii) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (i). We follow the analysis in the proof of Proposition 1, in that we prove that

$$T = I + \frac{c_1}{1!}D + \frac{c_2}{2!}D^2 + \cdots, \tag{3.3}$$

and

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} c_k x^k$$

has only positive zeros ($c_0 = 1$) for each $n$. From this latter fact we obtain (i) (see the proof of (iv) $\Rightarrow$ (i)). If $T$ has the form (3.3) then $P_n$ has only positive zeros since $Z_{[0,\infty)}(Tx^n) = n$.

It remains to prove that $T$ has the form (3.3). We follow the proof of Proposition 1. We first consider the case $k = 1$. We let $a$ therein satisfy $a \leqq 0$, and set $c = -a \geqq 0$ in (ii). The only relevant point is that $T\big((x+a)^n\big)$ has $n$ zeros in $[-a, \infty)$, and thus its $(n-2)$nd derivative evaluated at $y = x + a$ must have two nonnegative zeros for all $a \leqq 0$. This $(n-2)$nd derivative is given by (up to a constant)

$$\frac{n(n-1)}{2} y^2 + (n-1) b_{n,n-1} y + a(n b_{n-1,n-2} - (n-1) b_{n,n-1}) + b_{n,n-2}.$$

If $n b_{n-1,n-2} - (n-1) b_{n,n-1}$ is positive then taking $a$ sufficiently large we obtain a polynomial with a negative zero, a contradiction. If $n b_{n-1,n-2} -$

$(n-1)b_{n,n-1}$ is negative then taking $a$ sufficiently large we obtain a polynomial which is positive on $[0,\infty)$, again a contradiction. Therefore $nb_{n-1,n-2} = (n-1)b_{n,n-1}$.

Similar arguments can be applied to

$$\frac{n!}{(k+1)!}y^{k+1} + \binom{n}{1}c_1\frac{(n-1)!}{k!}y^k + \cdots + \binom{n}{k-1}c_{k-1}\frac{(n-k+1)!}{2!}y^2$$
$$+ b_{n,n-k}(n-k)!y + a(n-k-1)!\big[nb_{n-1,n-k-1} - (n-k)b_{n,n-k}\big]$$
$$+ b_{n,n-k-1}(n-k-1)!$$

to further the induction on $k$ as in the proof of Proposition 1. □

REMARK. Let us observe that the property $Z_{[c,\infty)}(Tp) \geqq Z_{[c,\infty)}(p)$ for every $p \in \Pi$ and $c \in \mathbf{R}$ is equivalent to the demand that if $x_1 \geqq \cdots \geqq x_r$ are the real zeros of $p$, then there exist $y_1 \geqq \cdots \geqq y_r$, zeros of $Tp$, such that $y_i \geqq x_i$, $i = 1, \ldots, r$.

Notwithstanding the above result, it is not true that every $T$ satisfying (3.2) is of the form $T = F(D)$ for some $F$ satisfying (3.1). Here are two examples.

EXAMPLE 1. The function $y^x$ is ETP on $(0,\infty) \times [0,\infty)$. (This follows since $y^x = e^{x \ln y}$, and $e^{st}$ is ETP on $\mathbf{R} \times \mathbf{R}$.) Set

$$d\psi(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\delta_k.$$

where $\delta_k$ is the point measure of mass one at $k$. The operator

$$(Sf)(y) = e^{-y}\int_0^\infty y^x f(x)\, d\psi(x)$$

satisfies $Z_{(0,\infty)}(Sf) \leqq Z_{[0,\infty)}(f)$ and for polynomials $p$ we in fact have $Z_{[0,\infty)}(Sp) \leqq Z_{[0,\infty)}(p)$. This operator is related to what Karlin [9, p. 446] calls the Poisson–Charlier polynomials, and what Iserles and Nørsett [6, p. 495] call the Charlier transformation.

It is easily checked that $S1 = 1$, $Sx = y$, and

$$\Big(S\big(x(x-1)(x-2)\cdots\big(x-(n-1)\big)\big)\Big)(y) = y^n.$$

Thus if we let $T$ be the operator satisfying $T1 = 1$, $Tx = y$, and

$$(Tx^n)(y) = y(y-1)(y-2)\cdots\big(y-(n-1)\big)$$

then $Z_{[0,\infty)}(Tp) \geqq Z_{[0,\infty)}(p)$, for every polynomial $p$.

