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Preface

Special issue on sparse approximate solution of linear systems

Recent years have witnessed an impressive research activity in the area of compressed sensing and sparse approximation. The basic mathematical problem consists in solving an underdetermined linear system, i.e., $y = Ax$, A being an $m \times n$ matrix with $m \ll n$, under certain a priori assumptions on the vector x. Within this theory the assumed prior information is that the vector x is approximately sparse, in the sense that the error of best k-term approximation $\min_{\{z:\ \|z\|_0 \le k\}} \|x - z\|$, with $\|z\|_0 = \#\{i : z_i \neq 0\}$, $k < n$ and some norm $\|\cdot\|$, is small. This assumption can be justified as reasonable, since the information content of a high-dimensional data vector is typically much lower than the dimension of the ambient space, sometimes only revealed by the application of a transform to the respective vector. Consider as an example the sparsity of the coefficient vector of a natural image with respect to a wavelet basis, which is key to the success of the JPEG2000 compression algorithm. While the ℓ_0-minimization approach to identifying such solutions x turns out to be NP hard, it came as a surprise that efficient alternatives such as convex relaxation (ℓ_1-minimization) and greedy algorithms are provably effective in identifying the sparsest solution – if one exists – under suitable (incoherence) assumptions on the matrix A. Hence, the success of compressed sensing methodology is always an interplay between sparsity assumptions, assumptions on the measurement matrix, and a careful selection of the recovery algorithm. If these assumptions are appropriately fulfilled, then the theory of compressed sensing shows that it is possible to recover an approximately sparse vector from incomplete linear, non-adaptive measurements with a high degree of accuracy. In the special case of an exactly sparse vector, it can even be recovered completely. This key observation has in fact led to a paradigm change, for instance, in the area of signal processing by now regarding sparsity as a natural assumption.

This exciting research area has by now reached a mature state with an established comprehensive core theory. We now see fascinating novel directions emerging, which we will highlight next. As already mentioned, classical results typically assume exact or approximate sparsity. To invoke additional known properties of the data, the notion of structured sparsity has recently become a focus of intense research. These studies include, for example, the analysis of vectors which are sparse under some transform given by a basis, or more generally a frame – predefined such as wavelets or derived by dictionary learning algorithms – or whose non-zero entries are geometrically clustered. The novel direction of matrix completion which considers sparsity in terms of the rank of a matrix can also be seen from this perspective. The design of measurement matrices has been another focus of recent research activities. A customarily assumed property of the measurement matrix is the restricted isometry property, which is in some sense optimally satisfied for random matrices such as Gaussian matrices. A fundamental open problem is how to build deterministic matrices with similar sparse recovery properties. One prominent direction of research are structured random matrices such as partial Fourier matrices with randomly selected rows. On the (recovery) algorithmic front, due to

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the interest in signals whose (analysis) frame coefficients are sparse, the recovery algorithms now take this into account by, for instance, placing the ℓ_1 norm on the analysis coefficients rather than the signal itself. However, a theoretical underpinning is almost completely missing. The novel theory of co-sparsity can be regarded as the first very promising approach in this direction. Finally, intense research on applications of the rich theory of compressed sensing and sparse approximation can be recently observed, achieving very promising advances in various areas such as communication theory, imaging sciences, optics, radar technology, sensor networks, and tomography.

As can already be seen, the area of compressed sensing and sparse approximation has close links to a variety of different research areas, and we would like to mention applied harmonic analysis, approximation theory, geometric functional analysis, numerical linear algebra, random matrix theory, and optimization theory. As varied as are the involved methods, as diverse are the researchers involved in advancing both theory and applications ranging from applied mathematicians to computer scientists to electrical engineers. Hence, this is a truly interdisciplinary research field.

One purpose of this special issue is to present a bouquet of publications focusing on different novel research directions in the area of compressed sensing and sparse approximation, in particular, from a linear algebra perspective. To our mind, the rich canon of methods from applied and numerical linear algebra could bring significant advances on several fronts of this area. We therefore regard a special issue on this topic in the journal Linear Algebra and its Applications as a particularly fortunate opportunity to bring this research field, its current research directions, and open problems to its reader's attention.

We now provide a brief survey of the content of this special issue. For this, we start with the broad topic of sparsity conditions. The article on "On unified view of nullspace-type conditions for recoveries associated with general sparsity structures" by Juditsky, Karzan, and Nemirovski introduces, for the first time, a unified view of several known structured sparsity models such as block-sparsity or low rank sparsity, also including the classical notion of exact sparsity. In particular, the authors derive nullspace-type sufficient conditions for recovery of signals being sparse within this general framework.

The design of the measurement matrix is an important question, also due to possible constraints posed by applications. This special issue contains a number of results in this area. The article entitled "Bounds of restricted isometry constants in extreme asymptotics: formulae for Gaussian matrices" by Bah and Tanner studies restricted isometry constants, which provide a means of measuring how far away from an isometry a matrix can be when applied to sparse vectors. The authors give very precise bounds on the range of such constants for Gaussian matrices as measurement matrices in different settings of the relation between the ambient dimension, the reduced dimension, and the sparsity. A second article, which is on "Stability and robustness of ℓ_1-minimizations with Weibull matrices and redundant dictionaries" authored by Foucart, analyzes Weibull random matrices as measurement matrices. In addition, considering sparsity within a frame – relating to structured sparsity as well – he proves that equality-constrained ℓ_1-minimization remains stable and robust. Interesting key ingredients of his analysis are the robust null space property and the quotient property. Another intriguing viewpoint is presented in the article on "RIPless compressed sensing from anisotropic measurements" by Kueng and Gross. The authors go beyond the customarily analyzed scenarios in which the covariance matrix associated with the measurement matrix is proportional to the identity matrix. In this work, they consider the setting of non-trivial covariance matrices, and prove by utilizing ideas from convex geometry that the required sampling rate for sparse recovery grows proportionally to the condition number of the covariance matrix. Another article focussing "On deterministic sketching and streaming for sparse recovery and norm estimation" in this series, authored by Nelson, Nguyen, and Woodruff, considers a specifically constructed deterministic measurement matrix for various recovery procedures such as ℓ_1/ℓ_1 and ℓ_∞/ℓ_1 sparse recovery problems aiming at studying the deterministic complexities of these problems. The very different article, entitled "Paved with good intentions: analysis of a randomized block Kaczmarz method", by Needell and Tropp shows, by comparison, the situation when solving overdetermined least-squares problems. In this work the authors consider the block Kaczmarz method, which is an iterative scheme for solving such problems, and introduce a variant which uses a randomized control scheme to choose the subset at each step. They then prove that the rate of convergence is linear, and that it can be expressed in terms of the geometric properties of the matrix and its submatrices.

Current research on recovery algorithms is represented by two articles in this special volume. With regards to the novel direction of co-sparsity theory, the article on “Greedy-like algorithms for the cosparse analysis model” by Giryes, Nam, Elad, Gribonval, and Davies introduces a new, promising family of pursuit algorithms for the cosparse analysis model, proves theoretical guarantees for success, and provides several empirical performance studies. Another contribution is the article entitled “Sparse polynomial interpolation in Chebyshev bases” by Potts and Tasche, which takes a very different point of view. In this work, the problem of reconstruction of sparse polynomials in a basis of Chebyshev polynomials from samples on a Chebyshev grid is considered. The authors employ the Prony method, which is in particular a completely deterministic approach, and derive theoretical recovery guarantees depending on the sparsity and the number of sample values.

Finally, focussing more on applicability of such methods, one immediate question concerns the robustness with respect to quantization. This is precisely the content of the article on “One-bit compressed sensing with non-Gaussian measurements” by Ai, Lapanowski, Plan, and Vershynin. They consider the even most extreme situation of 1-bit quantization in combination with a measurement matrix with entries given by a sub-Gaussian distribution. In this setting, it is proven that approximately sparse signals, which also satisfy a mild infinity-norm constraint, can be stably recovered.

It is our hope, that this special issue will whet the reader’s interest and encourage her/him to delve more deeply into this interesting and active research area, thereby also enhancing the field by inserting presumably different theoretical viewpoints and canons of methodologies.

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