This operator $T$ is not of the form $T = F(D)$ for some $F$ as in (3.1). For if we write

$$Tx^n = y^n + \sum_{k=0}^{n-1} b_{n,k} y^k$$

then $b_{1,0} = 0$ and $b_{n,n-1} \neq 0$ for all $n \geqq 2$. This property does not hold for any $F(D)$, where we must have $b_{n,n-1} = nb_{1,0}$. Another reason why $T$ cannot be of the form $F(D)$ is that $T$ does not commute with the factors of $F(D)$.

By a change of variable in $T$ or doing the same in the definition, it also easily follows that

$$(T_c x^n)(y) = y(y-c)(y-2c)\cdots\bigl(y-(n-1)c\bigr)$$

also has this same property for any $c > 0$. Furthermore, so does $T^2$, $T^3$, etc. Unfortunately $T^2$ is not $T_2$ or any $T_c$ (consider the polynomial $T^2x^3$ (and $T^2x^2$)).

EXAMPLE 2. Because $x^{y-1}$ is ETP on $(0,\infty) \times [0,\infty)$ (see Example 1), the operator

$$(Uf)(y) = \frac{1}{\Gamma(y)} \int_0^\infty x^{y-1} e^{-x} f(x)\, dx$$

(see Iserles and Nørsett [6, p. 493], and Iserles, Nørsett and Saff [7, p. 343]) satisfies $Z_{(0,\infty)}(Uf) \leqq Z_{(0,\infty)}(f)$. Now for $n = 1, 2, \ldots$

$$(Ux^n)(y) = \frac{1}{\Gamma(y)} \int_0^\infty x^{n+y-1} e^{-x}\, dx = \frac{\Gamma(y+n)}{\Gamma(y)} = y(y+1)\cdots(y+n-1)$$

(set $(U1)(y) = 1$).

For every polynomial $p$ we have $Z_{[0,\infty)}(Up) \leqq Z_{[0,\infty)}(p)$. The operator $V = U^{-1}$ is not of the form $F(D)$.

These examples demonstrate that we do not yet understand the full class of linear operators $L$ of the form

$$(Lx^n)(y) = y^n + \sum_{k=0}^{n-1} b_{n,k} y^k$$

which satisfy $Z_{[0,\infty)}(Lp) \geqq Z_{[0,\infty)}(p)$, for every polynomial $p$. All we can say is that this class seems to be rather complicated.

Note that the $S$ and $U$ we constructed above are still based on TP kernels, but they are not difference kernels (which is essentially what we obtained in Section 2).

REMARK. The appropriate analogue of Theorems 1 and 3 on any finite interval is elementary. The only linear operator $T$ of the form

$$Tx^n = x^n + \sum_{k=0}^{n-1} b_{n,k} x^k$$

satisfying $Z_{[a,b]}(Tp) \geqq Z_{[a,b]}(p)$ for every $p \in \Pi$ on any finite interval $[a,b]$, is the identity operator.

One proof of this result is by induction on $n$. For $n = 1$ and all $c \in [a,b]$ we have $T(x-c) = (x-c) + b_{1,0}$ and $Z_{[a,b]}\big((x-c)\big) = 1$. Thus $Z_{[a,b]}\big((x-c) + b_{1,0}\big) = 1$ for all $c \in [a,b]$, implying $b_{1,0} = 0$. Assume, by induction, that $b_{m,k} = 0$ for all $m \leqq n-1$ and all relevant $k$. Then

$$T\big[(x-c)^n\big] = (x-c)^n + \sum_{k=0}^{n-1} b_{n,k} x^k.$$

For $c \in [a,b]$, we again have $Z_{[a,b]}\big((x-c)^n\big) = n$ and thus $Z_{[a,b]}\big(T\big[(x-c)^n\big]\big) = n$. There are now various methods of proving that $b_{n,k} = 0$. For example, take consecutively the $(n-k)$th derivative, $k = 1, \dots, n$, to show that $b_{n,n-k} = 0$.

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